Lecture 4

PROPERTIES OF FLUIDS

1.1. INTRODUCTION

Hydraulics:

Hydraulics (this word has been derived from a Greek work 'Hudour' which means water) may be defined as follows :

"It is that branch of Engineering-science, which deals with water (at rest or in motion)."

or

"It is that branch of Engineering-science which is based on experimental observation of water flow."

Fluid Mechanics:

Fluid mechanics may be defined as *that branch of Engineering-science which deals with the behaviour of fluid under the conditions of rest and motion.*

The fluid mechanics may be divided into three parts: *Statics*, *kinematics* and *dynamics*.

Statics. The study of incompressible fluids under static conditions is called *hydrostatics* and that dealing with the compressible static gases is termed as *aerostatics*.

Kinematics. It deals with the velocities, accelerations and the patterns of flow only. Forces or energy causing velocity and acceleration are *not* dealt under this heading.

Dynamics. It deals with the relations between velocities, accelerations of fluid *with the forces or energy causing them*.

Properties of Fluids–General Aspects:

The matter can be classified on the basis of the *spacing between the molecules* of the matter as follows:

1. Solid state, and
2. Fluid state,
   (*i*) Liquid state, and (*ii*) Gaseous state.

In *solids*, the molecules are very closely spaced whereas in *liquids* the spacing between the different molecules is relatively large and in *gases* the spacing between the molecules is still large. It means that inter-molecular cohesive forces are *large* in solids, *smaller* in liquids and *extremely small* in gases, and on account of this fact, solids possess compact and rigid form, liquid molecules can move freely within the liquid mass and the molecules of gases have greater freedom of movement so that the gases fill the container completely in which they are placed.

A *solid* can resist tensile, compressive and shear stresses upto a certain limit whereas a fluid has no tensile strength or very little of it and it can resist the compressive forces only when it is kept in a container. When a fluid is subjected to a shearing force it deforms continuously as long as the force is applied. The amount of shear stress in a fluid depends on the magnitude of the rate of deformation of the fluid element.

Liquids and *gases* exhibit different characteristics. The liquids under ordinary conditions are quite difficult to compress (and therefore they may for most purposes be regarded as incompressible) whereas gases can be compressed much readily under the action of external pressure (and when the external pressure is removed the gases tend to expand indefinitely).

1.2. FLUID

A fluid may be defined as follows:

"A fluid is a substance which is capable of flowing."

or

"A fluid is a substance which deforms continuously when subjected to external shearing force."

A fluid has the following *characteristics*:

1. It has no definite shape of its own, but conforms to the shape of the containing vessel.
2. Even a small amount of shear force exerted on a liquid/fluid will cause it to undergo a deformation which continues as long as the force continues to be applied.

A fluid may be *classified* as follows:

A. (*i*) *Liquid*, (*ii*) *Gas*, (*iii*) *Vapour*.

B. (*i*) *Ideal fluids* (*ii*) *Real fluids*.

Liquid

- A liquid is a fluid which possesses a *definite volume* (which varies only slightly with temperature and pressure).
- Liquids have bulk elastic modulus when under compression and will store up energy in the same manner as a solid. As the contraction of volume of a liquid under compression is extremely small, it is usually ignored and the *liquid is assumed to be incompressible*. A liquid will withstand a slight amount of tension due to molecular attraction between the particles which will cause an apparent shear resistance, between two adjacent layers. This phenomenon is known as **viscosity**.
- All known liquids vaporise at narrow pressures above zero, depending on the temperature.

Gas. It possesses *no definite volume* and is *compressible*.

Vapour. It is a gas whose temperature and pressure are such that it is very near the liquid state (*e.g.*, steam).

Ideal fluids. An ideal fluid is one which has *no viscosity* and *surface tension* and is *incompressible*. In true sense no such fluid exists in nature. However fluids which have low viscosities such as water and air can be treated as ideal fluids under certain conditions. The assumption of ideal fluids helps in simplifying the mathematical analysis.

Real fluids. *A real practical fluid is one which has viscosity, surface tension and compressibility in addition to the density*. The real fluids are actually available in nature.

Continuum. *A continuous and homogeneous medium is called* ***continuum.*** From the *continuum view point, the overall properties and behaviour of fluids can be studied without regard for its atomic and molecular structure.*

1.3. LIQUIDS AND THEIR PROPERTIES

- Liquid can be easily distinguished from a solid or a gas.
- Solid has a definite shape.
- A liquid takes the shape of vessel into which it is poured.
- A gas completely fills the vessel which contains it.

The properties of water are of much importance because the subject of hydraulics is mainly concerned with it. Some important properties of water which will be considered are:

(*i*) Density,	(*ii*) Specific gravity,	(*iii*) Viscosity,
(*iv*) Vapour pressure,	(*v*) Cohesion,	(*vi*) Adhesion,
(*vii*) Surface tension,	(*viii*) Capillarity, and	(*ix*) Compressibility.

1.4. DENSITY

1.4.1 Mass Density

The *density* (also known as *mass density or specific mass*) of a liquid may be defined as *the mass per unit volume* $\left(\frac{m}{V}\right)$ *at a standard temperature and pressure*. It is usually denoted by ρ (rho).

Its units are kg/m^3, *i.e.*, $$\rho = \frac{m}{V} \qquad ...(1.1)$$

1.4.2 Weight Density

The weight density (also known as specific weight) is defined as the *weight per unit volume at the standard temperature and pressure*. It is usually denoted by *w*.

$$w = g \qquad ...(1.2)$$

For the purposes of all calculations, relating to Hydraulics and hydraulic machines, the specific weight of water is taken as follows:

In S.I. Units: $w = 9.81\ \text{kN/m}^3$ (or $9.81 \times 10^{-6}\ \text{N/mm}^3$)

In M.K.S. Units: $w = 1000\ \text{kg}_f/\text{m}^3$

1.4.3 Specific volume

It is defined as *volume per unit mass of fluid*. It is denoted by *v*.

Mathematically, $$v = \frac{V}{m} = \frac{1}{\rho} \qquad ...(1.3)$$

1.5. SPECIFIC GRAVITY

Specific gravity is the ratio of the specific weight of the liquid to the specific weight of a standard fluid. It is dimensionless and has no units. It is represented by *S*.

For liquids, the standard fluid is pure water at 4°C.

$$\therefore \qquad \text{Specific gravity} = \frac{\text{Specific weight of liquid}}{\text{Specific weight of pure water}} = \frac{w_{liquid}}{w_{water}}$$

Example 1.1. *Calculate the specific weight, specific mass, specific volume and specific gravity of a liquid having a volume of 6 m^3 and weight of 44 kN.*

Solution: Volume of the liquid = 6 m^3

Weight of the liquid = 44 kN

Specific weight, *w* :

$$w = \frac{\text{Weight of liquid}}{\text{Volume of liquid}} = \frac{44}{6} = \mathbf{7.333\ kN/m^3}\ \textbf{(Ans.)}$$

Specific mass or mass density, ρ :

$$\rho = \frac{w}{g} = \frac{7.333 \times 1000}{9.81} = \mathbf{747.5\ kg/m^3}\ \textbf{(Ans.)}$$

$$\text{Specific volume,} \quad v = \frac{1}{\rho} = \frac{1}{747.5} = \mathbf{0.00134\ m^3/kg}\ \textbf{(Ans.)}$$

Specific gravity, *S* :

$$S = \frac{w_{liquid}}{w_{water}} = \frac{7.333}{9.81} = \mathbf{0.747}\ \textbf{(Ans.)}$$

1.6. VISCOSITY

Viscosity may be defined as the *property of a fluid which determines its resistance to shearing stresses*. It is a measure of the internal fluid friction which causes resistance to flow. It is primarily *due to cohesion and molecular momentum exchange between fluid layers*, and as flow occurs, these effects appear as shearing stresses between the moving layers of fluid.

An *ideal fluid has no viscosity*. There is no fluid which can be classified as a perfectly ideal fluid. However, the fluids with very little viscosity are sometimes considered as ideal fluids.

Viscosity of fluids is due to *cohesion and interaction between particles*.

Refer to Fig 1.1. When two layers of fluid, at a distance '*dy*' apart, move one over the other at different velocities, say u and $u + du$, the viscosity together with relative velocity causes a shear stress acting between the fluid layers. The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. This shear stress is proportional to the rate of change of velocity with respect to y. It is denoted by τ (called Tau).

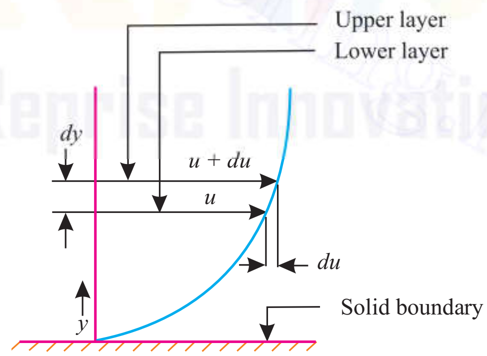


Fig. 1.1 Velocity variation near a solid boundary.

$$\text{Mathematically} \qquad \tau \propto \frac{du}{dy}$$

$$\text{or} \qquad \tau = \mu . \frac{du}{dy} \qquad ...(1.4)$$

where, μ = Constant of proportionality and is known as *co-efficient of* ***dynamic viscosity*** or *only viscosity*.

$\frac{du}{dy}$ = Rate of shear stress or rate of shear deformation or velocity gradient.

From Fig. 1.1, we have $$\mu = \frac{\tau}{\left[\frac{du}{dy}\right]} \quad \text{...(1.5)}$$

Thus viscosity may also be defined as the *shear stress required to produce unit rate of shear strain.*

Units of Viscosity:

In *S.I. Units*: N.s/m^2

In *M.K.S. Units:* kg$_f$.sec/m^2

$$\left[\because \ \mu = \frac{\text{force/area}}{(\text{length/time}) \times \frac{1}{\text{length}}} = \frac{\text{force/length}^2}{\frac{1}{\text{length}}} = \frac{\text{force} \times \text{time}}{(\text{length})^2}\right]$$

The unit of viscosity in C.G.S. is also called *poise* = $\frac{\text{dyne} - \text{sec}}{\text{cm}^2}$. One poise = $\frac{1}{10}$ N.s/m^2

Note. The viscosity of water at 20°C is $\frac{1}{100}$ poise or one centipoise.

Kinematic Viscosity :

Kinematic viscosity is defined as the *ratio between the dynamic viscosity and density of fluid.* It is denoted by ν (called nu).

$$\text{Mathematically,} \quad \nu = \frac{\text{Viscosity}}{\text{Density}} = \frac{\mu}{\rho} \quad \text{...(1.6)}$$

Units of kinematic viscosity:

In SI units: m^2/s

In M.K.S. units: m^2/sec.

In C.G.S. units the kinematic viscosity is also known as stoke (= cm^2/sec.)

One stoke = 10^{-4} m^2/s

Note: *Centistoke* means $\frac{1}{100}$ stoke.

1.6.1. Newton's Law of Viscosity

This law states that the *shear stress* (τ) *on a fluid element layer is directly proportional to the rate of shear strain*. The constant of proportionality is called the *co-efficient of viscosity*.

$$\text{Mathematically,} \quad \tau = \mu\frac{du}{dy} \quad \text{...(1.7)}$$

The fluids which follow this law are known as *Newtonian fluids*.

1.6.2. Types of Fluids

The fluids may be of the following types:

Refer to Fig.1.2.

1. Newtonian fluids. These fluids follow Newton's viscosity equation (*i.e.* eqn. 1.7). For such fluids μ *does not change with rate of deformation*.

Examples. Water, kerosene, air etc.

2. Non-Newtonian fluids. Fluids which *do not* follow the linear relationship between shear stress and rate of deformation (given by eqn. 1.7) are termed as *Non-Newtonian fluids*. Such fluids are relatively uncommon.

Examples. *Solutions or suspensions (slurries), mud flows, polymer solutions, blood* etc. These fluids are generally complex mixtures and are studied under *rheology*, a science of deformation and flow.

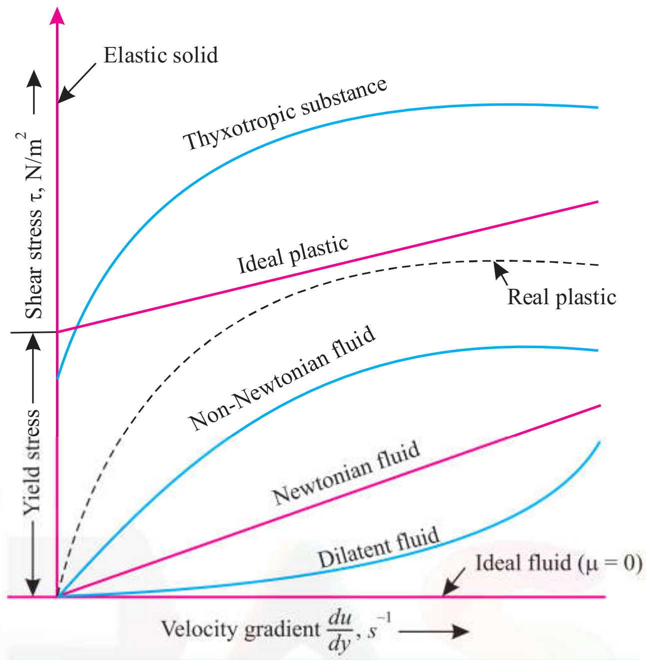


Fig. 1.2. Variation of shear stess with velocity gradient.

3. Plastic fluids. In the case of a plastic substance which is non-Newtonian fluid an initial yield stress is to be exceeded to cause a continuous deformation. These substances are represented by straight line intersecting the vertical axis at the "*yield stress*" (Refer to Fig. 1.2).

An ideal plastic (or Binigham plastic) has a definite yield stress and a constant linear relation between shear stress and the rate of angular deformation. Examples: *Sewage sludge, drilling muds etc*.

A *thyxotropic substance*, which is non-Newtonian fluid, has a non-linear relationship between the shear stress and the rate of angular deformation, beyond an initial yield stress. The *printer's ink* is an example of thyxotropic substance.

4. Ideal fluid. An ideal fluid is one which is incompressible and has zero viscosity (or in other words shear stress is always zero regardless of the motion of the fluid). Thus an ideal fluid is represented by the horizontal axis ($\tau = 0$).

A *true elastic solid* may be represented by the vertical axis of the diagram.

Summary of relations between shear stress (τ) *and rate of angular deformation for various types of fluids*:

(*i*) *Ideal fluids*: $\tau = 0$, (*ii*) *Newtonian fluids*: $\tau = \mu \cdot \frac{du}{dy}$,

(*iii*) *Ideal plastics*: $\tau = const. + \mu \cdot \frac{du}{dy}$, (*iv*) *Thyxotropic fluids*: $\tau = const. + \mu . \left(\frac{du}{dy}\right)^n$, *and*

(*v*) *Non-Newtonian fluids*: $\tau = \left(\frac{du}{dy}\right)^n$.

In case of non-Newtonian fluids, if n is *less* than unity then are called **pseudo-plastics** (*e.g., paper pulp, rubber suspension paints*) while fluids in which n is *greater* than unity are known as **dilatents.** (*e.g., Butter, printing ink*).

Ostwald-de-Waele Equation. It is an empirical solution to express steaty-state shear stress as a function of velocity gradient, and is given as

$$\tau_{yx} = \alpha \left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy}$$

If $n = 1$, this reduces to Newton's law of viscosity, with $\alpha = \mu$

Example 1.2. (*a*) *What are the characteristics of an ideal fluid ?*

(*b*) *The general relation between shear stress and velocity gradient of a fluid can be written as*

$$\tau = A\left(\frac{du}{dy}\right)^n + B$$

where A, B and n are constants that depend upon the type of fluid and conditions imposed on the flow. Comment on the value of these constants so that the fluid may behave as:

(*i*) *an ideal fluid*,

(*ii*) *a Newtonian fluid* and

(*iii*) *A non-Newtonian fluid.*

(*c*) *Indicate whether the fluid with the following characteristics is a Newtonian or non-Newtonian.*

(*i*) $\tau = Ay + B$ and $u = C_1 + C_2y + C_3y^2$

(*ii*) $\tau = Ay^{n(n-1)}$ and $u = Cy^n$

Solution. (*a*) An ideal fluid has the following *characteristics*:

- No viscosity (*i.e.*, $\mu = 0$)
- No surface tension.
- Incompressible (*i.e.*, ρ = constant)

An ideal fluid can slip near a solid boundary and cannot sustain any shear force however small it may be.

(*b*) $\tau = A\left(\frac{du}{dy}\right)^n + B$

(*i*) *An ideal fluid*:

Since an ideal fluid has zero viscosity (*i.e.*, shear stress is always zero regardless of the motion of the fluid), therefore.

$$A = B = 0$$

(*ii*) *A Newtonian fluid*:

Since a Newtonian fluid follows Newton's law of viscosity;

$$\tau = \mu\frac{du}{dy}, \text{ therefore:}$$

- $n = 1$ and $B = 0$
- The constant A takes the value of dynamic viscosity μ for the fluid.

Air, water, kerosene etc. behave as Newtonian fluids under normal working conditions.

(*iii*) *A non-Newtonian fluid*:

Depending on the value of power index n, the non-Newtonian fluids are classified as:

- If $n > 1$ and $B = 0$ ... **Dilatant fluids.**

Examples: *Sugar solution, aqueous suspension* and *printing ink*.

- If $n < 1$ and $B = 0$ .. ***Pseudo plastic fluids.***

Examples : *Blood, milk, liquid cement and clay.*

- If $n = 1$ and $B = \tau_0$ .... **Bingham fluid or ideal plastic.**

An ideal plastic fluid has a definite yield stress and a constant-linear relation between shear stress developed and rate of deformation:

i.e. $$\tau = \tau_0 + \mu\frac{du}{dy}$$

Examples: *Sewage sludge, water suspension of clay and flyash, etc.*

(c) **(i)** $\tau = Ay + B$ and $u = C_1 + C_2 y + C_3 y^2$

Now, $$\frac{du}{dy} = \frac{d}{dy}(C_1 + C_2 y + C_3 y_2) = C_2 + 2C_3 y$$

For Newtonian fluid, $$\tau = \mu\frac{du}{dy}$$

$\therefore$ $$\tau = \mu(C_2 + 2C_3 y) = 2\mu C_3 y + \mu C_2$$

which can be rewritten as

$$\tau = Ay + B \quad \text{where } A = 2\mu C_3 \text{ and } B = \mu C_2$$

Since this has the same form as the given shear stress, therefore the fluid characteristics correspond to that of an *ideal fluid*.

(ii) $\tau = Ay^{n(n-1)}$ and $u = Cy^n$

Now, $$\frac{du}{dy} = \frac{d}{dy}(Cy^n) = Cn(y)^{n-1}$$

For a Newtonian fluid $$\tau = \mu\frac{du}{dy} = \mu Cn(y)^{n-1}$$

This expression does not conform to the value of shear stress and as such the fluid is *non-Newtonian* in character.

1.6.3. Effect of Temperature on Viscosity

Viscosity is effected by temperature. The viscosity of *liquids decreases* but that of *gases* increases with *increase in temperature*. This is due to the reason that in *liquids* the shear stress is due to the inter-molecular cohesion which *decreases* with increase of temperature. In gases the inter-molecular cohesion is negligible and the shear stress is due to exchange of momentum of the molecules, normal to the direction of motion. The molecular activity increases with rise in temperature and so does the viscosity of gas.

For liquids: $$\mu_T = Ae^{\beta/T} \quad \text{...(1.8)}$$

For gases: $$\mu T = \frac{bT^{1/2}}{1 + a/T} \quad \text{...(1.9)}$$

where, μ_T = Dynamic viscosity at absolute temperature T,

A, β = Constants (for a given liquid), and

a, b = Constants (for a given gas).

1.6.4. Effect of Pressure on Viscosity

The viscosity under ordinary conditions is not appreciably affected by the changes in pressure. However, the viscosity of some oils has been found to increase with increase in pressure.

Example 1.3. *A plate 0.05 mm distant from a fixed plate moves at 1.2 m/s and requires a force of 2.2 N/m² to maintain this speed. Find the viscosity of the fluid between the plates.*

Solution: Velocity of the moving plate, $u = 1.2$ m/s

Distance between the plates, $dy = 0.05$ mm $= 0.05 \times 10^{-3}$ m

Force on the moving plate, $F = 2.2$ N/m^2

Viscosity of the fluid, μ:

We know, $\tau = \mu . \dfrac{du}{dy}$

where $\tau =$ shear stress or force per unit area $= 2.2$ N/m^2,

$du =$ change of velocity $= u - 0 = 1.2$ m/s and

$dy =$ change of distance $= 0.05 \times 10^{-3}$m.

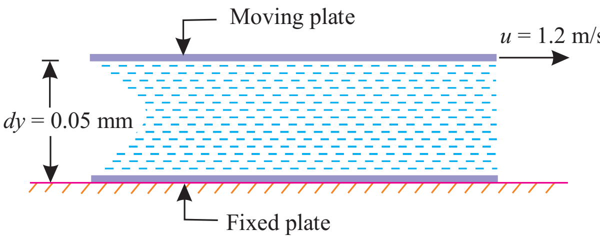


Fig. 1.3

$$\therefore \quad 2.2 = \mu \times \frac{1.2}{0.05 \times 10^{-3}}$$

$$\text{or,} \quad \mu = \frac{2.2 \times 0.05 \times 10^{-3}}{1.2} = 9.16 \times 10^{-5} \text{N.s/m}^2 \qquad \left[\because 1 \text{ poise} = \frac{1}{10}\frac{\text{N.s}}{\text{m}^2}\right]$$

$= 9.16 \times 10^{-4}$ poise (Ans.)

Example 1.4. *A plate having an area of 0.6 m^2 is sliding down the inclined plane at 30° to the horizontal with a velocity of 0.36 m/s. There is a cushion of fluid 1.8 mm thick between the plane and the plate. Find the viscosity of the fluid if the weight of the plate is 280 N.*

Solution: Area of plate, $A = 0.6$ m^2

Weight of plate, $W = 280$ N

Velocity of plate, $u = 0.36$ m/s

Thickness of film, $t = dy = 1.8$ mm $= 1.8 \times 10^{-3}$ m

Viscosity of the fluid, μ:

Component of W along the plate $= W \sin\theta = 280 \sin 30° = 140$ N

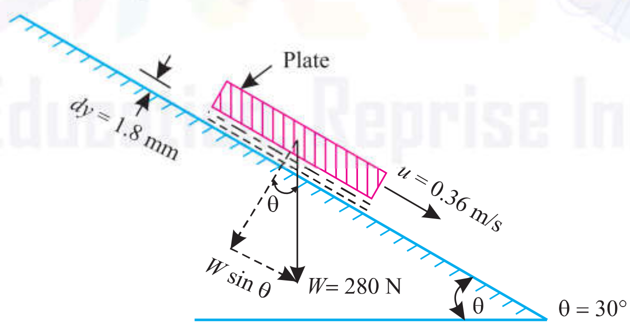


Fig. 1.4

$\therefore$ Shear force on the bottom surface of the plate, $F = 140$ N and shear stress,

$$\tau = \frac{F}{A} = \frac{140}{0.6} = 233.33 \text{N/m}^2$$

We know, $$\tau = \mu . \frac{du}{dy}$$

Where, $du =$ change of velocity $= u - 0 = 0.36$ m/s

$$dy = t = 1.8 \times 10^{-3} \text{ m}$$

$$\therefore \quad 233.33 = \mu \times \frac{0.36}{1.8 \times 10^{-3}}$$

or,
$$\mu = \frac{233.33 \times 1.8 \times 10^{-3}}{0.36} = 1.166 \text{ N.s/m}^2 = \mathbf{11.66 \text{ poise}} \text{ (Ans.)}$$

Example 1.5. *The space between two square flat parallel plates is filled with oil. Each side of the plate is 720 mm. The thickness of the oil film is 15 mm. The upper plate, which moves at 3 m/s requires a force of 120 N to maintain the speed. Determine*:

(*i*) *The dynamic viscosity of the oil*;

(*ii*) *The kinematic viscosity of oil if the specific gravity of oil is 0.95.*

Solution. Each side of a square plate = 720 mm = 0.72 m

The thickness of the oil, dy = 15 mm = 0.015 m

Velocity of the upper plate = 3 m/s

∴ Change of velocity between plates, du = 3 – 0 = 3 m/s

Force required on upper plate, F = 120 N

$$\therefore \quad \text{Shear stress, } \tau = \frac{\text{force}}{\text{area}} = \frac{120}{0.72 \times 0.72} = 231.5 \text{ N/m}^2$$

(*i*) Dynamic viscosity, μ:

We know that,

$$\tau = \mu . \frac{du}{dy}$$

$$231.5 = \mu . \frac{3}{0.015}$$

$$\therefore \quad \mu = \frac{231.5 \times 0.015}{3} = \mathbf{1.16 \text{ N.s/m}^2} \text{ (Ans.)}$$

(*ii*) Kinematic viscosity, ν:

Weight density of oil, $w = 0.95 \times 9.81 \text{ kN/m}^2 = 9.32 \text{ kN/m}^2 = \text{or } 9320 \text{ N/m}^3$

$$\text{Mass density of oil, } \rho = \frac{w}{g} = \frac{9320}{9.81} = 950$$

$$\text{Using the relation:} \quad \nu = \frac{\mu}{\rho} = \frac{1.16}{950} = 0.00122 \text{ m}^2\text{/s}$$

Hence $\nu = \mathbf{0.00122 \text{ m}^2\text{/s}}$ **(Ans.)**

Example 1.6. *The velocity distribution for flow over a plate is gives by $u = 2y - y^2$ where u is the velocity in m/s at a distance y metres above the plate. Determine the velocity gradient and shear stress at the boundary and 1.5 m from it.*

Take dynamic viscosity of fluid as 0.9 N.s/m^2.

Soluton. $u = 2y - y^2$ ...(given) $\quad \therefore \frac{du}{dy} = 2 - 2y$

(*i*) Velocity gradient, $\frac{du}{dy}$:

At the boundary : At $y = 0, \left(\frac{du}{dy}\right)_{y=0} = \mathbf{2 \text{ s}^{-1}}$ **(Ans.)**

At 0.15 m from the boundary:

$$\text{At } y = 0.15 \text{ m}, \left(\frac{du}{dy}\right)_{y=0.15} = 2 - 2 \times 0.15 = \mathbf{1.7 \text{ s}^{-1}} \text{ (Ans.)}$$

(*ii*) Shear stress, τ:

$$(\tau)_{y=0} = \mu \cdot \left(\frac{du}{dy}\right)_{y=0} = 0.9 \times 2 = \mathbf{1.8\ N/m^2}\ \textbf{(Ans.)}$$

and,
$$(\tau)_{y=0.15} = \mu\left(\frac{du}{dy}\right)_{y=0.15} = 0.9 \times 1.7 = \mathbf{1.53\ N/m^2}\ \textbf{(Ans.)}$$

[Where $\mu = 0.9$ N.s/m^2 ... (given)]

Example 1.7. *A lubricating oil of viscosity μ undergoes steady shear between a fixed lower plate and an upper plate moving at speed V. The clearance between the plates is t. Show that a linear velocity profile results if the fluid does not slip at either plate.*

Solution. For the given geometry and motion, the shear stress τ is constant throughout. From Newton's law of viscosity, we have

$$\frac{du}{dy} = \frac{\tau}{\mu} = \text{constant}$$

or $u = ly + m$

The constantS l and m are evaluated from the no slip conditions at the upper and lower plates.

At $y = 0, \mu = 0 \quad \therefore m = 0$

At $y = t, u = V$

$\therefore \quad V = lt + 0$ or $l = \dfrac{V}{t}$

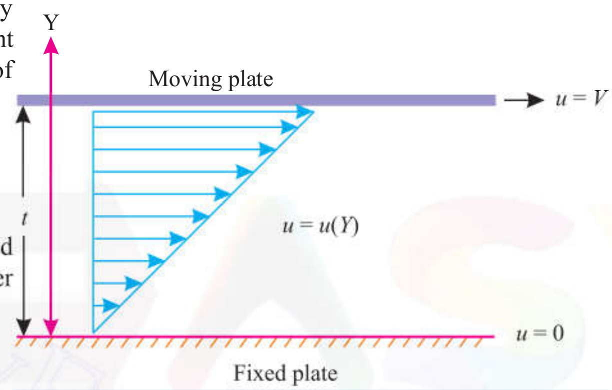


Fig. 1.5

$\therefore$ The velocity profile between plates is then given by:

$u = \dfrac{Vy}{t}$ and is *linear* as indicated in Fig 1.5 **(Ans.)**

Example 1.8. *The velocity distribution of flow over a plate is parabolic with vertex 30 cm from the plate, where the velocity is 180 cm/s. If the viscosity of the fluid is 0.9 N.s/m² find the velocity gradients and shear stresses at distances of 0, 15 cm and 30 cm from the plate.*

Solution. Distance of the vertex from the plate = 30 cm.

Velocity at vertex, $u = 180$ cm/s

Viscosity of the fluid = 0.9 N.s/m^2

The equation of velocity profile, which is parabolic, is given by

$$u = ly^2 + my + n \qquad ...(1)$$

where l, m and n are constants. The values of these constants are found from the following boundary conditions:

(*i*) At $y = 0, u = 0,$

(*ii*) At $y = 30$ cm,

$u = 180$ cm/s and

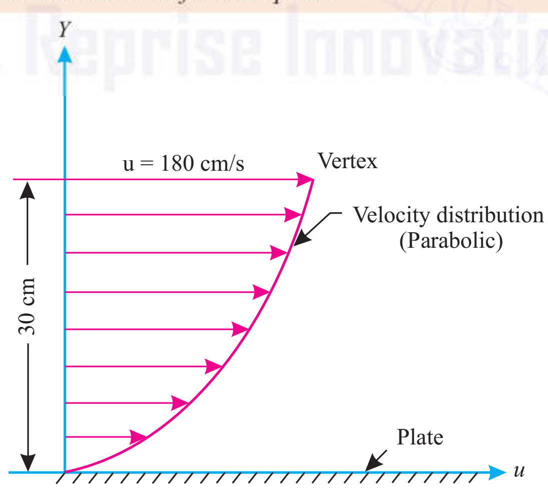


Fig. 1.6

(iii) At $y = 30$ cm, $\frac{du}{dy} = 0$.

Substituting boundary conditions (i) in eqn. (1), we get

$$0 = 0 + 0 + n \quad \therefore n = 0$$

Substituting boundary conditions (ii) in eqn. (1), we get

$$180 = l \times (30)^2 + m \times 30 \quad \text{or} \quad 180 = 900\,l + 30\,m \qquad ...(2)$$

Substituting boundary conditions (iii) in eqn. (1), we get

$$\frac{du}{dy} = 2ly + m \quad \therefore \; 0 = 2l \times 30 + m \quad \text{or} \quad 0 = 60l + m \qquad ...(3)$$

Solving eqns. (2) and (3), we have $l = -0.2$ and $m = 12$.

Substituting the values of l, m and n in eqn. (1), we get $u = -0.2\,y^2 + 12y$

Velocity gradients, $\frac{du}{dy}$:

$$\frac{du}{dy} = -0.2 \times 2y + 12 = -0.4y + 12$$

At $y = 0, \left(\frac{du}{dy}\right)_{y=0} = \mathbf{12/s}$ **(Ans.)**

At $y = 15$ cm, $\left(\frac{du}{dy}\right)_{y=15} = -0.4 \times 15 + 12 = \mathbf{6/s}$ **(Ans.)**

At $y = 30$ cm, $\left(\frac{du}{dy}\right)_{y=30} = -0.4 \times 30 + \mathbf{12} = \mathbf{0}$ **(Ans.)**

Shear stresses, τ:

We know, $\tau = \mu \frac{du}{dy}$

At $y = 0, (\tau)_{y=0} = \mu.\left(\frac{du}{dy}\right)_{y=0} = 0.9 \times 12 = \mathbf{10.8\ N/m^2}$ **(Ans.)**

At $y = 15, (\tau)_{y=15} = \mu.\left(\frac{du}{dy}\right)_{y=15} = 0.9 \times 6 = \mathbf{5.4\ N/m^2}$ **(Ans.)**

At $y = 30, (\tau)_{y=30} = \mu.\left(\frac{du}{dy}\right)_{y=30} = 0.9 \times 0 = \mathbf{0}$ **(Ans.)**

Example 1.9. *A fluid has an absolute viscosity of 0.048 Pa-s and a specific gravity of 0.913. For flow of such a fluid over a solid flat surface, the velocity at a point 75 mm away from the surface is 1.125 m/s. Calculate the shear stresses at the solid boundary and also at points 25 mm, 50 mm and 75 mm away from the surface in normal direction, if the velocity distribution across the surface is* ***(i)*** *linear,* ***(ii)*** *parabolic with vertex at the point 75 mm away from the surface.*

Solution. (*i*) Linear velocity distribution:

If velocity distribution is *linear*, $\frac{du}{dy}$ *is same at every point within the boundary layer and is equal to* $\frac{du}{dy} = \frac{1.125}{0.075}$ per s.

Shear stress for all the locations,

$$\tau = \mu \frac{du}{dy} = 0.048 \times \frac{1.125}{0.075} = \mathbf{0.72\ N/m}\ \textbf{(Ans.)}$$

(*ii*) Parabolic velocity distribution:

For parabolic velocity distribution, let the velocity profile be $u = ly^2 + my + n$ where the constants, l, m, and n are found from the boundary conditions.

At $y = 0$, $u = 0$, giving $n = 0$

At $y = 0.075$ m, $u = 1.125$ m/s, giving

$$1.125 = (0.075)^2 l + 0.075\ m \qquad \ldots(i)$$

or $1.125 = 5.625 \times 10^{-3}\ l + 0.075\ m$

At $y = 0.075$ m, $\frac{du}{dy} = 0 = 2ly + m$

or $$0 = 2l \times 0.075 + m \quad \text{or} \quad m = -0.15\ l \qquad \ldots(ii)$$

Substituting (*ii*) in (*i*), we get

$$1.125 = 5.625 \times 10^{-3} l - 0.075 \times 0.15\ l$$

$$= l\,(5.625 \times 10^{-3} - 0.075 \times 0.15) = -0.005625\ l$$

$$\therefore \quad l = -\frac{1.125}{0.005625} = -200$$

and from (*ii*), we have $m = 30$.

Hence the velocity distribution becomes $u = -200y^2 + 30y$, and $\frac{du}{dy} = 30 - 400\,y$

Hence the shear stresses at the required locations, y, are determined in the table below:

y (m)	0	0.025	0.05	0.075
$\frac{du}{dy}$ (per second)	30	20	10	0
Shere stress = $\mu \frac{du}{dy}$ N/m^2	1.44	0.96	0.48	0

(Ans.)

Example 1.10. *A 400 mm diameter shaft is rotating at 200 r.p.m. in a bearing of length 120 mm. If the thickness of oil film is 1.5 mm and the dynamic viscosity of the oil is 0.7 N.s/m^2, determine:*

(*i*) *Torque required to overcome friction in bearing;*

(*ii*) *Power utilised in overcoming viscous resistance.*

Assume a linear velocity profile.

Solution. Diameter of the shaft, $d = 400$ mm $= 0.4$ m

Speed of the shaft, $N = 200$ r.p.m.

Thickness of the oil film, $t = 1.5$ mm $= 1.5 \times 10^{-3}$ m

Length of the bearing, $l = 120$ mm $= 0.12$ m

Viscosity, $\mu = 0.7$ N.s/m^2

Tangential velocity of the shaft, $u = \frac{\pi d N}{60} = \frac{\pi \times 0.4 \times 200}{60} = 4.19$ m/s

(i) Torque required to overcome friction, T :

We know, $\tau = \mu . \dfrac{du}{dy}$

where du = change of velocity = $u - 0 = 4.19$ m/s

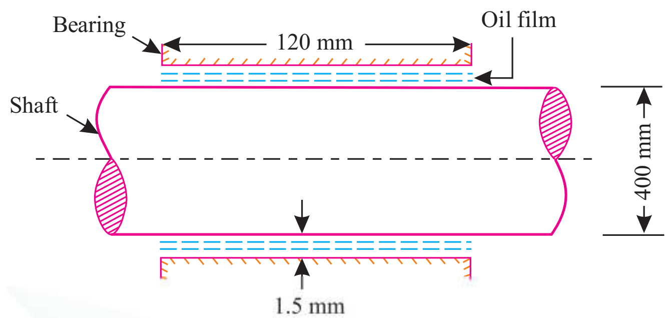


Fig. 1.7

$$dy = t = 1.5 \times 10^{-3} \text{ m}$$

$$\therefore \quad \tau = 0.7 \times \frac{4.19}{1.5 \times 10^{-3}}$$

$$= 1955.3 \text{ N/m}^2.$$

$\therefore$ Shear force, F = shear stress × area

$$= \tau \cdot \pi dl$$

$$= 1955.3 \times \pi \times 0.4 \times 0.12$$

$$= 294.85 \text{ N}$$

Hence, viscous torque $= F \times d/2 = 294.85 \times \dfrac{0.4}{2}$

= **58.97 Nm (Ans.)**

(ii) Power utilised, P:

$$P = T \times \frac{2\pi N}{60} \text{ watts, where } T \text{ is in Nm}$$

$$P = 58.97 \times \frac{2\pi \times 200}{60} = 1235 \text{ W}$$ or **1.235 kW (Ans.)**

Example 1.11. *A 150 mm diameter shaft rotates at 1500 r.p.m. in a 200 mm long journal bearing with 150.5 mm internal diameter. The uniform annular space between the shaft and the bearing is filled with oil of dynamic viscosity 0.8 poise. Calculate the power dissipated as heat.*

Solution. *Given*: d_{shaft} = 150 mm; $d_{bearing}$ = 150.5 mm; l = 200 mm = 0.2 m

N = 1500 r.p.m.; μ = 0.8 poise = 0.8 × 0.1 = 0.08 Ns/m^2

Power dissipated as heat:

Radial thickness of the oil, $dy = \dfrac{(150.5 - 150)/2}{1000} \text{ m} = 0.00025 \text{ m}$

Tangential velocity of the shaft, $u = \dfrac{\pi dN}{60} = \dfrac{\pi \times (150 \times 10^{-3}) \times 1500}{60} = 11.78 \text{ m/s}$

$\therefore$ Change of velocity, $du = u - 0 = 11.78$ m/s

Tangential stress in the oil layer,

$$\tau = \mu . \frac{du}{dy}$$

$$\therefore \qquad \tau = 0.08 \times \frac{11.78}{0.00025} = 3769.6 \text{ N/m}^2$$

Power dissipated as heat = shear force × tangential velocity of this shaft

$$= [\tau \times (\pi dl)] \times u$$

$$= 769.6 \times \pi \times (150 \times 10^{-3}) \times 0.2 \times 11.78$$

$$= 4185 \text{ W or } \mathbf{4.185\ kW}$$ **(Ans.)**

Example 1.12. *A vertical cylinder of diameter 180 mm rotates concentrically inside another cylinder of diameter 181.2 mm. Both the cylinders are 300 mm high. The space between the cylinders is filled with a liquid whose viscosity is unknown. Determine the viscosity of the fluid if a torque of 20 Nm is required to rotate the inner cylinder at 120 r.p.m.*

Solution. *Given*: Diameter of inner cylinder, $d = 180$ mm = 0.18 m

Diameter of outer cylinder, $D = 181.2$ mm = 0.1812 m

Length of each cylinder, $l = 300$ mm = 0.3 m

Speed of the inner cylinder, $N = 120$ r.p.m.

Torque, $T = 20$ Nm.

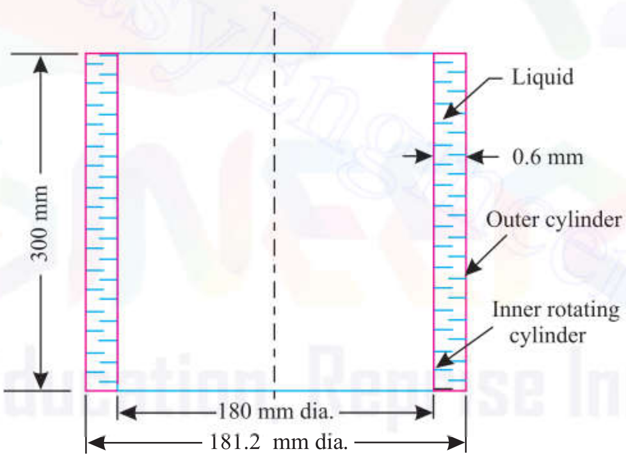


Fig. 1.8

Viscosity of the liquid, μ:

Tangential velocity of the inner cylinder

$$u = \frac{\pi d N}{60} = \frac{\pi \times 0.18 \times 120}{60} = 1.13 \text{ m/s}$$

Surface area of the inner cylinder,

$$A = \pi dl = \pi \times 0.18 \times 0.3$$

$$= 0.1696 \text{ m}^2$$

Using the relation:

$$\tau = \mu . \frac{du}{dy}$$

$$\text{where,} \quad du = u - 0 = 1.13 - 0 = 1.13 \text{ m/s}$$

$$\text{and} \quad dy = \frac{0.1812 - 0.180}{2} = 0.0006 \text{ m}$$

$$\tau = \mu \times \frac{1.13}{0.0006} = 1883.33\mu$$

$$\therefore \quad \text{Shear force, } F = \tau \times A = 1883.33\ \mu \times 0.1696 \text{ N}$$

$$\therefore \quad \text{Torque, } T = F \times F \times \frac{d}{2}$$

$$= 1883.33\ \mu \times 0.1696 \times \frac{0.18}{2}$$

$$\text{or} \quad 20 = 1883.33\ \mu \times 0.1696 \times 0.09$$

$$\text{or} \quad \mu = \frac{20}{1883.33 \times 0.1696 \times 0.09} = 0.696 \text{ Ns/m}^2$$

i.e., μ = **6.96 poise (Ans.)**

Example 1.13. *A circular disc of diameter D is slowly rotated in a liquid of large viscosity* (μ) *at a small distance* (h) *from a fixed surface. Derive an expression of torque* (T) *necessary to maintain an angular velocity* (ω).

Solution. The arrangement is shown in Fig. 1.9.

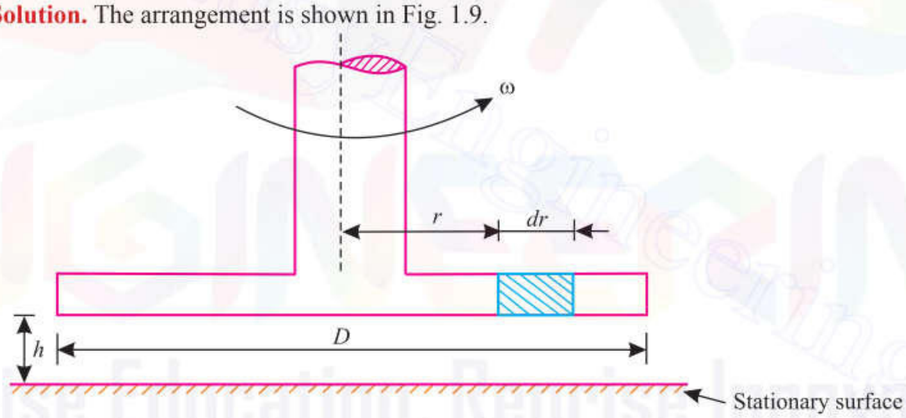


Fig. 1.9

Consider an elementary ring of disc at radius r and having a width dr. Linear velocity at this radius is ωr.

$$\text{Shear stress, } \tau = \mu \frac{du}{dy}$$

$$\text{Torque} = \text{shear stress} \times \text{area} \times r$$

$$= \tau \times 2\pi r\ dr \times r$$

$$= \mu \frac{du}{dy} \times 2\pi r^2 \times dr$$

Assuming the gap h to be *small so that the velocity distribution may be assumed linear.*

$$\frac{du}{dy} = \frac{\omega r}{h}$$

$\therefore$ Torque on the element

$$dT = \mu \frac{\omega r}{h} \times 2\pi r^2 \times dr = \frac{2\pi\mu\omega}{h} r^3 \times dr$$

$$\therefore \quad \text{Total torque, } T = \int_0^{\pi/2} \frac{2\pi\mu\omega}{h} r^3 \times dr$$

or

$$T = \frac{2\pi\mu\omega}{h}\left[\frac{r^4}{4}\right]_0^{D/2} = \frac{2\pi\mu\omega}{h} \cdot \frac{1}{4}\left(\frac{D}{2}\right)^4$$

or

$$T = \frac{\pi\mu\omega D^4}{32h}$$, which is the *required expression.* **(Ans.)**

Example 1.14. *A 120 mm disc rotates on a table separated by an oil film of 1.8 mm thickness. Find the viscosity of oil if the torque required to rotate the disc at 60 r.p.m is 3.6×10^{-4} Nm.*

Assume the velocity gradient in the oil film to be linear.

Solution. *Given*: Diameter of the disc, D = 120 mm = 0.12 m

Thickness of oil film, t = 1.8 mm = 1.8×10^{-3} m

Torque, T = 3.6×10^{-4} Nm

Speed of the disc, N = 60 r.p.m.

$\therefore$ Angular speed of the disc, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 60}{60} = 2\pi$ rad/s

Viscosity, μ:

We know that when the velocity gradient is linear,

$$\frac{du}{dy} = \frac{u}{t}$$

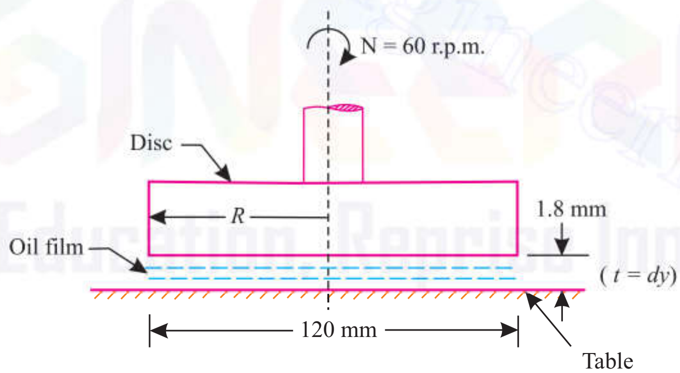


Fig. 1.10

Shearing stress, $\tau = \mu.\frac{u}{t}$.

Shearing force = Shearing stress × Area

$= \mu.\frac{u}{t}.2\pi r\,dr$ (considering an element at radius r and thickness dr)

$= \mu.\frac{\omega r}{t}.2\pi r\,dr = \frac{2\pi\mu\omega r^2.dr}{t}$ (where $u = \omega r$, ω being the angular velocity)

$\therefore$ Viscous torque $=$ Shearing force $\times\, r$

$$= \frac{2\pi\mu\omega r^2 \,.\, dr}{t} \,.\, r = \frac{2\pi\mu\omega r^3 \,.\, dr}{t}$$

$\therefore$ Total viscous torque,

$$T = \int_0^R \frac{2\pi\mu\omega r^3\, dr}{t} = \frac{2\pi\mu\omega}{t}\int_0^R r^3 dr = \frac{\pi\mu\omega R^4}{2t}$$

i.e.,

$$T = \frac{\pi\mu\omega R^4}{2t}$$

Substituting the values, we get:

$$3.6 \times 10^{-4} = \frac{\pi \times \mu \times 2\pi \times (0.12/2)^4}{2 \times 1.8 \times 10^{-3}}$$

or

$$\mu = \frac{3.6 \times 10^{-4} \times 2 \times 1.8 \times 10^{-3}}{\pi \times 2\pi \times (0.06)^4} = 0.00506 \text{ N.s/m}^2 = 0.0506 \text{ poise.}$$

Hence, $\mu =$ **0.0506 poise (Ans.)**

Example 1.15. *A solid cone of maximum radius R and vertex angle 2θ is to rotate at angular velocity ω. An oil of viscosity μ and thickness t fills the gap between the cone and the housing. Derive an expression for the torque required and the rate of heat dissipation in the bearing.*

Solution. *Given*: Maximum radius of the cone $= R$

Vertex angle $= 2\theta$

Viscosity the oil $= \mu$

Thickness of oil $= t$

Refer Fig. to 1.11.

Consider an elementary area dA at radius r of the cone.

$$dA = 2\pi r \text{ ds} \times \frac{dr}{\sin\theta}$$

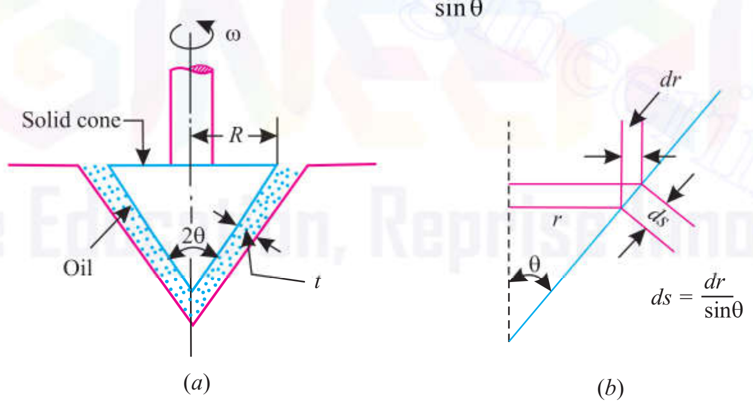


Fig. 1.11

$$\text{Shear stress } \tau = \mu\frac{du}{dy} = \mu\frac{u}{t}$$

$$\text{Shear force} = \text{shear stress} \times \text{area of the element}$$

$$= \mu\frac{u}{t}\left(2\pi r \times \frac{dr}{\sin\theta}\right)$$

$$\text{Viscous torque on the element, } dT = \mu\frac{u}{t}\left(2\pi r \times \frac{dr}{\sin\theta}\right) \times r$$

Since the cone rotates with angular velocity ω rad/sec., the tangential velocity, $u = \omega r$

or, $$dT = \mu \frac{\omega r}{t}\left(2\pi r \times \frac{dr}{\sin\theta}\right) \times r = \frac{2\pi\mu\omega}{t\sin\theta} r^3\, dr$$

∴ *Total torque*, $$T = \frac{2\pi\mu\omega}{t\sin\theta}\int_0^R r^3\, dr$$

i.e., $$\boldsymbol{T} = \frac{2\pi\mu\omega}{t\sin\theta} \times \frac{R^4}{4} = \frac{\boldsymbol{\pi\mu\omega}}{\boldsymbol{2t\sin\theta}}\boldsymbol{R^4} \quad \textbf{(Ans.)}$$

Power utilised in overcoming the resistance (*or rate of heat dissipation in the bearing*),

$$\boldsymbol{P} = T\omega = \left(\frac{\boldsymbol{\pi\mu\omega^2}}{\boldsymbol{2t\sin\theta}}\boldsymbol{R^4}\right) \quad \textbf{(Ans.)}$$

Lecture 5

Example 1.16. *Two large fixed parallel planes are 12 mm apart. The space between the surfaces is filled with oil of viscosity 0.972 N.s/m^2. A flat thin plate 0.25 m^2 area moves through the oil at a velocity of 0.3 m/s. Calculate the drag force:*

(*i*) *When the plate is equidistant from both the planes, and*

(*ii*) *When the thin plate is at a distance of 4 mm from one of the plane surfaces.*

Solution. Given: Distance between the fixed parallel planes = 12 mm = 0.012 m

Area of *thin* plate, $A = 0.25 \text{ m}^2$

Velocity of plate, $u = 0.3$ m/s

Viscosity of oil $= 0.972 \text{ N.s/m}^2$

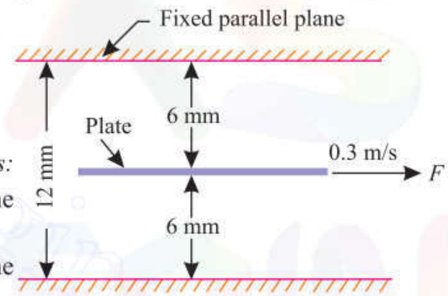


Fig. 1.12

Drag force, *F*:

(*i*) *When the plate is equidistant from both the planes:*

Let, F_1 = Shear force on the upper side of the thin plate,

F_2 = Shear force on the lower side of the thin plate,

F = Total force required to drag the plate $(= F_1 + F_2)$.

The shear τ_1, on the upper side of the thin plate is given by:

$$t_1 = \mu \, . \left(\frac{du}{dy}\right)_1$$

where, $du = 0.3$ m/s (relative velocity between upper fixed plane and the plate), and dy = 6 mm = 0.006 m (distance between the upper fixed plane and the plate)

(Thickness of the plate neglected).

$$\therefore \quad \tau_1 = 0.972 \times \frac{0.3}{0.006} = 48.6 \text{ N/m}^2$$

$$\therefore \quad \text{Shear force, } F_1 = \tau_1 \cdot A = 48.6 \times 0.25 = 12.15 \text{ N}$$

Similarly shear stress (τ_2) on the lower side of the thin plate is given by

$$\tau_2 = u \, . \left(\frac{du}{dy}\right)_2 = 0.972 \times \frac{0.3}{0.06} = 48.6 \text{ N/m}^2$$

and $\quad F_2 = \tau_2 \, . A = 48.6 \times 0.25 = 12.15$ N

$\therefore \quad F = F_1 + F_2 = 12.15 + 12.15 =$ **24.30 N (Ans.)**

(*ii*) *When the thin plate is at a distance of* 40 *mm from one of the plane surfaces*: Refer to Fig. 1.13.

The shear force on the upper side of the thin plate,

$$F_1 = \tau_1 \cdot A = \mu \cdot \left(\frac{du}{dy}\right)_1 \times A$$

$$= 0.972 \times \frac{0.3}{0.008} \times 0.25 = 9.11 \text{N}$$

The shear force on the lower side of the thin plate,

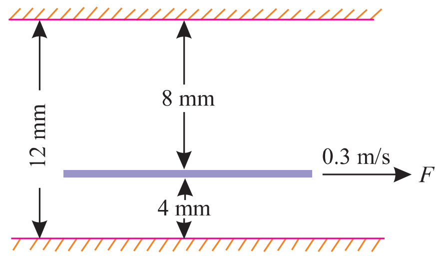


Fig. 1.13

$$F_2 = \tau_2 \times A = \mu \cdot \left(\frac{du}{dy}\right)_2 \times A$$

$$= 0.972 \times \left(\frac{0.3}{0.004}\right) \times 0.25 = 18.22 \text{N}$$

$\therefore$ Total force $F = F_1 + F_2 = 9.11 + 18.22 =$ **27.33 N (Ans.)**

Example 1.17. *In the Fig. 1.14 is shown a central plate of area 6 m² being pulled with a force of 160 N. If the dynamic viscosities of the two oils are in the ratio of 1:3 and the viscosity of top oil is 0.12 N.s/m² determine the velocity at which the central plate will move.*

Solution: Area of the plate, $A = 6 \text{ m}^2$

Force applied to the plate, $F = 160$ N

Viscosity of top oil, $\mu = 0.12 \text{ N.s/m}^2$

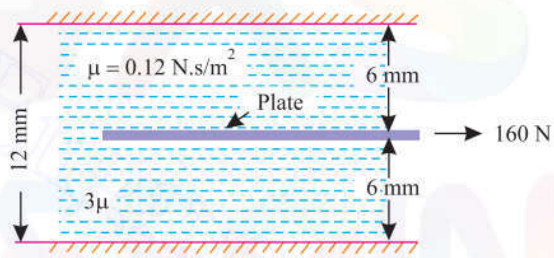


Fig. 1.14

Velocity of the plate, *u*:

Let F_1 = Shear force in the upper side of *thin* (assumed) plate,

F_2 = Shear force on the lower side of the thin plate, and

F = Total force required to drag the plate $(= F_1 + F_2)$

Then, $F = F_1 + F_2 = \tau_1 \times A + \tau_2 \times A$

$$= \mu\left(\frac{\partial u}{\partial y}\right)_1 \times A + 3\mu\left(\frac{du}{dy}\right)_2 \times A$$

(where τ_1 and τ_2 are the shear stresses on the two sides of the plate)

$$160 = 0.12 \times \frac{u}{6 \times 10^{-3}} \times 6 + 3 \times 0.12 \times \frac{u}{6 \times 10^{-3}} \times 6$$

or $\quad 160 = 120u + 360u = 480u \quad$ or $\quad u = \dfrac{160}{480} =$ **0.333 m/s (Ans.)**

Example 1.18. *A metal plate 1.25 m × 1.25 m × 6 mm thick and weighing 90 N is placed midway in the 24 mm gap between the two vertical plane surfaces as shown in the Fig. 1.15. The gap is filled with an oil of specific gravity 0.85 and dynamic viscosity 3.0 N.s/m². Determine the force required to lift the plate with a constant velocity of 0.15 m/s.*

Solution. *Given*: Dimensions of the plate = 1.25 m × 1.25 m × 6 mm

$\therefore$ Area of the plate, $A = 1.25 \times 1.25 = 1.5625 \text{ m}^2$

Thickness of the plate = 6 mm

$\therefore \qquad t_1 = t_2 = \dfrac{24-6}{2} = 9$ mm

(Since the plate is situated midway in the gap)

Specific gravity of oil = 0.85

Dynamic viscosity of oil = 3 N.s/m^2

Velocity of the plate = 0.15 m/s

Weight of the plate = 90 N

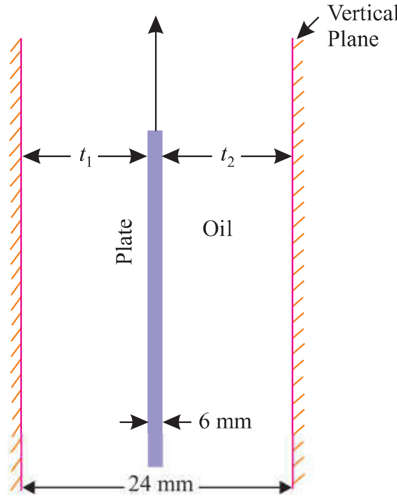


Fig. 1.15

Force required to lift the plate:

Drag force (or viscous resistance) against the motion of the plate,

$$F = \tau_1 . A + \tau_2 . A$$

(where τ_1 and τ_2 are the shear stresses on two sides of the plate)

$$= \mu . \left(\frac{du}{dy}\right)_1 \times A + \mu \left(\frac{du}{dy}\right)_2 \times A$$

$$= \mu . \frac{u}{t_1} \times A + \mu . \frac{u}{t_2} \times A$$

$$= \mu A u . \left(\frac{1}{t_1} + \frac{1}{t_2}\right)$$

or $$F = 3 \times 1.5625 \times 0.15 \left(\frac{1}{9 \times 10^{-3}} + \frac{1}{9 \times 10^{-3}}\right)$$

$$= 3 \times 1.5625 \times 0.15 \times \frac{2}{9 \times 10^{-3}} = 156.25 \text{ N}$$

Upward thrust or buoyant force on the plate = specific weight × volume of oil displaced

$= 0.85 \times 9810 \times (1.25 \times 1.25 \times 0.006) = 78.17$ N

Effective weight of the plate = 90 – 78.17 = 11.83 N

$\therefore$ Total force required to lift the plate at velocity of 0.15 m/s = F + effective weight of the plate

$= 156.25 + 11.83 =$ **168.08 N (Ans.)**

Example 1.19. *A square metal plate 1.8 m side and 1.8 mm thick weighing 60 N is to be lifted through a vertical gap of 30 mm of infinite extent. The oil in the gap has a specific gravity of 0.95 and viscosity of 3 N.s/m^2. If the metal plate is to be lifted at a constant speed of 0.12 m/s, find the force and power required.*

Solution. Area of metal plate, $A = 1.8 \times 1.8 = 3.24$ m^2

Thickness of the oil film, $t = dy = \dfrac{30-1.8}{2 \times 1000} = 0.0141$

Speed of the metal plate, $u = 0.12$ m/s.

Change of speed,

$$du = 0.12 - 0 = 0.12 \text{ m/s}$$

Viscosity, $\mu = 3 \text{ N.s/m}^2$

We know, shear stress,

$$\tau = \mu . \frac{du}{dy}$$

$$\therefore \qquad \tau = 3 \times \frac{0.12}{0.0141} = 25.53 \text{ N/m}^2$$

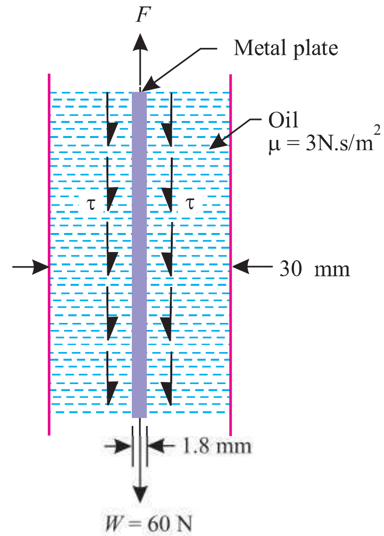


Fig. 1.16

Force required, F:

$$F = W + 2 (\tau . A)$$

[where W = weight of the plate

$= 60$ N (given)]

$= 60 + 2 \times 25.53 \times 3.24 = 225.4$ N

Hence $\quad F = $ **225.4 N (Ans.)**

Power required, P:

$$P = F \times u = 225.4 \times 0.12 = 27.05 \ W$$

Hence $\quad P = $ **27.05 (Ans.)**

Example 1.20. *A thin plate of very large area is placed in a gap of height h with oils of viscosities μ' and μ'' on the two sides of the plate. The plate is pulled at a constant velocity V. Calculate the position of plate so that :*

(*i*) *The shear force on the two sides of the plate is equal*

(*ii*) *The force required to drag the plate is minimum.*

Assume viscous flow and neglect all end effects.

Solution. *Given* : Height of the gap = h

Viscosities of oils = u' and u''

Velocity of the plate = V

Position of the plate, y:

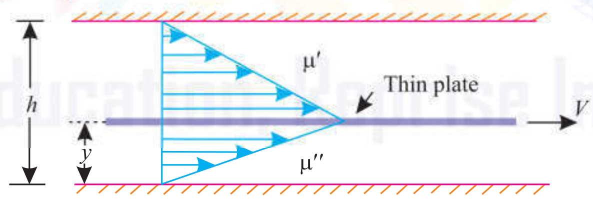


Fig. 1.17

Let y = The distance of the thin plate from one of the surfaces of the gap.

Force on the upper side of the plate,

$$F_{upper} = \mu \frac{du}{dy} = \mu' \times \frac{V}{(h-y)} A$$

Force on the lower side of the plate, $F_{lower} = \mu'' \times \dfrac{V}{y} A$

(*i*) Since the forces on the two sides of the plate are equal (given) we have,

i.e., $\qquad F_{upper} = F_{lower}$

$$\therefore \qquad \mu' \times \frac{V}{(h-y)} A = \mu'' \times \frac{V}{y} A$$

$$\text{or,} \qquad \frac{\mu'}{h-y} = \frac{\mu''}{y} \quad \text{or} \quad \mu' y = \mu'' \, h - \mu'' \, y$$

$$\therefore \qquad y = \frac{\mu'' h}{\mu' + \mu''} \quad \textbf{(Ans.)}$$

(*ii*) Total drag force = sum of the forces on the upper and lower surfaces of the plate.

i.e., $$F = F_{upper} + F_{lower}$$

or, $$F = \mu' \times \frac{V}{h-y} \times A + \mu'' \times \frac{V}{y} A$$

For the drag force to be *minimum* $\dfrac{dF}{dy} = 0$

i.e., $$\frac{d}{dy}\left[\mu' \times \frac{V}{h-y} \times A + \mu'' \times \frac{V}{y} A\right] = 0$$

or, $$\frac{\mu' VA}{(h-y)^2} - \frac{\mu'' VA}{y^2} = 0$$

or, $$\frac{\mu'}{\mu''} = \frac{(h-y)^2}{y^2} = \frac{h^2 + y^2 - 2hy}{y^2} = \frac{h^2}{y^2} + 1 - \frac{2h}{y}$$

$$\therefore \qquad = \frac{h^2}{y^2} - \frac{2h}{y} + \left(1 + \frac{\mu'}{\mu''}\right)$$

or, $$\frac{h}{y} = \frac{2 \pm \sqrt{4 - 4(1 - \mu'/\mu'')}}{2} = 1 \pm \sqrt{(\mu'/\mu'')}$$

Since, $\dfrac{h}{y}$ cannot be less than unity, therefore

$$\frac{h}{y} = 1 + \sqrt{\mu'/\mu''} \quad \text{or} \quad y = \frac{h}{1 + \sqrt{\mu'/\mu''}} \quad \textbf{(Ans.)}$$

1.7. THERMODYNAMIC PROPERTIES

The thermodynamic properties need to be considered when a fluid is influenced by change of temperature. The following equation, known as the *characteristic equation of a state of a perfect gas,* is used for this purpose.

$$pV = mRT \qquad \text{...(1.10)}$$

where, p = Absolute pressure, m = Mass of gas,

V = Volume of m kg of gas, R = Characteristic gas constant, and

T = Absolute temperature.

The characteristic equation in *another form,* can be derived by using *kilogram-mole as a unit.* The *kilogram-mole* is defined as a quantity of a gas equivalent to M kg of the gas, where M is the molecular weight of the gas (*i.e.,* since the molecular weight of oxygen is 32, then 1 kg mole of oxygen is equivalent to 32 kg of oxygen).

As per definition of the kilogram-mole, for m kg of a gas, we have:

$$m = nM \qquad \text{...(1.11)}$$

where, n = No. of moles.

Note. Since the standard of mass is the kg, kilogram-mole will be written simply as mole.

Substituting for m from eqn. 1.11 in Eqn. 1.10 gives:

$$pV = nMRT \quad \text{or} \quad \text{MR} = \frac{pV}{nT}$$

According to Avogadro's hypothesis the volume of 1 mole of any gas is the same as the volume of 1 mole of any other gas, when the gases are at same temperature and pressure. Therefore, $\frac{V}{n}$ is the same for all gases at the same value of p and T. That is the quantity $\frac{pV}{nT}$ is a constant for all gases. This constant is called '*universal gas constant*', and is given the symbol, R_0,

i.e., $$MR = R_0 = \frac{pV}{nT} \quad \text{or} \quad pV = nR_0T \qquad ...(1.12)$$

Since, $$MR = R_0, \text{ then } R = \frac{R_0}{M} \qquad ...(1.13)$$

It has been found experimentally that the volume of 1 mole of any perfect gas at 1 bar and 0°C is approximately 22.71 m^3. Therefore from eqn. 1.12,

$$R_0 = \frac{pV}{nT} = \frac{1 \times 10^5 \times 22.71}{1 \times 273.15} = 8314.3 \text{ Nm/mole K}$$

Using eqn. 1.13, the gas constant for any gas can be found when the molecular weight is known.

Example. *For oxygen which has a molecular weight of 32, the gas constant*

$$R = \frac{R_0}{M} = \frac{8314}{32} = 259.8 \text{ Nm/kg K.}$$

If the value of R is known, the specific weight of any gas can be computed at any temperature. The density can be changed by changing temperature or pressure.

(i) When the change in the state of the fluid system is affected at *constant pressure* the process is known as **isobaric** or **constant pressure process.**

Here $\frac{V}{T}$ = constant; (Charle's law) or $\frac{v}{T}$ = constant or $\frac{v}{T} = \frac{1}{\rho T}$ = constant ...(1.14)

(ii) When the change in the state of the fluid system is affected at *constant temperature* the process is known as **isothermal process.**

Here pv^γ = constant; (Boyle's Law) or $pv = \frac{p}{\rho}$ = constant ...(1.15)

(iii) *When no heat is transferred* to or from the fluid during the change in the state of fluid system, the process is called **adiabatic process.**

Here, pv^γ = constant or $pv^\gamma = \frac{p}{\rho^\gamma}$ = constant ...(1.16)

where $$\gamma = \frac{c_p}{c_v},$$

c_p = Specific heat of gas at constant pressure, and

c_v = Specific heat of gas at constant volume.

γ depends upon the molecular structure of the gas.

Example 1.21. *The pressure and temperature of carbon-dioxide in a vessel are 600 kN/m² abs. and 30°C respectively. Find its mass density, specific weight and specific volume.*

Solution. *Given*: Pressure of CO_2 = 600 kN/m^2 abs.

Temperature of CO_2 = 30 + 273 = 303 K

Molecular weight of CO_2 = 12 + 2 × 16 = 44

Universal gas constant, R_0 = 8314.3 Nm/mole K

$$\therefore \text{ Characteristic gas constant, } R = \frac{R_0}{M} = \frac{8314.3}{44} = 189 \text{ Nm/kg K}$$

(*i*) Mass density, ρ:

We know, $pV = mRT \qquad \therefore \quad \frac{m}{V} = \frac{p}{RT}$

or, $$\rho = \frac{p}{RT} = \frac{600 \times 10^3}{189 \times 313} = 10.14 \text{ kg/m}^3$$

i.e., ρ = **10.14 kg/m³ (Ans.)**

(*ii*) Specific weight, *w*:

$$w = \rho g = 10.14 \times 9.81 = \mathbf{99.47 \ N/m^3} \text{ (Ans.)}$$

(*iii*) Specific volume *v*:

$$v = \frac{1}{\rho} = \frac{1}{10.14} = \mathbf{0.0986 \ m^3/kg} \text{ (Ans.)}$$

1.8. SURFACE TENSION AND CAPILLARITY

1.8.1. Surface Tension

Cohesion. Cohesion means intermolecular attraction between *molecules of the same liquid.* It enables a liquid to resist small amount of tensile stresses. Cohesion is a tendency of the liquid to remain as one *assemblage of particles.* "*Surface tension*" *is due to cohesion between particles at the free surface.*

Adhesion. *Adhesion* means *attraction between the molecules of a liquid and the molecules of a solid boundary surface in contact with the liquid.* This property enables a liquid to stick to *another body.*

Capillary action is due to both *cohesion* and *adhesion.*

Surface tension is caused by the *force of cohesion at the free surface.* A liquid molecule in the interior of the liquid mass is surrounded by other molecules all around and is in equilibrium. At the free surface of the liquid, there are no liquid molecules above the surface to balance the force of the molecules below it. Consequently, as shown in Fig. 1.18, there is a net inward force on the molecule. The force is normal to the liquid surface. At the free surface a thin layer of molecules is formed. This is because of this *film that a thin small needle can float on the free surface* (the layer acts as a *membrane*).

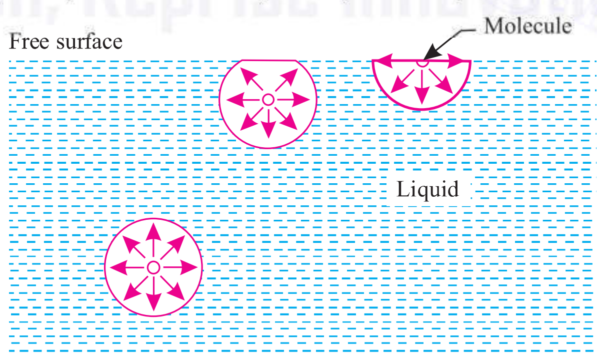


Fig. 1.18

Some *important examples of phenomenon of surface tension* are as follows:

(*i*) Rain drops (A falling rain drop becomes spherical due to cohesion and surface tension).

(*ii*) Rise of sap in a tree.

(*iii*) Bird can drink water from ponds.

(*iv*) Capillary rise and capillary siphoning.

(*v*) Collection of dust particles on water surface.

(*vi*) Break up of liquid jets.

Dimensional formula for surface tension:

The dimensional formula for surface tension is given by:

$$\left[\frac{E}{L}\right] \text{ or } \left[\frac{\mathrm{M}}{\mathrm{T}^2}\right]$$

It is usually expressed in N/m. The value of surface tension depends upon the following *factors*:

(*i*) Nature of the liquid,

(*ii*) Nature of the surrounding matter (*e.g.*, solid, liquid or gas), and

(*iii*) Kinetic energy (and hence the temperature of the liquid molecules).

Note. As compared to pressure and gravitational forces surface tension forces are generally negligible but *become quite significant when there is a free surface and the boundary conditions are small as in the case of small scale models of hydraulic engineering structures.*

Surface tension of water and mercury when in contact with air:

Water-air ... 0.073 N/m at 20°C; Water-air ... 0.058 N/m at 100°C;

Mercury-air ... 0.1 N/m length.

1.8.1.1. Pressure Inside a Water Droplet, Soap Bubble and a Liquid Jet

Case I. Water droplet:

Let, p = Pressure inside the droplet above outside pressure (*i.e.*, $\Delta p = p - 0 = p$ above atmospheric pressure)

d = Diameter of the droplet and

σ = Surface tension of the liquid.

From free body diagram (Fig. 1.19 *d*), we have:

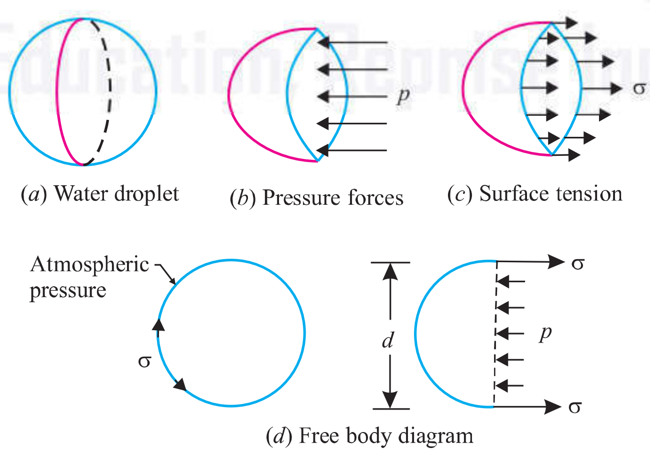


Fig. 1.19. Pressure inside a water droplet.

(i) Pressure force = $p \times \frac{\pi}{4} d^2$, and

(ii) Surface tension force acting around the circumference = $\sigma \times \pi\, d$.

Under equilibrium conditions these two forces will be equal and opposite,

i.e.,
$$p \times \frac{\overleftarrow{\pi}}{4} d^2 = \sigma \times \overrightarrow{\pi} d$$

$$\therefore \qquad p = \frac{\sigma \times \pi d}{\frac{\pi}{4} d^2} = \frac{4\sigma}{d} \qquad \text{...(1.17)}$$

Eqn. 1.17 shows that *with an increase in size of the droplet the pressure intensity decreases*.

Case II. Soap (or hollow) bubble:

Soap bubbles have two surfaces on which surface tension σ acts.

From the free body diagram (Fig. 1.20), we have

$$p \times \frac{\overleftarrow{\pi}}{4} d^2 = 2 \times (\overrightarrow{\sigma} \times \pi d)$$

$$\therefore \qquad p = \frac{2\sigma \times \pi d}{\frac{\pi}{4} d^2} = \frac{8\sigma}{d} \qquad \text{...(1.18)}$$

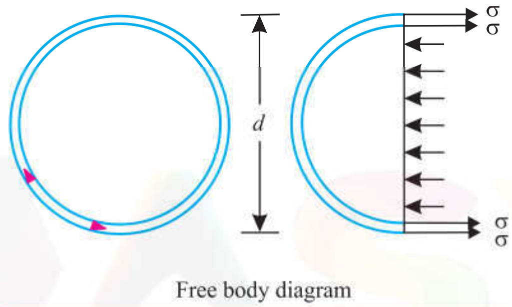


Fig. 1.20. Pressure inside a soap bubble.

Since the soap solution has a high value of surface tension σ, even with small pressure of blowing a soap bubble will tend to grow larger in diameter (hence formation of large soap bubbles).

Case III. A Liquid jet:

Let us consider a cylindrical liquid jet of diameter d and length l. Fig. 1.21 shows a semi-jet.

$$\text{Pressure force} = p \times l \times d$$
$$\text{Surface tension force} = \sigma \times 2l$$

Equating the two forces, we have:

$$p \times l \times d = \sigma \times 2l$$

$$\therefore \qquad p = \frac{\sigma \times 2l}{l \times d} = \frac{2\sigma}{d} \qquad \text{...(1.19)}$$

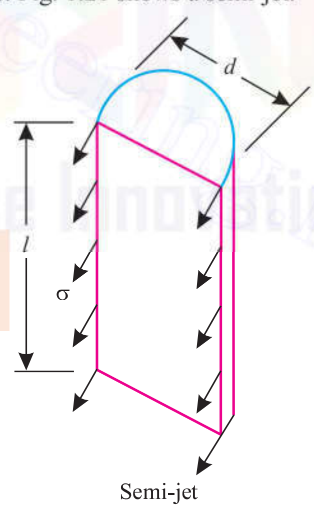


Fig. 1.21. Forces on liquid jet.

Example 1.22. *If the surface tension at air-water interface is 0.069 N/m, what is the pressure difference between inside and outside of an air bubble of diameter 0.009 mm?*

Solution. *Given*: $\sigma = 0.069$ N/m; $d = 0.009$ mm

An air bubble has only one surface. Hence,

$$p = \frac{4\sigma}{d}$$
$$= \frac{4 \times 0.069}{0.009 \times 10^{-3}} = 30667 \text{ N/m}^2$$
$$= \mathbf{30.667\ kN/m^2\ or\ kPa} \text{ (Ans.)}$$

Example 1.23. *If the surface tension at the soap-air interface is 0.09 N/m, calculate the internal pressure in a soap bubble of 28 mm diameter.*

Solution. *Given*: $\sigma = 0.09$ N/m; $d = 28$ mm.

In a soap bubble there are two interfaces. Hence,

$$p = \frac{8\sigma}{d} = \frac{8 \times 0.09}{28 \times 10^{-3}}$$

$$= \mathbf{25.71\ N/m^2}\ \textbf{(above atmospheric pressure)}\ \textbf{(Ans.)}$$

Example 1.24. *In order to form a stream of bubbles, air is introduced through a nozzle into a tank of water at 20°C. If the process requires 3.0 mm diameter bubbles to be formed, by how much the air pressure at the nozzle must exceed that of the surrounding water?*

What would be the absolute pressure inside the bubble if the surrounding water is at 100.3 kN/m²?

Take surface tension of water at 20°C = 0.0735 N/m.

Solution. Diameter of a bubble, $d = 3.0\text{ mm} = 3 \times 10^{-3}\text{ m}$

Surface tension of water at 20°C, $\sigma = 0.0735$ N/m

The excess pressure intensity of air over that of surrounding water, $\Delta p = p$.

We know,
$$p = \frac{4\sigma}{d} = \frac{4 \times 0.0735}{3 \times 10^{-3}} = \mathbf{98\ N/m^2}\ \textbf{(Ans.)}$$

Absolute pressure inside the bubble, p_{abs}:

$$p_{abs} = p + p_{atm}$$
$$= 98 \times 10^{-3} + 100.3$$
$$= 0.098 + 100.3 = \mathbf{100.398\ kN/m^2}\ \textbf{(Ans.)}$$

Example 1.25. *A soap bubble 62.5 mm diameter has an internal pressure in excess of the outside pressure of 20 N/m². What is tension in the soap film?*

Solution. *Given*: Diameter of the bubble, $d = 62.5\text{ mm} = 62.5 \times 10^{-3}$ m;

Internal pressure in excess of the outside pressure, $p = 20$ N/m².

Surface tension, σ:

Using the relation,
$$p = \frac{8\sigma}{d}$$

i.e.,
$$20 = \frac{8\sigma}{62.5 \times 10^{-3}} \quad \therefore \quad \sigma = 20 \times \frac{62.5 \times 10^{-3}}{8} = \mathbf{0.156\ N/m}\ \textbf{(Ans.)}$$

Example 1.26. *What do you mean by surface tension? If the pressure difference between the inside and outside of the air bubble of diameter 0.01 mm is 29.2 kPa, what will be the surface tension at air-water interface?*

Solution. Surface tension is defined *as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension.* The magnitude of this force per unit length of the free surface will have the same value as the surface energy per unit area. It is denoted by the letter σ and is expressed as N/m.

$$p \times \frac{\pi}{4} d^2 = \sigma\,(\pi d)$$

or
$$\sigma = p \times \frac{d}{4}$$

Substituting the values; $d = 0.01 \times 10^{-3}$ m; $p = 29.2 \times 10^3$ Pa (or N/m²), we get

$$\sigma = 29.2 \times 10^3 \times \frac{0.01 \times 10^{-3}}{4} = \mathbf{0.073\ N/m}\ \textbf{(Ans.)}$$

1.8.2. Capillarity

Capillarity *is a phenomenon by which a liquid (depending upon its specific gravity) rises into a thin glass tube above or below its general level.* This phenomenon is due to the *combined effect of cohesion and adhesion of liquid particles.*

Fig. 1.22 shows the phenomenon of rising water in the tube of *smaller* diameters.

Let,
d = Diameter of the capillary tube,
θ = Angle of contact of the water surface,

σ = Surface tension force for unit length, and

w = Weight density (ρg).

Now, upward surface tension force (lifting force) = weight of the water column in the tube (*gravity force*)

$$\pi d.\sigma \cos\theta = \frac{\pi}{4} d^2 \times h \times w$$

$$\therefore \qquad h = \frac{4\sigma \cos\theta}{wd} \qquad ...(1.20)$$

For water and glass: $\theta \simeq 0$.

Hence the capillary rise of water in the glass tube,

$$h = \frac{4\sigma}{wd} \qquad ...(1.21)$$

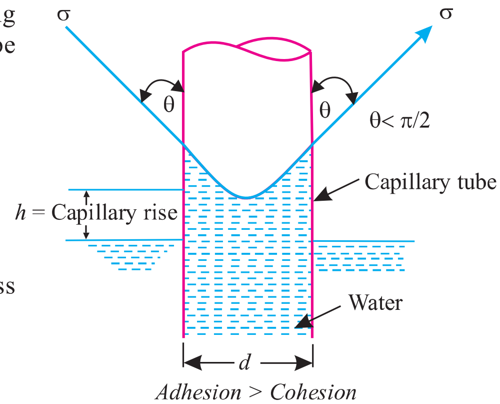


Fig. 1.22. Effect of capillarity.

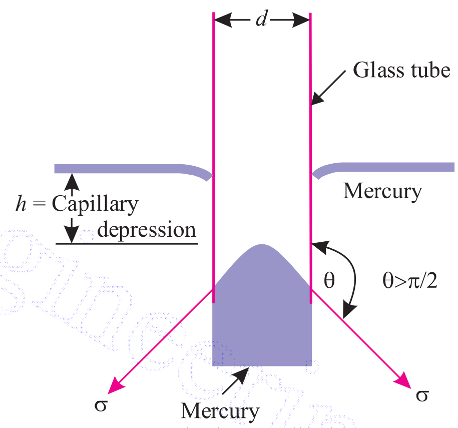


Fig. 1.23

Example 1.27. *A clean tube of diameter 2.5 mm is immersed in a liquid with a coefficient of surface tension = 0.4 N/m. The angle of contact of the liquid with the glass can be assumed to be 135°. The density of the liquid = 13600 kg/m^3.*

What would be the level of the liquid in the tube relative to the free surface of the liquid inside the tube.

Solution. *Given:* $d = 2.5$ mm ; $\sigma = 4$ N/m, $\theta = 135°$; $\rho = 13600$ kg/m^3

Level of the liquid in the tube, *h*:

The liquid in the tube rises (or falls) due to capillarity. The capillary rise (or fall),

$$h = \frac{4\sigma \cos\theta}{wd} \qquad \text{...[Eqn. (1.20)]}$$

$$= \frac{4 \times 0.4 \times \cos 135°}{(9.81 \times 13600) \times 2.5 \times 10^{-3}} \qquad (\because w = \rho g)$$

$$= -3.39 \times 10^{-3} \text{m or } -3.39 \text{ mm}$$

Negative sign indicates that there is a capillary **depression (fall) of 3.39 mm. (Ans.)**

Example 1.28. *Assuming that the interstices in a clay are of size equal to one tenth the mean diameter of the grain, estimate the height to which water will rise in a clay soil of average grain diameter of 0.048 mm. Assume surface tension at air-water interface as 0.074 N/m.*

Solution. *Given:* Diameter of the pores, $d = \frac{1}{10} \times 0.048 = 0.0048$ mm; $\sigma = 0.074$ N/m

Assuming $\theta = 0°$

$$h = \frac{4\sigma}{wd} = \frac{4 \times 0.074}{(9.81 \times 1000) \times 0.0048 \times 10^{-3}} = \mathbf{6.286 \text{ m}} \text{ (Ans.)}$$

Example 1.29. *Calculate the work done in blowing a soap bubble of diameter 100 mm. Assume the surface tension of soap solution = 0.038 N/m.*

Solution. *Given:* $d = 100$ mm or 0.1 m; $\sigma = 0.038$ N/m.

The soap bubble has two interfaces.

$\therefore$ Work done = Surface tension × total surface area

$$= 0.038 \times 4\pi \times \left(\frac{0.1}{2}\right)^2 \times 2$$

$$= \mathbf{0.002388\ Nm}\ \textbf{(Ans.)}$$

Example 1.30. *Determine the minimum size of glass tubing that can be used to measure water level, if the capillary rise in the tube is not to exceed 0.3 mm. Take surface tension of water in contact with air as 0.0735 N/m.*

Solution. *Given* :Capillary rise, $h = 0.3 \text{ mm} = 0.3 \times 10^{-3}$ m

Surface tension, $\sigma = 0.0735$ N/m

Specific weight of water, $w = 9810$ N/m^3.

Size of glass tubing, *d*:

$$\text{Capillary rise, } h = \frac{4\sigma\cos\theta}{wd} = \frac{4\sigma}{wd}$$

(Assuming $\theta = 0$ for water)

$$0.3 \times 10^{-3} = \frac{4 \times 0.0735}{9810 \times d}$$

$$\therefore \quad d = \frac{4 \times 0.0735}{0.3 \times 10^{-3} \times 9810} = 0.1 \text{ m } = \mathbf{100\ mm}\ \textbf{(Ans.)}$$

Example 1.31. *A U-tube is made up of two capillaries of bores 1.2 m and 2.4 mm respectively. The tube is held vertical and partially filled with liquid of surface tension 0.06 N/m and zero contact angle. If the estimated difference in the level of two menisci is 15 mm, determine the mass density of the liquid.*

Solution. *Given*: Bores of the capillaries:

$$d_1 = 1.2 \text{ mm} = 0.0012 \text{ m}$$

$$d_2 = 2.4 \text{ mm} = 0.0024 \text{ m}$$

Difference of level, $h_1 - h_2 = 15 \text{ mm} = 0.015$ m; Angle of contact, $\theta = 0$

Mass density of the liquid, ρ:

$$h_1 = \frac{4\sigma\cos\theta}{wd_1}, \quad \text{and} \quad h_2 = \frac{4\sigma\cos\theta}{wd_2}$$

[where w (= ρg) = weight density of the liquid)]

$$\therefore \quad h_1 - h_2 = \frac{4\sigma}{w}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] \quad (\because \theta = 0)$$

$$0.015 = \frac{4 \times 0.06}{\rho \times 9.81}\left[\frac{1}{0.0012} - \frac{1}{0.0024}\right] = \frac{0.02446}{\rho} \times 416.67$$

$$\therefore \quad \rho = \frac{0.02446 \times 416.67}{0.015} = \mathbf{679.45\ kg/m^3}\ \textbf{(Ans.)}$$

Lecture 6

PRESSURE AND FLUID STATICS

Objectives

- Pressure
- Pressure Measurement Devices
- Introduction to Fluid Statics

1- Pressure:

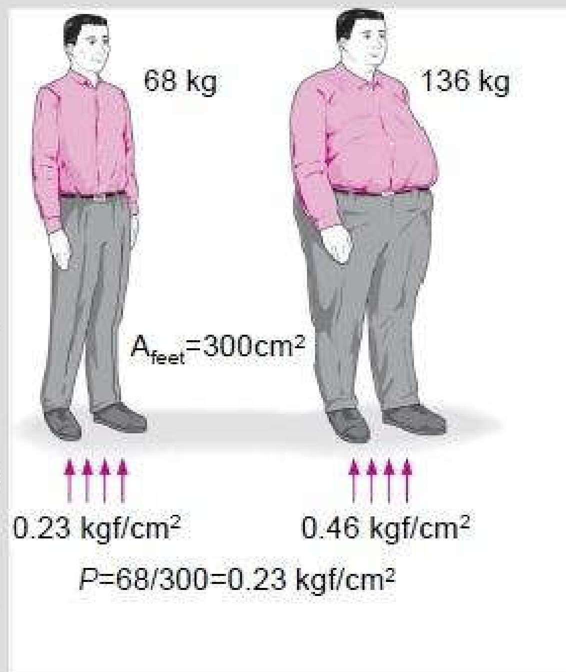
Pressure: A normal force exerted by a fluid per unit area

$$1\ \text{Pa} = 1\ \text{N/m}^2$$

$$1\ \text{bar} = 10^5\ \text{Pa} = 0.1\ \text{MPa} = 100\ \text{kPa}$$

$$1\ \text{atm} = 101{,}325\ \text{Pa} = 101.325\ \text{kPa} = 1.01325\ \text{bars}$$

$$1\ \text{kgf/cm}^2 = 9.807\ \text{N/cm}^2 = 9.807 \times 10^4\ \text{N/m}^2 = 9.807 \times 10^4\ \text{Pa}$$
$$= 0.9807\ \text{bar}$$
$$= 0.9679\ \text{atm}$$



The normal stress (or “pressure”) on the feet of a chubby person is much greater than on the feet of a slim person.

Some basic pressure gages.

Absolute pressure: The actual pressure at a given position. It is measured relative to absolute vacuum (i.e., absolute zero pressure).

Gage pressure: The difference between the absolute pressure and the local atmospheric pressure. Most pressure-measuring devices are calibrated to read zero in the atmosphere, and so they indicate gage pressure.

Vacuum pressures: Pressures below atmospheric pressure.

$$P_{gage} = P_{abs} - P_{atm}$$

$$P_{vac} = P_{atm} - P_{abs}$$

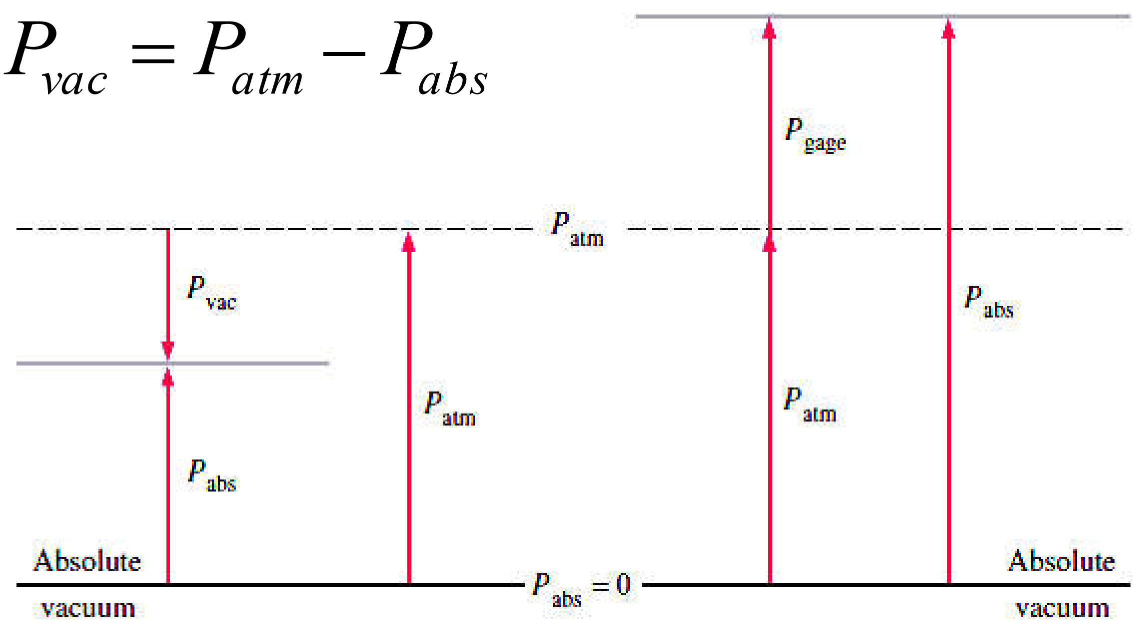


Figure1. Throughout this text, the pressure P will denote absolute pressure unless specified otherwise.

EXAMPLE 3–1 Absolute Pressure of a Vacuum Chamber

A vacuum gage connected to a chamber reads 5.8 psi at a location where the atmospheric pressure is 14.5 psi. Determine the absolute pressure in the chamber.

SOLUTION The gage pressure of a vacuum chamber is given. The absolute pressure in the chamber is to be determined.

Analysis The absolute pressure is easily determined from

$$P_{abs} = P_{atm} - P_{vac} = 14.5 - 5.8 = \mathbf{8.7\ psi}$$

Discussion Note that the *local* value of the atmospheric pressure is used when determining the absolute pressure.

1-1. Pressure at a Point:

Pressure is the *compressive force* per unit area but it is not a vector. Pressure at any point in a fluid is the same in all directions. Pressure has magnitude but not a specific direction, and thus it is a scalar quantity.

- Pressure is a scalar quantity, not a vector; the pressure at a point in a fluid is the same in all directions.

This can be domenstrated by considering a small wedge-shaped fluid element of unit length as shown in figure 2. The mean pressures at the three surfaces are P1,P2, and P3, and the force acting on asurface is the product of mean pressure and the surface area. From Newton's second law, a force balance in the x- and z- directions gives:

Newton's second law: The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\sum F_x = ma_x = 0 \qquad P_1 \Delta y \Delta z - P_3 \Delta y l \sin\theta = 0$$

$$\sum F_z = ma_z = 0 \qquad P_2\Delta y\Delta x - P_3\Delta y l\cos\theta - \frac{1}{2}\rho g\Delta x\Delta y\Delta z = 0$$

$$W = mg = \rho g\Delta x\Delta y\Delta z/2$$

$$\Delta z = l\sin\theta$$

$$\Delta x = l\cos\theta$$

Substituting these geometric relations and dividing by **Δz** and by **Δx** gives:

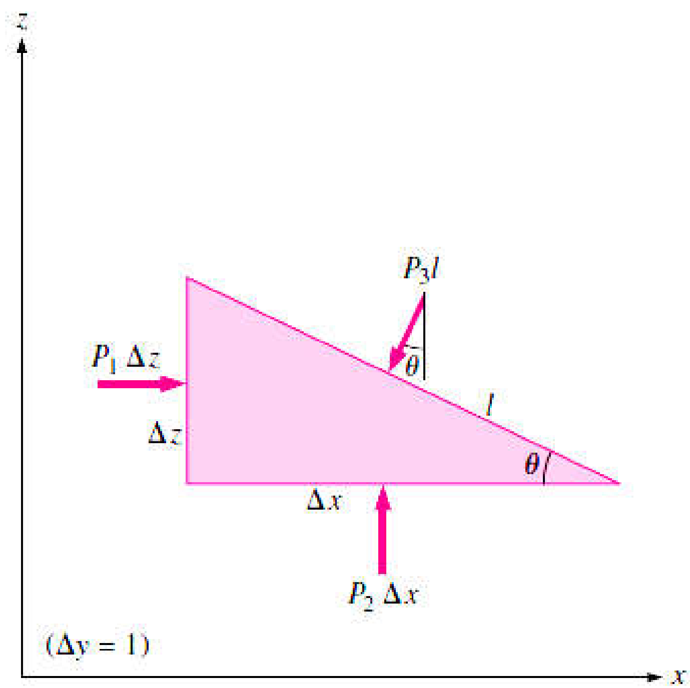


Figure2. Forces acting on a wedge-shaped fluid element in equilibrium

$$P_1 - P_3 = 0$$

$$P_2 - P_3 - \frac{1}{2}\rho g \Delta z = 0$$

Because of $\mathbf{\Delta z} \rightarrow$ zero the wedge becomes infinitesimal, and thus the fluid element shrinks to a point.

$$P_1 = P_2 = P_3 = P$$

We can repeat the analysis for an element in the yz-palne and obtain a similar result. Thus we conclude that *Pressure at any point in a fluid is the same in all directions*.

1-2. Variation of Pressure with Depth:

It will come no surprise to you that pressure in a fluid at rest does not change in the horizontal direction. However, this is not the case in the vertical direction in a gravity field. Pressure in a fluid increase with depth because more fluid rests on deeper layers and the effect of this "extra weight" on a deeper layer is balanced by an increase in pressure (Figure 3).

To obtain a relation for variation of pressure with depth, consider a rectangular fluid element of height Δz, length Δx, and unit depth as shown in figure 4.

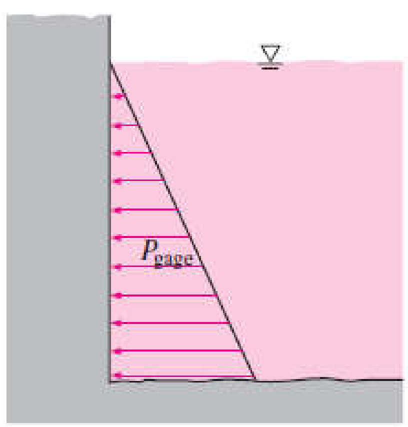


Figure3. The pressure of a fluid at rest increases with depth (as a result of added weight).

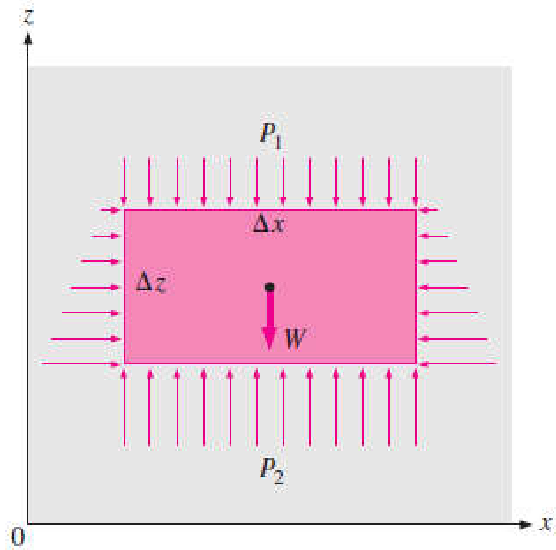


Figure4.Free-body diagram of a rectangular fluid element in equilibrium

Assuming the density of the fluid to be constant, a force balance in the vertical Z- direction gives:

$$\sum F_z = ma_z = 0 \qquad P_2\,\Delta x - P_1\,\Delta x - \rho g\,\Delta z\,\Delta x = 0$$

Where $\Delta y = 1$

And $W = mg = \rho g\,\Delta z\,\Delta x$ is the weight of the fluid element. Dividing by Δx and rearranging gives:

$$\Delta P = P_2 - P_1 = \rho g\,\Delta z = \Upsilon_s\,\Delta z$$

For a given fluid, the variation distance Δz is sometimes used as a measure of pressure, and it is called the pressure head (***h***).

We also conclude that for small to moderate distance, the variation of pressure with height is negligible for gases because of their low density. For example the pressure in a room filled with air can be assumed to be constant (Figure5).

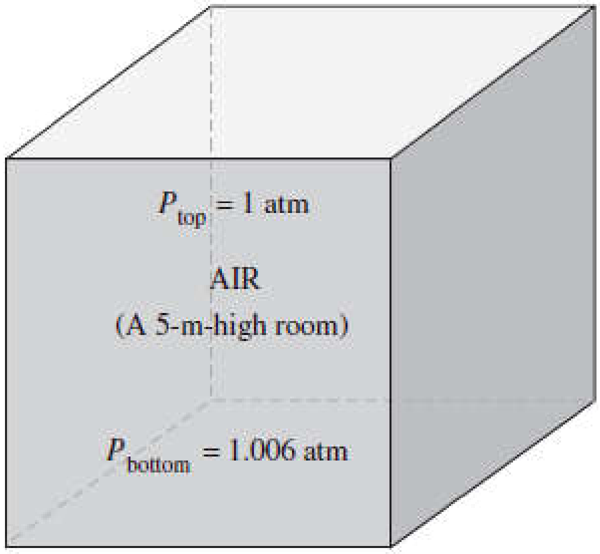


Figure5. In a room filled with a gas, the variation of pressure with height is negligible.

If we take point 1 to be at free surface of a liquid open to atmosphere (Figure 6), where the pressure is the atmospheric pressure P_{atm}, then the pressure at a depth ***h*** from the free surface becomes:

$$P = P_{atm} + \rho g h \quad \text{or} \quad P_{gage} = \rho g h$$

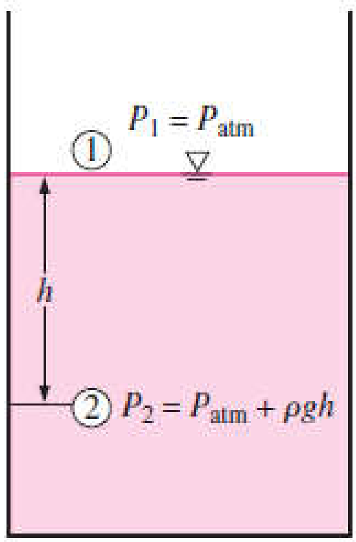


Figure6.Pressure in a liquid at rest increases linearly with distance from the free surface.

Liquids are essentially incompressible substances, and thus the variation of density with depth is negligible. This is also the case for gases when the elevation change is not very large. However, the variation of density of liquids or gases with temperature can be significant.

The gravitational acceleration g varies from 9.807m/s^2 at sea level to 9.764 m/s^2 at an elevation of 14000 m where large passenger planes cruise. This is a change of just 0.4 percent in this extreme case. Therefore, g can be assumed to be constant with negligible error.

When the variation of density with elevation is known, the pressure difference between points 1 and 2 can be determined by integration to be:

$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g dz$$

Pressure in a fluid at rest is independent of the shape or cross section of the container. It changes with the vertical distance, but remains constant in in other directions. Therefore, the pressure is the same at all points on a horizontal plane in a given fluid.

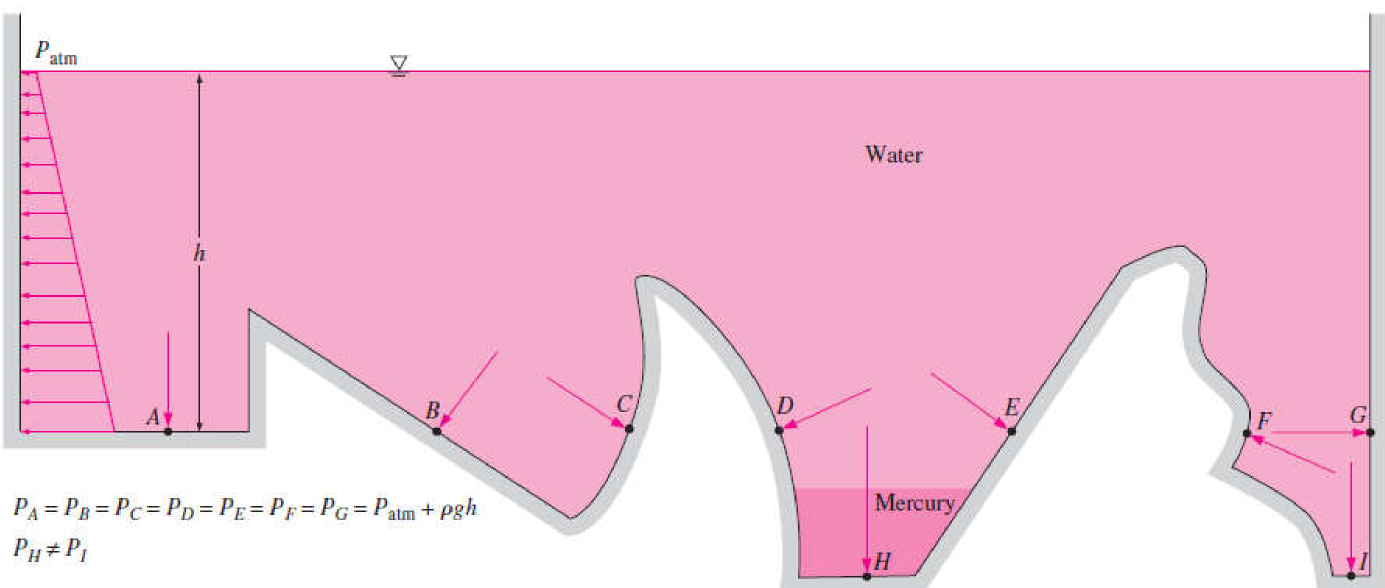


Figure7.The pressure is the same at all points on a horizontal plane in a given fluid regardless of geometry, provided that the points are interconnected by the same fluid.

Pascal's law: The pressure applied to a confined fluid increases the pressure throughout by the same amount.

$$P_1 = P_2 \rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

The area ratio A_2/A_1 is called the ideal mechanical advantage of the hydraulic lift.

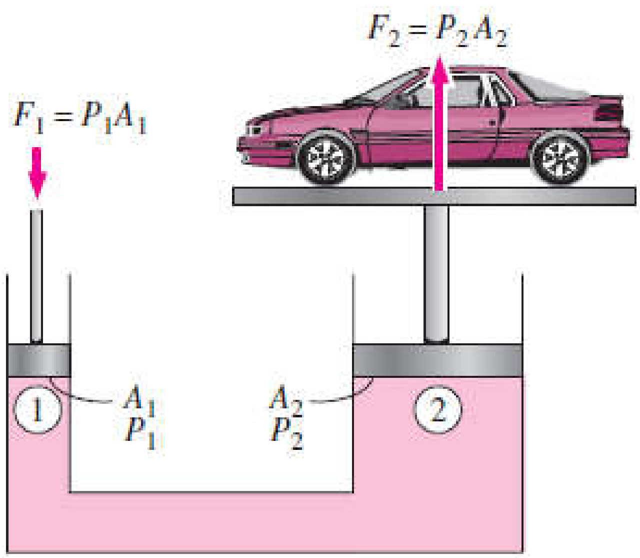


Figure8.Lifting of a large weight by a small force by the application of Pascal's law

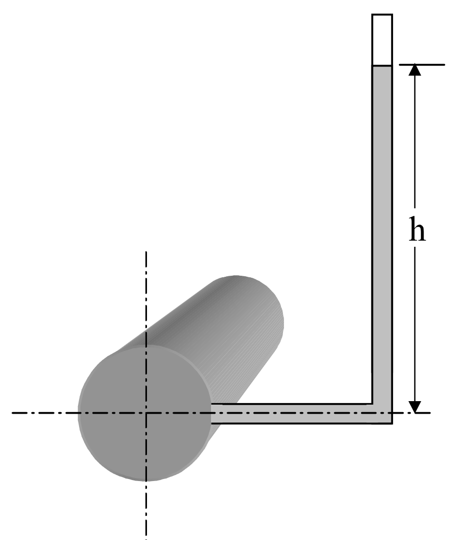
Lecture Seven

Measurement of Fluid Pressure

In chemical and other industrial processing plants it is often to measure and control the pressure in vessel or process and/or the liquid level vessel.

The pressure measuring devices are: -

- **Piezometer tube**



The piezometer consists a tube open at one end to atmosphere, the other end is capable of being inserted into vessel or pipe of which pressure is to be measured. The height to which liquid rises up in the vertical tube gives the pressure head directly.

i.e. $\boxed{P = h\,\rho\,g}$

Piezometer is used for measuring moderate pressures. It is meant for measuring *gauge pressure* **only** as the end is open to atmosphere. It cannot be used for *vacuum pressures*.

- **Manometers**

The manometer is an improved (modified) form of a piezometer. It can be used for measurement of comparatively *high pressures* and of both *gauge and vacuum pressures*.

Following are the various types of manometers: -

a- Simple manometer

b- The well type manometer

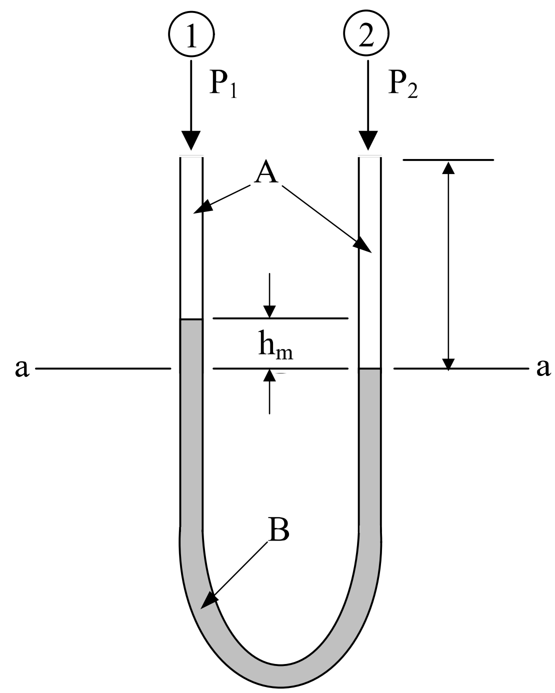
c- Inclined manometer

d- The inverted manometer

e- The two-liquid manometer

a- Simple manometer

It consists of a transparent U-tube containing the fluid A of density (ρ_A) whose pressure is to be measured and an immiscible fluid (B) of higher density (ρ_B). The limbs are connected to the two points between which the pressure difference (P_2 - P_1) is required; the connecting leads should be completely full of fluid A. If P_2 is greater than P_1, the interface between the two liquids in limb ② will be depressed a distance (h_m) (say) below that in limb ①.



The pressure at the level a — a must be the same in each of the limbs and, therefore:

$$P_2 + Z_m \rho_A g = P_1 + (Z_m - h_m) \rho_A g + h_m \rho_B g$$

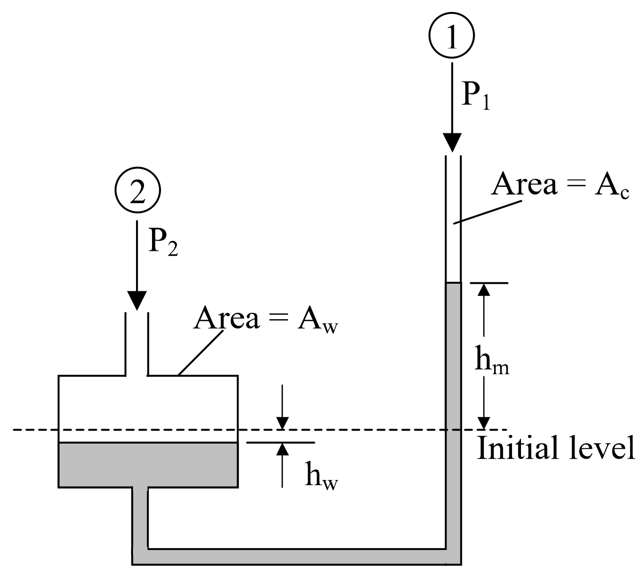
$$\Rightarrow \quad \Delta p = P_2 - P_1 = h_m (\rho_B - \rho_A) g$$

If fluid A is a gas, the density ρ_A will normally be small compared with the density of the manometer fluid pm so that:

$$\Delta p = P_2 - P_1 = h_m \rho_B g$$

b- The well-type manometer

In order to avoid the inconvenience of having to read two limbs, and in order to measure low pressures, where accuracy id of much importance, the well-type manometer shown in Figure (5) can be used. If A_w and A_c are the cross-sectional areas of the well and the column and h_m is the increase in the level of the column and h_w the decrease in the level of the well, then:



$$P_2 = P1 + (h_m + h_w) \rho g$$

or: $$\Delta p = P_2 - P_1 = (h_m + h_w) \rho g$$

The quantity of liquid expelled from the well is equal to the quantity pushed into the column so that:

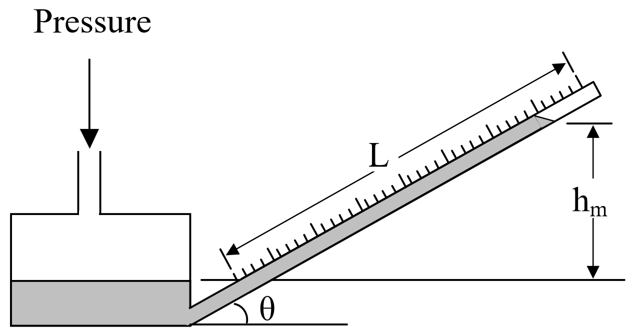
$$A_w h_w = A_c h_m \Rightarrow h_w = (A_c/A_w) h_m$$

$$\Rightarrow \Delta p = P_2 - P_1 = \rho g h_m (1 + A_c/A_w)$$

If the well is large in comparison to the column then:

i.e. $(A_c/A_w) \rightarrow \approx 0 \Rightarrow \quad \Delta p = P_2 - P_1 = \rho g h_m$

c- The inclined manometer



Shown in Figure (6) enables the sensitivity of the manometers described previously to be increased by measuring the length of the column of liquid. If θ is the angle of inclination of the manometer (typically about 10-20°) and L is the movement of the column of liquid along the limb, then:

$$h_m = L \sin \theta$$

If $\theta = 10°$, the manometer reading L is increased by about 5.7 times compared with the reading h_m which would have been obtained from a simple manometer.

d- The inverted manometer

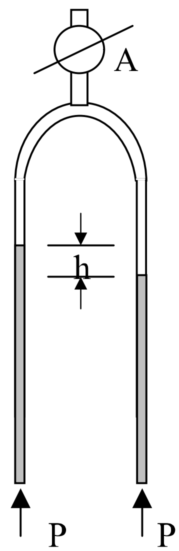


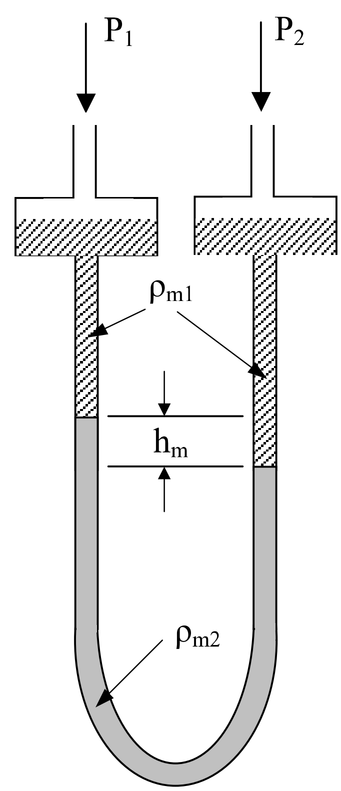
Figure (7) is used for measuring *pressure differences in liquids*. The space above the liquid in the manometer is filled with air, which can be admitted or expelled through the tap A in order to adjust the level of the liquid in the manometer.

e- The two-liquid manometer

Small differences in pressure in gases are often measured with a manometer of the form shown in Figure 6.5. The reservoir at the top of each limb is of a sufficiently large cross-section for the liquid level to remain approximately
the on each side of the manometer.
The difference in pressure is then given by:

$$\Delta p = P_2 - P_1 = h_m (\rho_{m1} - \rho_{m2})\, g$$

where ρ_{m1} and ρm2 are the densities of the two manometer liquids. The sensitivity of the instrument is very high if the densities of the two liquids are nearly the same. To obtain accurate readings it is necessary to choose liquids, which give sharp interfaces: paraffin oil and industrial alcohol are commonly used.



2- PRESSURE MEASUREMENT DEVICES

2-1. The Manometer:

It is commonly used to measure small and moderate pressure differences. A manometer contains one or more fluids such as mercury, water, alcohol, or oil. Fluid column can be used to measure pressure differences. To keep the size of the manometer to a manageable level, heavy fluids such as mercury are used if large pressure differences are anticipated.

Consider the manometer shown in figure9. That used to measure the pressure in the tank.

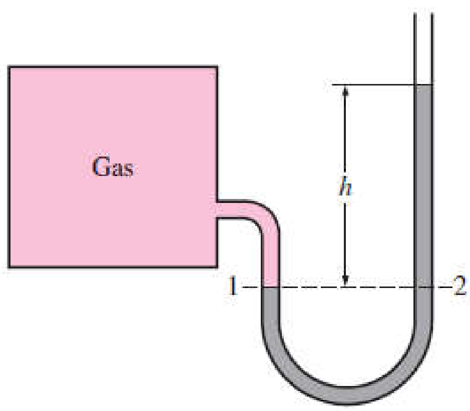


Figure9.The basic manometer

As shown in the figure above the pressure in a fluid does not vary in the horizontal direction with in the fluid, the pressure at point 2 is the same as the pressure at the point 1. $P_1=P_2$

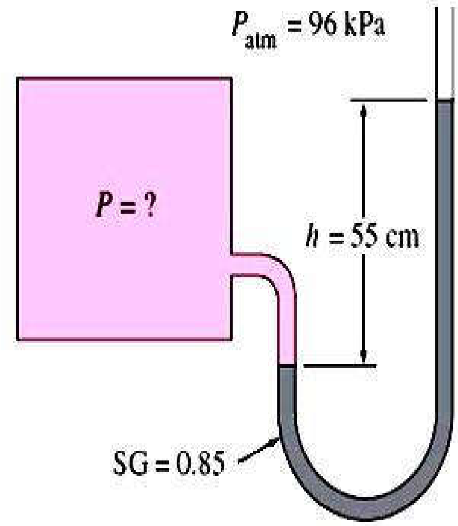
Because the fluid at point 2 is opened to the atmosphere, the pressure at point to can be measured from:

$$P_2 = P_{atm} + \rho g h$$

We should note that, the diameter of the tube is more than 1 cm to make ensure that the surface tension and the capillary rise will not effect.

Measuring Pressure with a Manometer

A manometer is used to measure the pressure of a gas in a tank. The fluid used has a specific gravity of 0.85, and the manometer column height is 55 cm, as shown in **the figure**. If the local atmospheric pressure is 96 kPa, determine the absolute pressure within the tank.



SOLUTION The reading of a manometer attached to a tank and the atmospheric pressure are given. The absolute pressure in the tank is to be determined.

Assumptions The density of the gas in the tank is much lower than the density of the manometer fluid.

Properties The specific gravity of the manometer fluid is given to be 0.85. We take the standard density of water to be 1000 kg/m^3.

Analysis The density of the fluid is obtained by multiplying its specific gravity by the density of water,

$$\rho = \text{SG}\,(\rho_{H_2O}) = (0.85)(1000\ \text{kg/m}^3) = 850\ \text{kg/m}^3$$

Then from Eq. 3–13,

$$P = P_{atm} + \rho g h$$

$$= 96\ \text{kPa} + (850\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.55\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right)$$

$$= \mathbf{100.6\ kPa}$$

Discussion Note that the gage pressure in the tank is 4.6 kPa.

Some manometers involve multiple immiscible fluids of different densities stacked on top of each other. The system can be analyzed easily by remembering that:

a- The pressure change across a fluid column of height $\boldsymbol{h}$ is $\boldsymbol{\Delta P=\rho g h}$.

b- Pressure increases down ward in a given fluid and decreases upward ($\boldsymbol{P_{battom}>P_{top}}$).

c- Two points at same elevation in a continuous fluid at rest are at the same pressure.

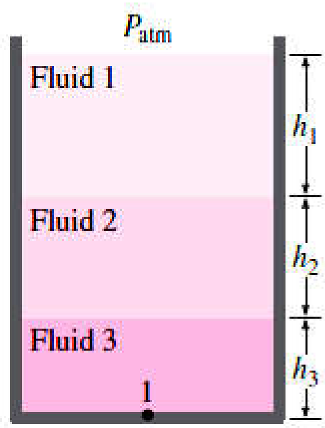


Figure10. In stacked-up fluid layers, the pressure change across a fluid layer of density $\boldsymbol{\rho}$ and height $\boldsymbol{h}$ is $\boldsymbol{\rho g h}$

The pressure at the (figure 10) can be determined by starting at the free surface where the pressure is $\boldsymbol{P_{atm}}$, moving downward until we reach point 1 at the bottom, and setting the result equal to $\boldsymbol{P_1}$. It gives:

$$P_{atm} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1$$

Furthermore, if all fluids have the same density the equation will be:

$$P_1 = P_{atm} + \rho g h$$

Manometers are particularly well-suited to measure pressure drops across a horizontal flow section between two specified points due to the presence of a device such as a valve or heat exchanger or any resistance to flow. This is done by connecting the two legs of manometer to these two points, as shown in (figure11). The working fluid can be either a gas or a liquid whose density is $\boldsymbol{\rho_1}$. The density of the manometer fluid is $\boldsymbol{\rho_2}$, and the differential fluid height is $\boldsymbol{h}$. A relation for the pressure difference P_1-P_2 can be obtained be by starting at point 1 with P_1, moving along the tube by adding or subtracting the $\rho g h$ terms until we reach point P_2, and setting the result equal to P_2 :

$$P_1 + \rho_1 g(a+h) - \rho_2 g h - \rho_1 g a = P_2$$

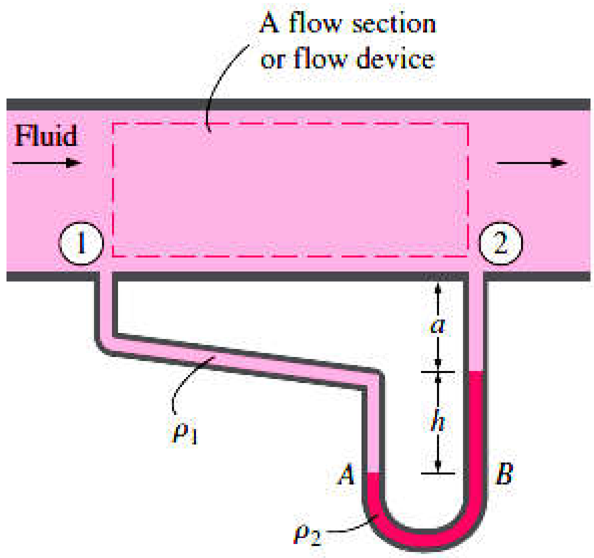


Figure11. Measuring the pressure drop across a flow section or a flow device by a differential manometer

Note that we jumped from point A horizontally to point B and ignored the part underneath since the pressure at both points is the same. Simplifying

$$P_1 - P_2 = (\rho_2 - \rho_1) g h$$

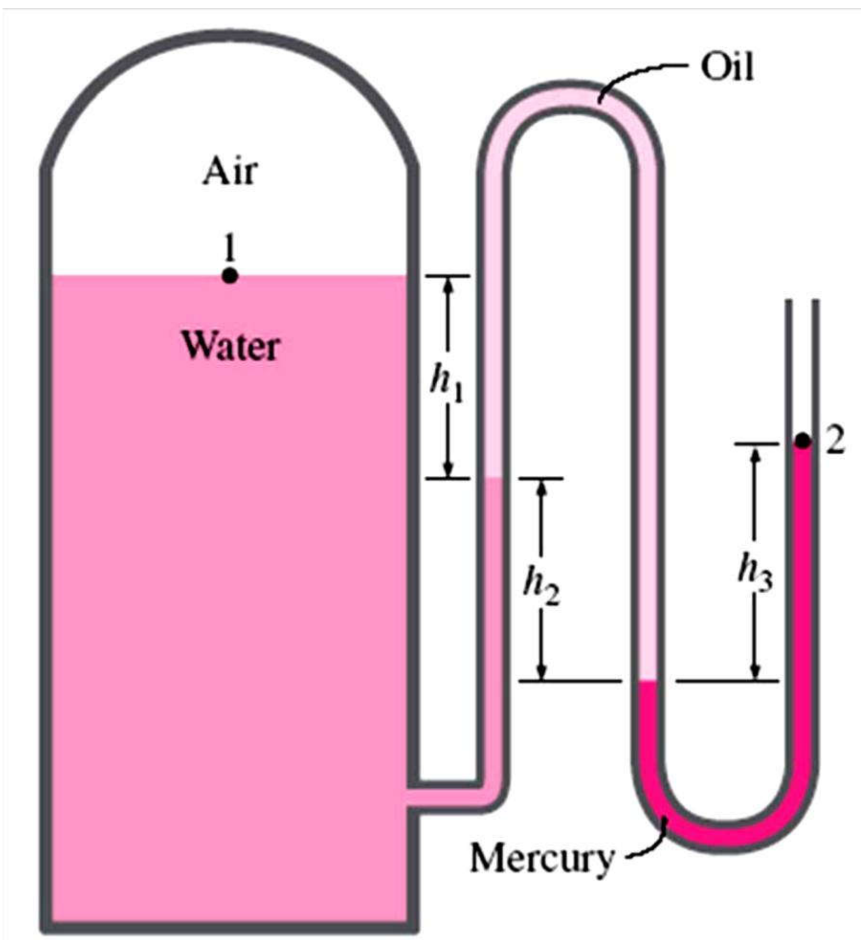
When the fluid flowing in the pipe is a gas, then $\rho_1 << \rho_2$ and the relation simplifying to

$$P_1 - P_2 \cong \rho_2 g h$$

Measuring Pressure with a Multifluid Manometer

The water in a tank is pressurized by air, and the pressure is measured by a multifluid manometer as shown in the figure . The tank is located on a mountain at an altitude of 1400 m where the atmospheric pressure is 85.6 kPa. Determine the air pressure in the tank if $h_1 = 0.1$ m, $h_2 = 0.2$ m, and $h_3 = 0.35$ m. Take the densities of water, oil, and mercury to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.

SOLUTION The pressure in a pressurized water tank is measured by a multifluid manometer. The air pressure in the tank is to be determined.



Assumption The air pressure in the tank is uniform (i.e., its variation with elevation is negligible due to its low density), and thus we can determine the pressure at the air–water interface.

Properties The densities of water, oil, and mercury are given to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.

Analysis Starting with the pressure at point 1 at the air–water interface, moving along the tube by adding or subtracting the ρgh terms until we reach point 2, and setting the result equal to P_{atm} since the tube is open to the atmosphere gives

$$P_1 + \rho_{\text{water}} g h_1 + \rho_{\text{oil}} g h_2 - \rho_{\text{mercury}} g h_3 = P_2 = P_{\text{atm}}$$

Solving for P_1 and substituting,

$$\begin{aligned} P_1 &= P_{\text{atm}} - \rho_{\text{water}} g h_1 - \rho_{\text{oil}} g h_2 + \rho_{\text{mercury}} g h_3 \\ &= P_{\text{atm}} + g(\rho_{\text{mercury}} h_3 - \rho_{\text{water}} h_1 - \rho_{\text{oil}} h_2) \\ &= 85.6\text{ kPa} + (9.81\text{ m/s}^2)[(13{,}600\text{ kg/m}^3)(0.35\text{ m}) - (1000\text{ kg/m}^3)(0.1\text{ m}) \\ &\quad - (850\text{ kg/m}^3)(0.2\text{ m})]\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right)\left(\frac{1\text{ kPa}}{1000\text{ N/m}^2}\right) \\ &= \mathbf{130\ kPa} \end{aligned}$$

2-2. The Barometer:

- Atmospheric pressure is measured by a device called a barometer; thus, the atmospheric pressure is often referred to as the barometric pressure.
- The atmospheric pressure can be measured by inverting a mercury-filled tube into a mercury container that is open in to atmosphere as shown the figure (12). The pressure at point **B** is equal to atmospheric pressure, and the pressure at point **C** can be taken to be zero since there is only mercury vapor above the point **C** and the pressure is very low relative to **P_{atm}** and can be neglected to an excellent approximation. Writing a force balance in the vertical direction give:

$$P_{atm} = \rho g h$$

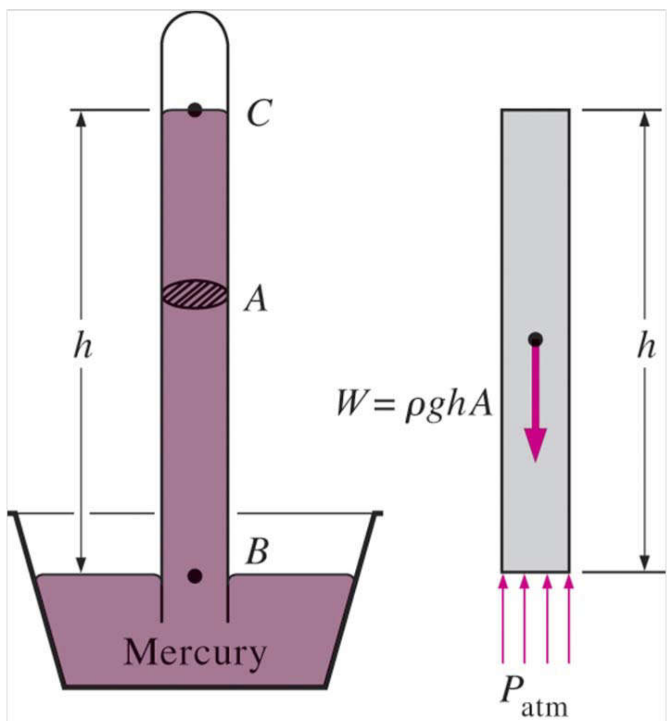


Figure 12. The basic barometer.

- Note that the length and the cross sectional area of the tube have no effect on the height of the fluid column of a barometer, figure 13

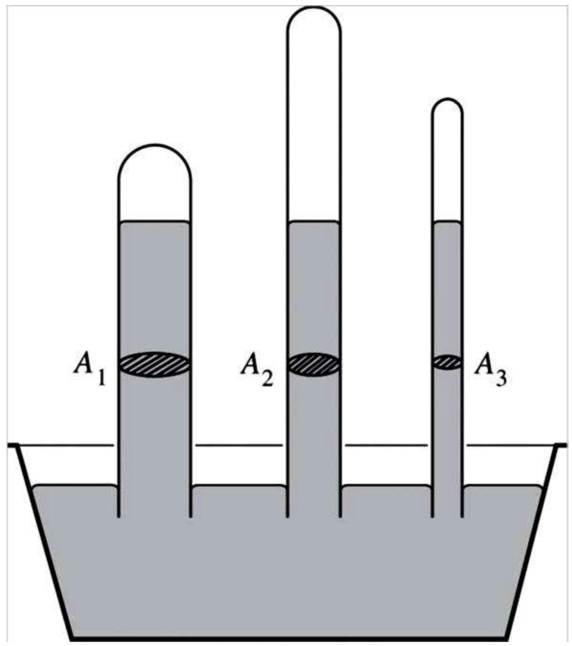


Figure 13.The length or the cross-sectional area of the tube has no effect on the height of the fluid column of a barometer, provided that the tube diameter is large enough to avoid surface tension (capillary) effects.

- A frequently used pressure unit is the standard atmosphere, which is defined as the pressure produced by a column of mercury 760 mm in height at 0°C (ρ_{Hg} = 13,595 kg/ m^3) under standard gravitational acceleration (g = 9.807 m/s^2).

EXAMPLE 3–2 **Measuring Atmospheric Pressure with a Barometer**

Determine the atmospheric pressure at a location where the barometric reading is 740 mm Hg and the gravitational acceleration is $g = 9.805\ \text{m/s}^2$. Assume the temperature of mercury to be 10°C, at which its density is 13,570 kg/m^3.

SOLUTION The barometric reading at a location in height of mercury column is given. The atmospheric pressure is to be determined.
Assumptions The temperature of mercury is assumed to be 10°C.
Properties The density of mercury is given to be 13,570 kg/m^3.
Analysis From Eq. 3–12, the atmospheric pressure is determined to be

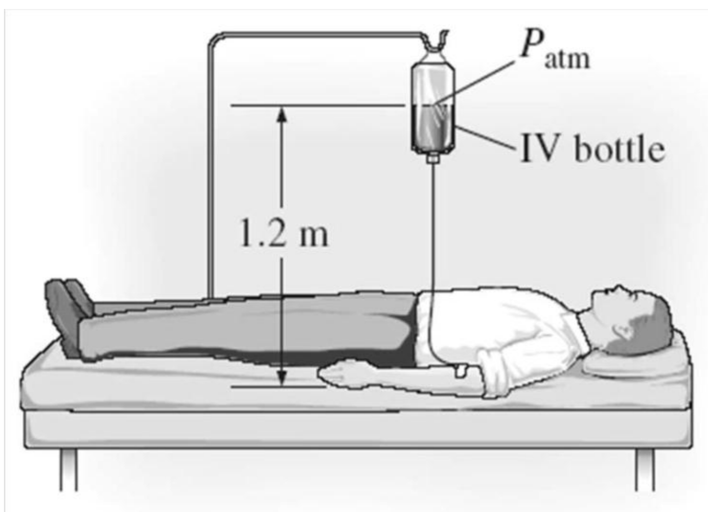
$$
\begin{aligned}
P_{\text{atm}} &= \rho g h \\
&= (13{,}570\ \text{kg/m}^3)(9.805\ \text{m/s}^2)(0.740\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right) \\
&= \mathbf{98.5\ kPa}
\end{aligned}
$$

Discussion Note that density changes with temperature, and thus this effect should be considered in calculations.

Gravity Driven Flow from an IV Bottle

Intravenous infusions usually are driven by gravity by hanging the fluid bottle at sufficient height to counteract the blood pressure in the vein and to force the fluid into the body (Fig. below) The higher the bottle is raised, the higher the flow rate of the fluid will be. (*a*) If it is observed that the fluid and the blood pressures balance each other when the bottle is 1.2 m above the arm level, determine the gage pressure of the blood. (*b*) If the gage pressure of the fluid at the arm level needs to be 20 kPa for sufficient flow rate, determine how high the bottle must be placed. Take the density of the fluid to be 1020 kg/m^3.

SOLUTION It is given that an IV fluid and the blood pressures balance each other when the bottle is at a certain height. The gage pressure of the blood and elevation of the bottle required to maintain flow at the desired rate are to be determined.



Assumptions 1 The IV fluid is incompressible. 2 The IV bottle is open to the atmosphere.

Properties The density of the IV fluid is given to be $\rho = 1020 \text{ kg/m}^3$.

Analysis (*a*) Noting that the IV fluid and the blood pressures balance each other when the bottle is 1.2 m above the arm level, the gage pressure of the blood in the arm is simply equal to the gage pressure of the IV fluid at a depth of 1.2 m,

$$\begin{aligned} P_{\text{gage, arm}} &= P_{\text{abs}} - P_{\text{atm}} = \rho g h_{\text{arm - bottle}} \\ &= (1020 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(1.20 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)\left(\frac{1 \text{ kPa}}{1 \text{ kN/m}^2}\right) \\ &= \mathbf{12.0 \text{ kPa}} \end{aligned}$$

(*b*) To provide a gage pressure of 20 kPa at the arm level, the height of the surface of the IV fluid in the bottle from the arm level is again determined from $P_{\text{gage, arm}} = \rho g h_{\text{arm - bottle}}$ to be

$$\begin{aligned} h_{\text{arm - botttle}} &= \frac{P_{\text{gage, arm}}}{\rho g} \\ &= \frac{20 \text{ kPa}}{(1020 \text{ kg/m}^3)(9.81 \text{ m/s}^2)}\left(\frac{1000 \text{ kg} \cdot \text{m/s}^2}{1 \text{ kN}}\right)\left(\frac{1 \text{ kN/m}^2}{1 \text{ kPa}}\right) \\ &= \mathbf{2.00 \text{ m}} \end{aligned}$$

Discussion Note that the height of the reservoir can be used to control flow rates in gravity-driven flows. When there is flow, the pressure drop in the tube due to frictional effects also should be considered. For a specified flow rate, this requires raising the bottle a little higher to overcome the pressure drop.

<u>Other Pressure Measurement Devices:</u>

• **Bourdon tube:** Consists of a hollow metal tube bent like a hook whose end is closed and connected to a dial indicator needle.

• **Pressure transducers:** Use various techniques to convert the pressure effect to an electrical effect such as a change in voltage, resistance, or capacitance.

• Pressure transducers are smaller and faster, and they can be more sensitive, reliable, and precise than their mechanical counterparts.

• **Strain-gage pressure transducers:** Work by having a diaphragm deflect between two chambers open to the pressure inputs.

• **Piezoelectric transducers:** Also called solid state pressure transducers, work on the principle that an electric potential is generated in a crystalline substance when it is subjected to mechanical pressure.

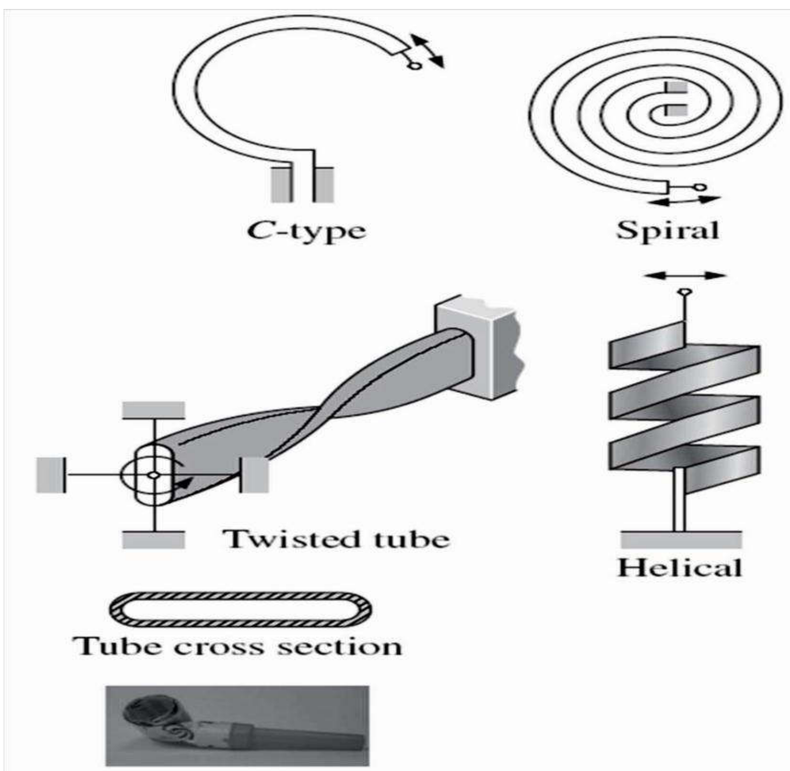


Figure14. Various types of Bourdon tubes used to measure pressure.

Lecture 8

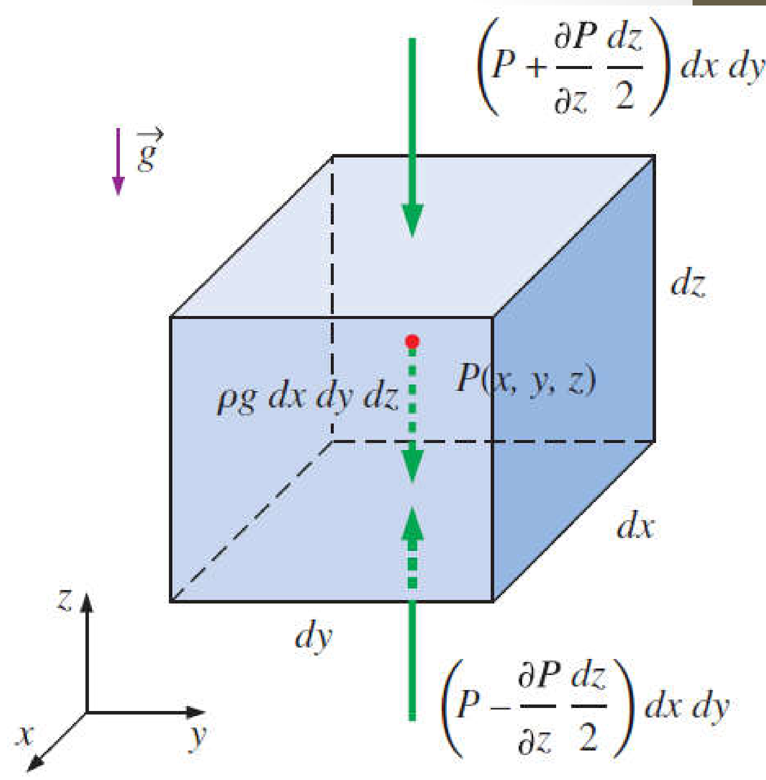
FLUIDS IN RIGID-BODY MOTION

Newton's second law of motion **for this element can be expressed as**

$$\delta\vec{F} = \delta m \cdot \vec{a}$$

$$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$$

The forces acting on the fluid element consist of *body forces* such as gravity and *surface forces* such as the pressure forces



$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z} dx\, dy\, dz$$

$$\delta F_{S,x} = -\frac{\partial P}{\partial x} dx\, dy\, dz \quad \text{and} \quad \delta F_{S,y} = -\frac{\partial P}{\partial y} dx\, dy\, dz \quad \delta F_{S,z} = -\frac{\partial P}{\partial z} dx\, dy\, dz$$

$$\delta\vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$

$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\,dy\,dz = -\vec{\nabla}P\,dx\,dy\,dz$$

The surface force (the pressure force) acting on the entire element expressed by vector form

$$\vec{\nabla}P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$

∇: Del: The pressure Gradient

$$\delta\vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\,dx\,dy\,dz\vec{k}$$

The weight is only **body force** acting on the fluid element

$$\delta\vec{F} = \delta\vec{F}_S + \delta\vec{F}_B = -(\vec{\nabla}P + \rho g\vec{k})\,dx\,dy\,dz$$

Rigid-body motion of fluids: $\boxed{\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}}$

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

Important

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$

Special Case 1: Fluids at Rest

For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero, and the relations reduce to

$$\textit{Fluids at rest:} \qquad \frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

The pressure remains constant in any horizontal direction (P is independent of x and y) and varies only in the vertical direction as a result of gravity [and thus $P = P(z)$]. These relations are applicable for both compressible and incompressible fluids.

A glass of water at rest is a special case of a fluid in rigid-body motion. If the glass of water were moving at constant velocity in any direction, the hydrostatic equations would still apply.

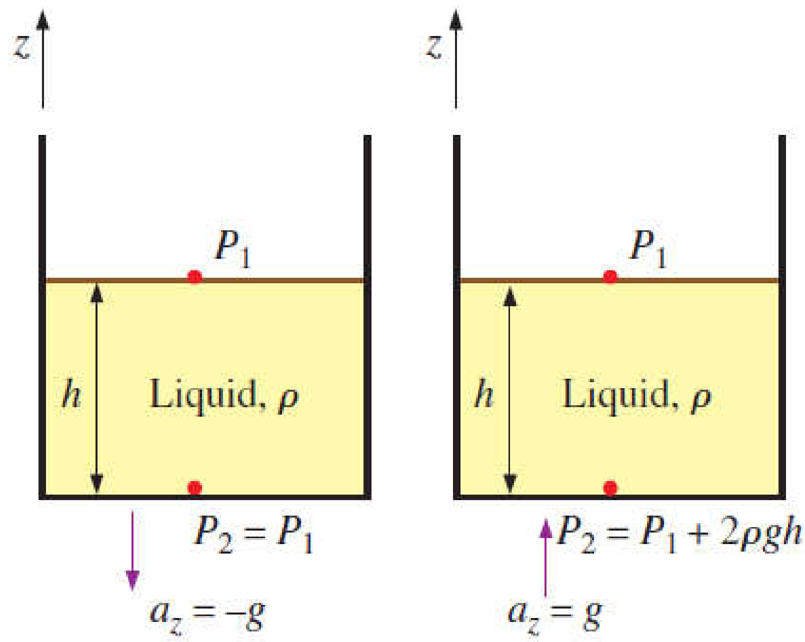
Special Case 2: Free Fall of a Fluid Body

A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero. Therefore, $a_x = a_y = 0$ and $a_z = -g$.

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$

Free-falling fluids: $$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$$

In a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout.



(*a*) Free fall of a liquid

(*b*) Upward acceleration of a liquid with $a_z = +g$

The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

Acceleration on a Straight Path

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

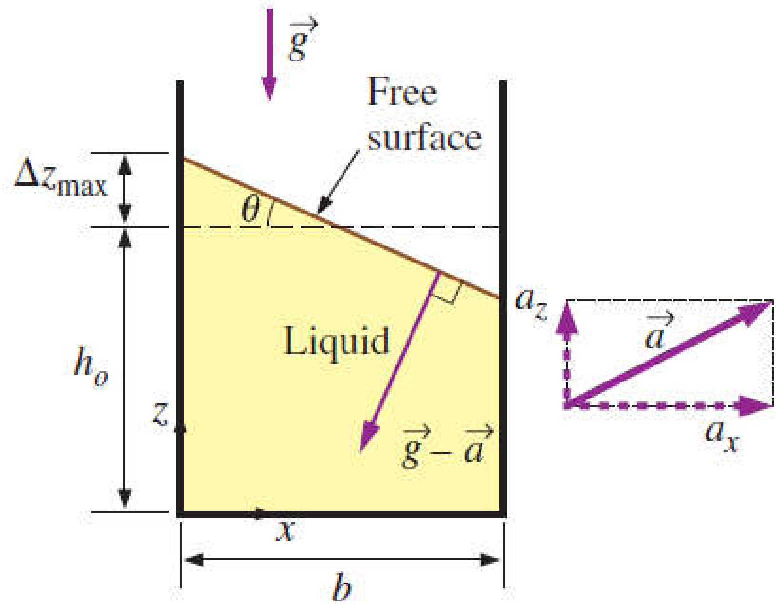
$$P = P(x, z) \qquad dP = (\partial P/\partial x)\, dx + (\partial P/\partial z)\, dz$$

$$dP = -\rho a_x\, dx - \rho(g + a_z)\, dz$$

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Taking P_1 to be the original (x=0, z=0) where the pressure is P_0 and the point 2 to be any point in the fluid (no subscript) then the pressure distribution is:

Pressure variation: $\boxed{P = P_0 - \rho a_x x - \rho(g + a_z)z}$



Rigid-body motion of a liquid in a linearly accelerating tank.
(The container is moving on a straight path with a constant acceleration)

The vertical rise (or drop) of the free surface at point 2 relative to point 1 is determined by choosing both 1 and 2 on the free surface so (P_1=P_2).

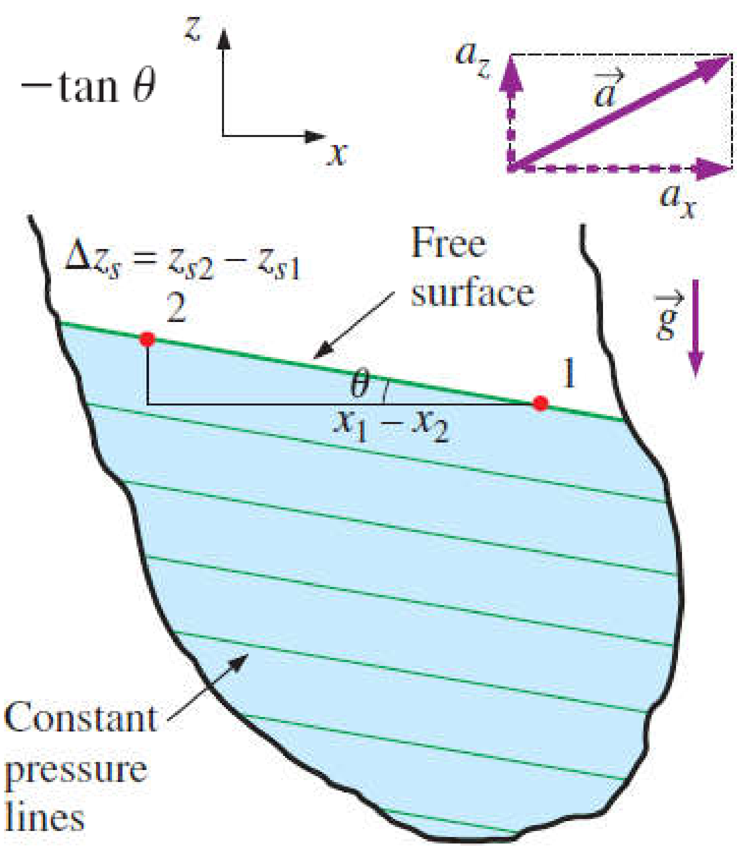
$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Vertical rise of surface: $\Delta z_s = z_{s2} - z_{s1} = -\dfrac{a_x}{g + a_z}(x_2 - x_1)$

Zs: is the z coordinate of the liquid's free surface. Setting dp=o:

Surfaces of constant pressure: $\dfrac{dz_{\text{isobar}}}{dx} = -\dfrac{a_x}{g + a_z} = \text{constant}$

Slope of isobars: $\text{Slope} = \dfrac{dz_{\text{isobar}}}{dx} = -\dfrac{a_x}{g + a_z} = -\tan\theta$



Lines of constant pressure (which are the projections of the surfaces of constant pressure on the *xz*-plane) in a linearly accelerating liquid. Also shown is the vertical rise.

EXAMPLE 3–12 **Overflow from a Water Tank During Acceleration**

An 80-cm-high fish tank of cross section 2 m × 0.6 m that is partially filled with water is to be transported on the back of a truck (Fig. 3–58). The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?

SOLUTION A fish tank is to be transported on a truck. The allowable water height to avoid spill of water during acceleration and the proper orientation are to be determined.

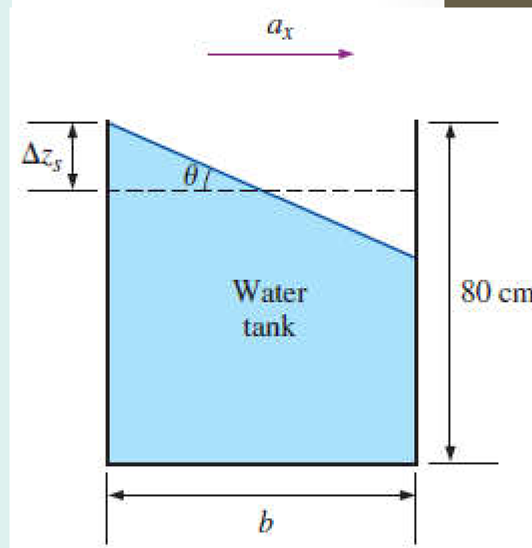
Assumptions **1** The road is horizontal during acceleration so that acceleration has no vertical component ($a_z = 0$). **2** Effects of splashing, braking, shifting gears, driving over bumps, climbing hills, etc., are assumed to be secondary and are not considered. **3** The acceleration remains constant.

Analysis We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90 - 0)\text{ km/h}}{10\text{ s}}\left(\frac{1\text{ m/s}}{3.6\text{ km/h}}\right) = 2.5\text{ m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \qquad (\text{and thus } \theta = 14.3°)$$



The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical midplane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the midplane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = \mathbf{25.5\ cm}$$

Case 2: The short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = \mathbf{7.6\ cm}$$

Therefore, assuming tipping is not a problem, **the tank should definitely be oriented such that its short side is parallel to the direction of motion.** Emptying the tank such that its free surface level drops just 7.6 cm in this case will be adequate to avoid spilling during acceleration.

Discussion Note that the orientation of the tank is important in controlling the vertical rise. Also, the analysis is valid for any fluid with constant density, not just water, since we used no information that pertains to water in the solution.

Rotation in a Cylindrical Container

Consider a vertical cylindrical container partially filled with a liquid. The container is now rotated about its axis at a constant angular velocity of ω. After initial transients, the liquid will move as a rigid body together with the container. There is no deformation, and thus there can be no shear stress, and every fluid particle in the container moves with same angular velocity. (cylindrical coordinates (r, θ, z), $a_r = -r\,\omega^2$, $a_y = 0$, $a_z = 0$

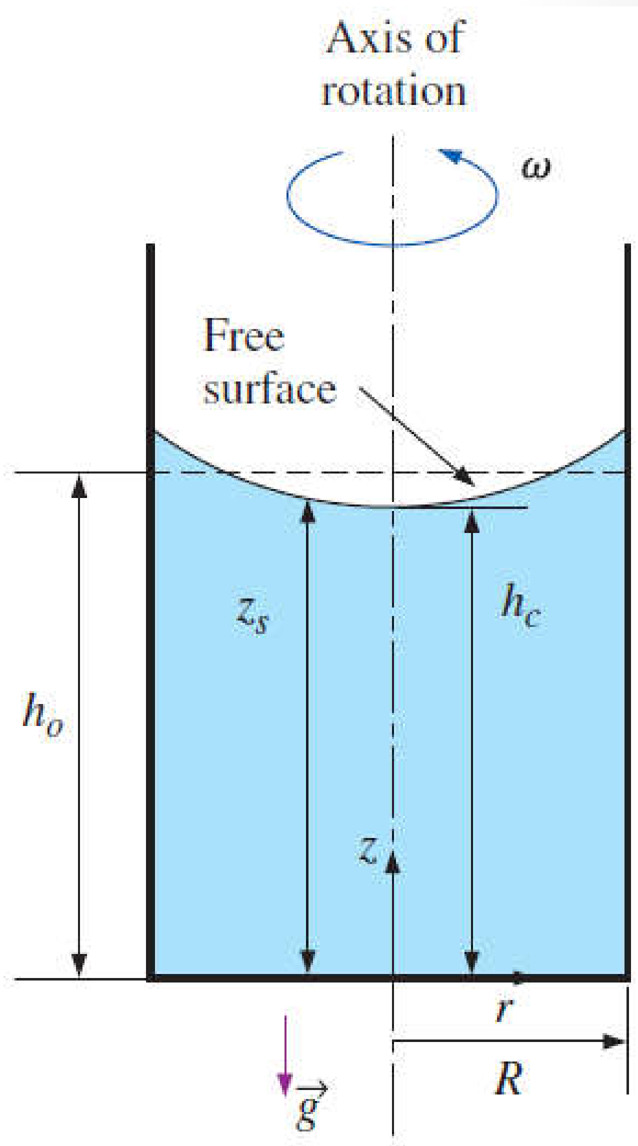
$$\frac{\partial P}{\partial r} = \rho r \omega^2, \qquad \frac{\partial P}{\partial \theta} = 0, \qquad \text{and} \qquad \frac{\partial P}{\partial z} = -\rho g$$

$$P = P(r, z) \quad dP = (\partial P/\partial r)dr + (\partial P/\partial z)dz$$

$$dP = \rho r \omega^2\, dr - \rho g\, dz$$

Surface of constant pressure (dP=0) $\dfrac{dz_{\text{isobar}}}{dr} = \dfrac{r\omega^2}{g}$

Rigid-body motion of a liquid in a rotating vertical cylindrical container.



$$\frac{dz_{\text{isobar}}}{dr} = \frac{r\omega^2}{g}$$

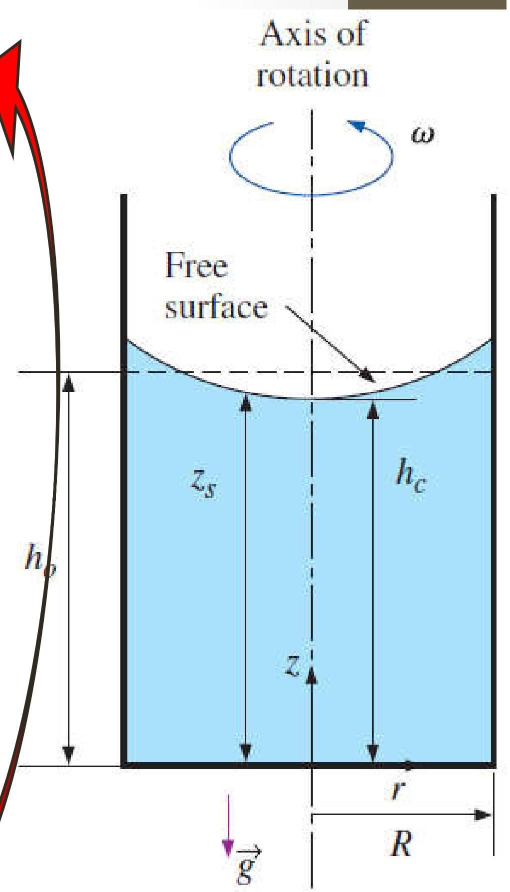
By integrating Surface of constant pressure including the free surface⇒, are paraboloids of revolution

Surface of constant pressure

$$z_{\text{isobar}} = \frac{\omega^2}{2g} r^2 + C_1$$

$$z_s = \frac{\omega^2}{2g} r^2 + h_c$$

When r=0, Z=0; at the free surface



$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

$$V = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^3 dr + r dr\right) = 2\pi\left(\frac{\omega^2}{4 \times 2g} r^4 + \frac{r^2}{2} + h_c\right) \Big|_{r=0}^{R}$$

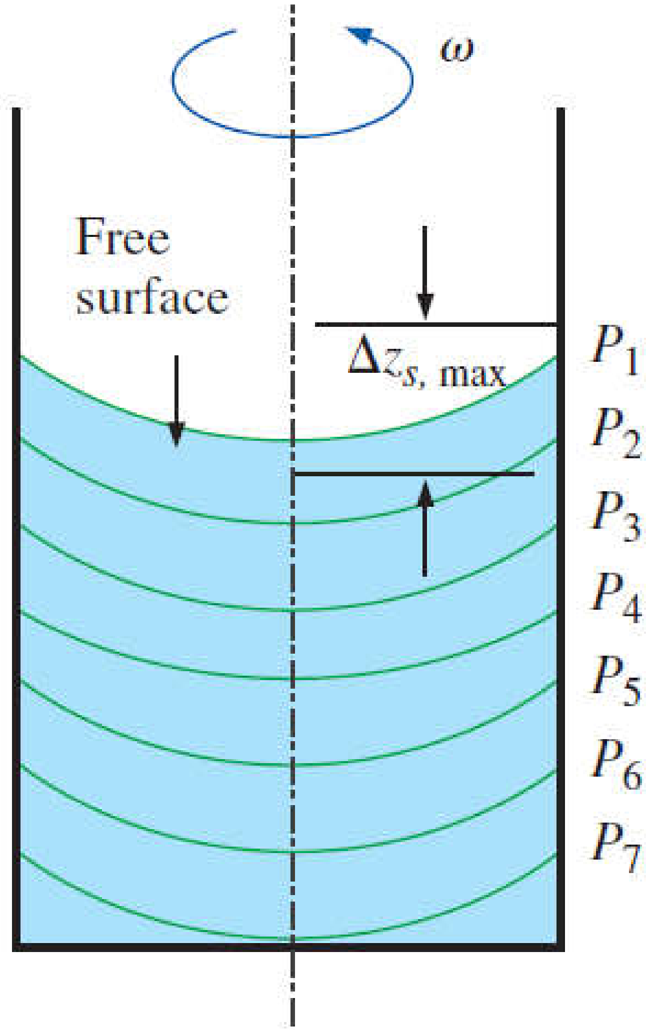
$$= 2\pi \frac{R^2}{2}\left(\frac{\omega^2}{4g} R^2 + h_c\right) = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

This should= the original volume with h_0 $\quad V = \pi R^2 h_0$

$$\pi R^2 h_0 = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

$$h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

Free surface: $\quad z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$



Surfaces of constant pressure in a rotating liquid.

From: $z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$

Maximum height difference: $\Delta z_{s,\,\max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g}R^2$

By integrating: $dP = \rho r\omega^2\,dr - \rho g\,dz$ $\quad P_2 - P_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$

Taking P_1 to be the original (r=0, z=o) where the pressure is P_0 and the point 2 to be any point in the fluid (no subscript) then the pressure distribution is:

Pressure variation: $P = P_0 + \frac{\rho\omega^2}{2}r^2 - \rho g z$

Note that at a fixed radius, the pressure varies hydrostatically in the vertical direction, as in a fluid at rest.

For a fixed vertical distance *z*, the pressure varies with the square of the radial distance *r*, increasing from the centerline toward the outer edge.

In any horizontal plane, the pressure difference between the center and edge of the container of radius *R* is

$$\Delta P = \rho\omega^2 R^2/2$$

EXAMPLE 3–13 **Rising of a Liquid During Rotation**

A 20-cm-diameter, 60-cm-high vertical cylindrical container, shown in Fig. 3–62, is partially filled with 50-cm-high liquid whose density is 850 kg/m^3. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container.

SOLUTION A vertical cylindrical container partially filled with a liquid is rotated. The angular speed at which the liquid will start spilling is to be determined.

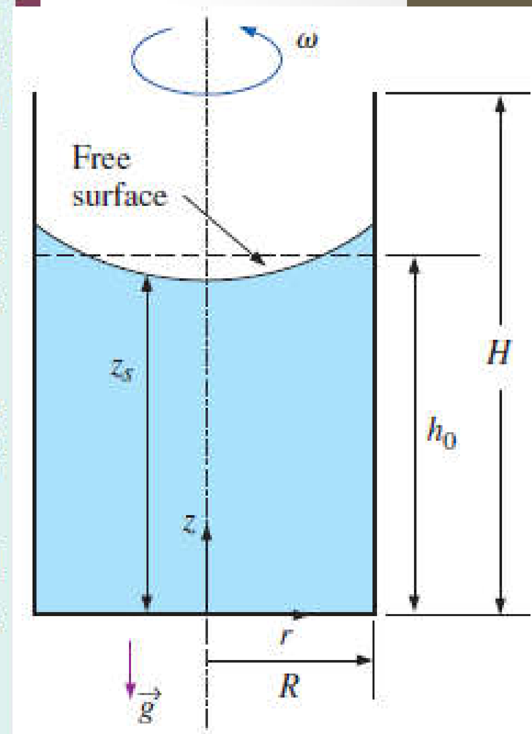
Assumptions **1** The increase in the rotational speed is very slow so that the liquid in the container always acts as a rigid body. **2** The bottom surface of the container remains covered with liquid during rotation (no dry spots).

Analysis Taking the center of the bottom surface of the rotating vertical cylinder as the origin ($r = 0$, $z = 0$), the equation for the free surface of the liquid is given as

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Then the vertical height of the liquid at the edge of the container where $r = R$ becomes

$$z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$



where $h_0 = 0.5$ m is the original height of the liquid before rotation. Just before the liquid starts spilling, the height of the liquid at the edge of the container equals the height of the container, and thus $z_s(R) = H = 0.6$ m. Solving the last equation for ω and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g(H - h_0)}{R^2}} = \sqrt{\frac{4(9.81\ \text{m/s}^2)[(0.6 - 0.5)\ \text{m}]}{(0.1\ \text{m})^2}} = \mathbf{19.8\ rad/s}$$

Noting that one complete revolution corresponds to 2π rad, the rotational speed of the container can also be expressed in terms of revolutions per minute (rpm) as

$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8\ \text{rad/s}}{2\pi\ \text{rad/rev}}\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = \mathbf{189\ rpm}$$

Therefore, the rotational speed of this container should be limited to 189 rpm to avoid any spill of liquid as a result of the centrifugal effect.

Discussion Note that the analysis is valid for any liquid since the result is independent of density or any other fluid property. We should also verify that our assumption of no dry spots is valid. The liquid height at the center is

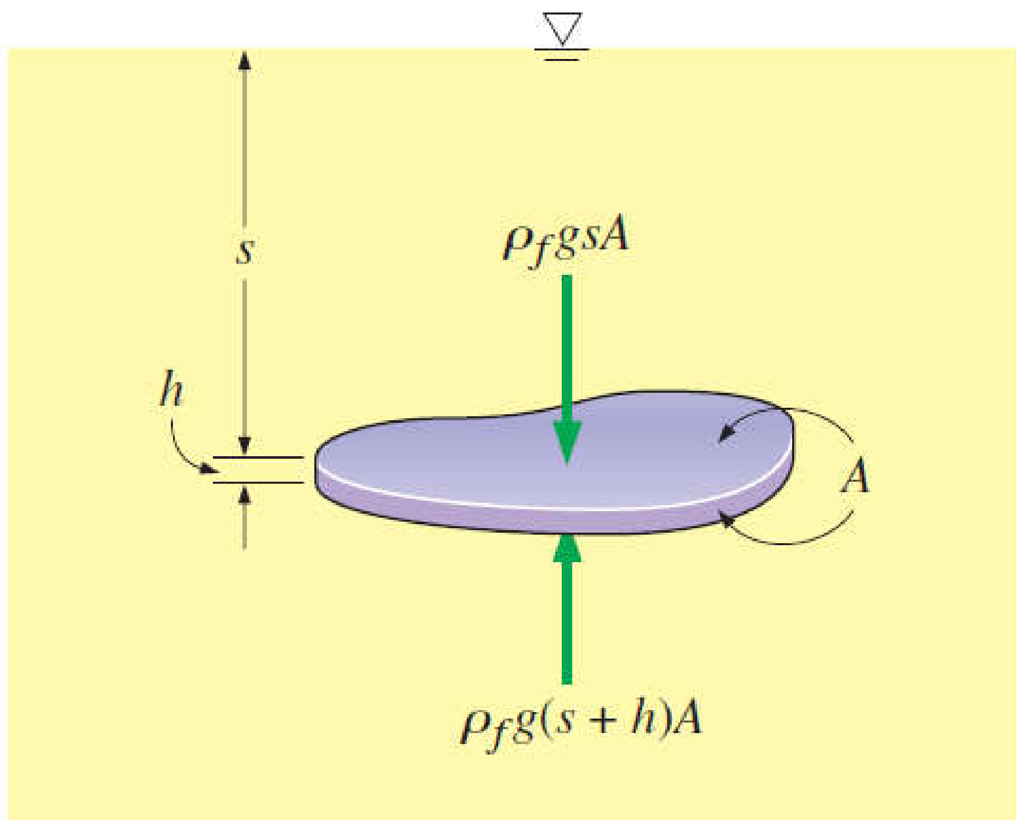
$$z_s(0) = h_0 - \frac{\omega^2 R^2}{4g} = 0.4\ \text{m}$$

Since $z_s(0)$ is positive, our assumption is validated.

Lecture 9

BUOYANCY AND STABILITY

Buoyant force: The upward force a fluid exerts on a body immersed in it. The buoyant force is caused by the increase of pressure with depth in a fluid.



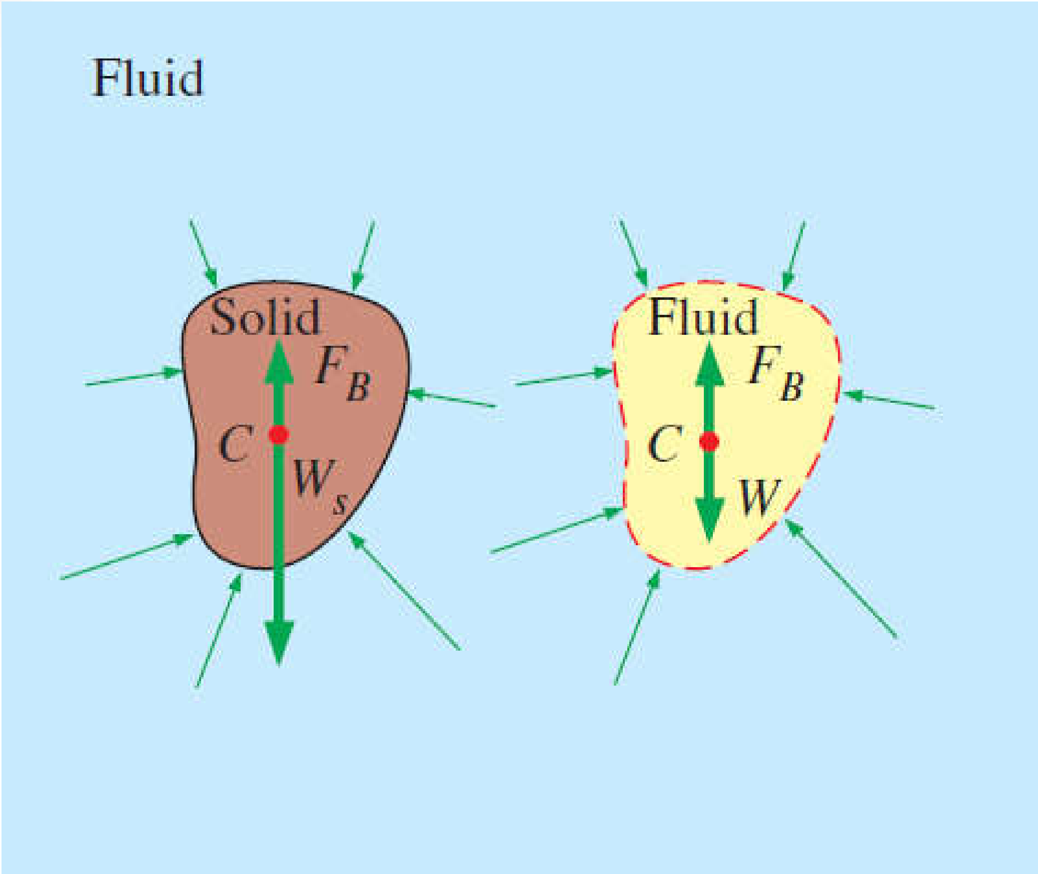
A flat plate of uniform thickness *h* submerged in a liquid parallel to the free surface.

The buoyant force acting on the plate is equal to the weight of the liquid displaced by the plate.

For a fluid with constant density, the buoyant force is independent of the distance of the body from the free surface.

It is also independent of the density of the solid body.

$$F_B = F_{\text{bottom}} - F_{\text{top}} = \rho_f g(s + h)A - \rho_f g s A = \rho_f g h A = \rho_f g V$$

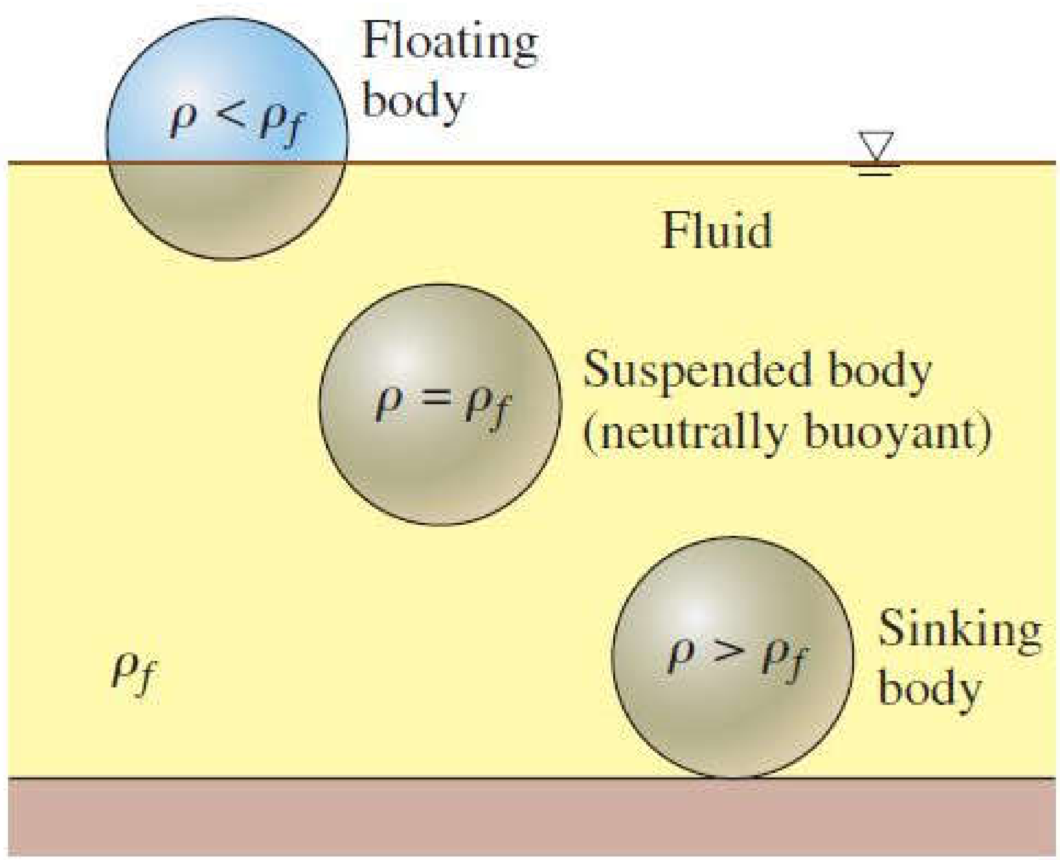


The buoyant forces acting on a solid body submerged in a fluid and on a fluid body of the same shape at the same depth are identical. The buoyant force F_B acts upward through the centroid C of the displaced volume and is equal in magnitude to the weight W of the displaced fluid, but is opposite in direction. For a solid of uniform density, its weight W_s also acts through the centroid, but its magnitude is not necessarily equal to that of the fluid it displaces. (Here $W_s > W$ and thus $W_s > F_B$; this solid body would sink.)

Archimedes' principle: The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

For *floating* bodies, the weight of the entire body must be equal to the buoyant force, which is the weight of the fluid whose volume is equal to the volume of the submerged portion of the floating body:

$$F_B = W \rightarrow \rho_f g V_{\text{sub}} = \rho_{\text{avg, body}} g V_{\text{total}} \rightarrow \frac{V_{\text{sub}}}{V_{\text{total}}} = \frac{\rho_{\text{avg, body}}}{\rho_f}$$



A solid body dropped into a fluid will sink, float, or remain at rest at any point in the fluid, depending on its average density relative to the density of the fluid.

The altitude of a hot air balloon is controlled by the temperature difference between the air inside and outside the balloon, since warm air is less dense than cold air. When the balloon is neither rising nor falling, the upward buoyant force exactly balances the downward weight.

EXAMPLE 3–10 Measuring Specific Gravity by a Hydrometer

If you have a seawater aquarium, you have probably used a small cylindrical glass tube with a lead-weight at its bottom to measure the salinity of the water by simply watching how deep the tube sinks. Such a device that floats in a vertical position and is used to measure the specific gravity of a liquid is called a *hydrometer* (Fig. 3–45). The top part of the hydrometer extends above the liquid surface, and the divisions on it allow one to read the specific gravity directly. The hydrometer is calibrated such that in pure water it reads exactly 1.0 at the air–water interface. (*a*) Obtain a relation for the specific gravity of a liquid as a function of distance Δz from the mark corresponding to pure water and (*b*) determine the mass of lead that must be poured into a 1-cm-diameter, 20-cm-long hydrometer if it is to float halfway (the 10-cm mark) in pure water.

SOLUTION The specific gravity of a liquid is to be measured by a hydrometer. A relation between specific gravity and the vertical distance from the reference level is to be obtained, and the amount of lead that needs to be added into the tube for a certain hydrometer is to be determined.

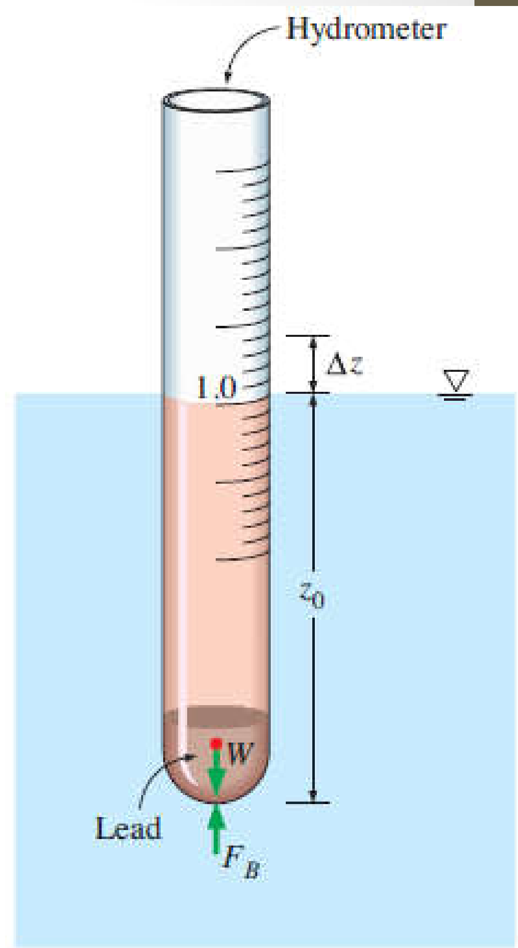
Assumptions **1** The weight of the glass tube is negligible relative to the weight of the lead added. **2** The curvature of the tube bottom is disregarded.

Properties We take the density of pure water to be 1000 kg/m^3.

Analysis (*a*) Noting that the hydrometer is in static equilibrium, the buoyant force F_B exerted by the liquid must always be equal to the weight W of the hydrometer. In pure water (subscript w), we let the vertical distance between the bottom of the hydrometer and the free surface of water be z_0. Setting $F_{B,\,w} = W$ in this case gives

$$W_{\text{hydro}} = F_{B,w} = \rho_w g V_{\text{sub}} = \rho_w g A z_0 \qquad (1)$$

where A is the cross-sectional area of the tube, and ρ_w is the density of pure water.



In a fluid lighter than water ($\rho_f < \rho_w$), the hydrometer will sink deeper, and the liquid level will be a distance of Δz above z_0. Again setting $F_B = W$ gives

$$W_{\text{hydro}} = F_{B,f} = \rho_f g V_{\text{sub}} = \rho_f g A(z_0 + \Delta z) \qquad \textbf{(2)}$$

This relation is also valid for fluids heavier than water by taking Δz to be a negative quantity. Setting Eqs. (1) and (2) here equal to each other since the weight of the hydrometer is constant and rearranging gives

$$\rho_w g A z_0 = \rho_f g A(z_0 + \Delta z) \quad \rightarrow \quad \text{SG}_f = \frac{\rho_f}{\rho_w} = \frac{z_0}{z_0 + \Delta z}$$

which is the relation between the specific gravity of the fluid and Δz. Note that z_0 is constant for a given hydrometer and Δz is negative for fluids heavier than pure water.

(*b*) Disregarding the weight of the glass tube, the amount of lead that needs to be added to the tube is determined from the requirement that the weight of the lead be equal to the buoyant force. When the hydrometer is floating with half of it submerged in water, the buoyant force acting on it is

$$F_B = \rho_w g V_{\text{sub}}$$

Equating F_B to the weight of lead gives

$$W = mg = \rho_w g V_{\text{sub}}$$

Solving for m and substituting, the mass of lead is determined to be

$$m = \rho_w V_{\text{sub}} = \rho_w(\pi R^2 h_{\text{sub}}) = (1000\ \text{kg/m}^3)[\pi(0.005\ \text{m})^2(0.1\ \text{m})] = \mathbf{0.00785\ kg}$$

Discussion Note that if the hydrometer were required to sink only 5 cm in water, the required mass of lead would be one-half of this amount. Also, the assumption that the weight of the glass tube is negligible is questionable since the mass of lead is only 7.85 g.

EXAMPLE 3–11 Weight Loss of an Object in Seawater

A crane is used to lower weights into the sea (density = 1025 kg/m³) for an underwater construction project (Fig. 3–46). Determine the tension in the rope of the crane due to a rectangular 0.4-m × 0.4-m × 3-m concrete block (density = 2300 kg/m³) when it is (*a*) suspended in the air and (*b*) completely immersed in water.

SOLUTION A concrete block is lowered into the sea. The tension in the rope is to be determined before and after the block is in water.

Assumptions **1** The buoyant force in air is negligible. **2** The weight of the ropes is negligible.
Properties The densities are given to be 1025 kg/m³ for seawater and 2300 kg/m³ for concrete.
Analysis (*a*) Consider a free-body diagram of the concrete block. The forces acting on the concrete block in air are its weight and the upward pull action (tension) by the rope. These two forces must balance each other, and thus the tension in the rope must be equal to the weight of the block:

$$V = (0.4\text{ m})(0.4\text{ m})(3\text{ m}) = 0.48\text{ m}^3$$

$$F_{T,\text{air}} = W = \rho_{\text{concrete}}\, g V$$

$$= (2300\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.48\text{ m}^3)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = \mathbf{10.8\text{ kN}}$$

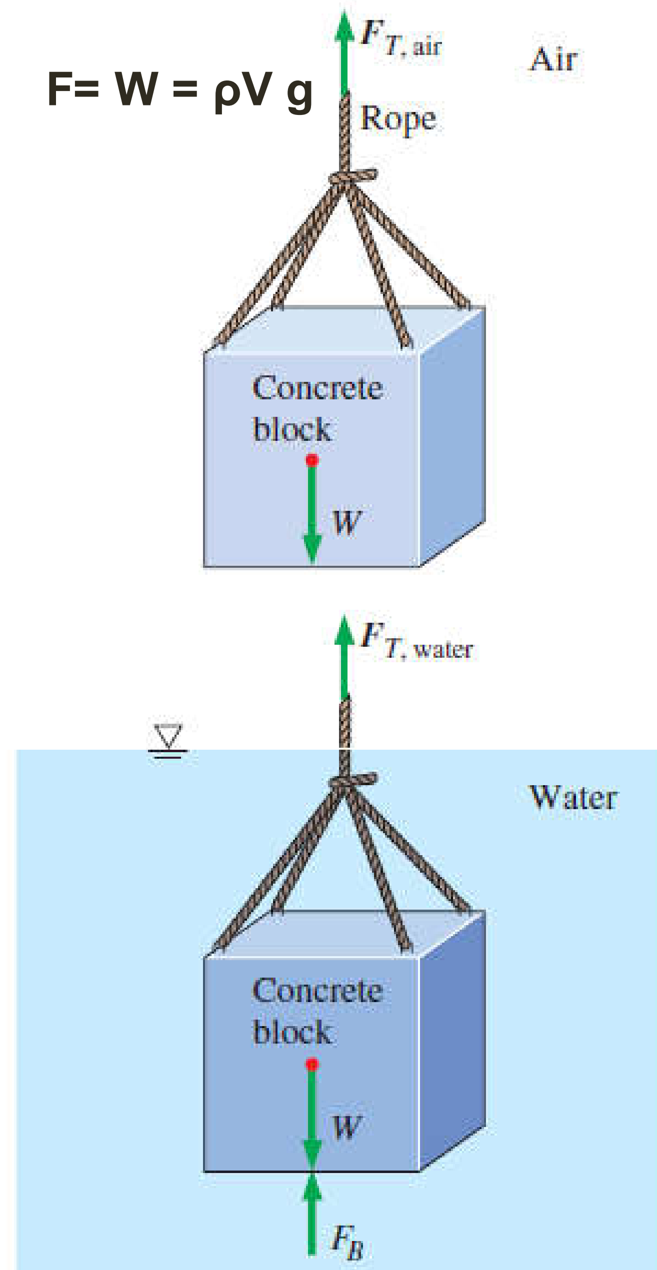
(*b*) When the block is immersed in water, there is the additional force of buoyancy acting upward. The force balance in this case gives

$$F_B = \rho_f\, g V = (1025\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.48\text{ m}^3)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = 4.8\text{ kN}$$

$$F_{T,\text{water}} = W - F_B = 10.8 - 4.8 = \mathbf{6.0\text{ kN}}$$

Discussion Note that the weight of the concrete block, and thus the tension of the rope, decreases by (10.8 − 6.0)/10.8 = 55 percent in water.

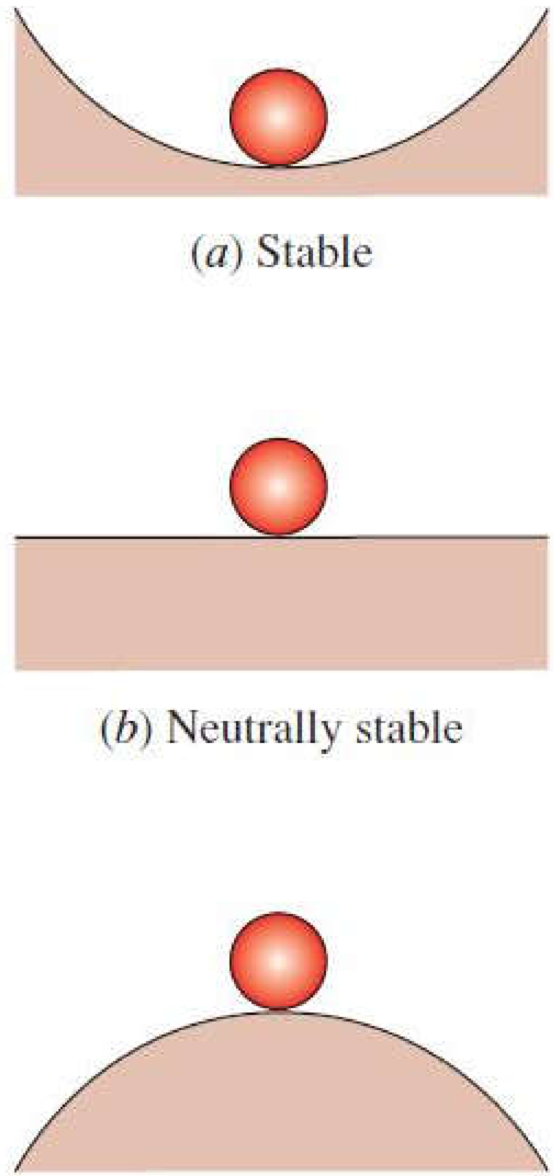
F= W = ρV g



Stability of Immersed and Floating Bodies

For floating bodies such as ships, stability is an important consideration for safety.

Stability is easily understood by analyzing a ball on the floor.

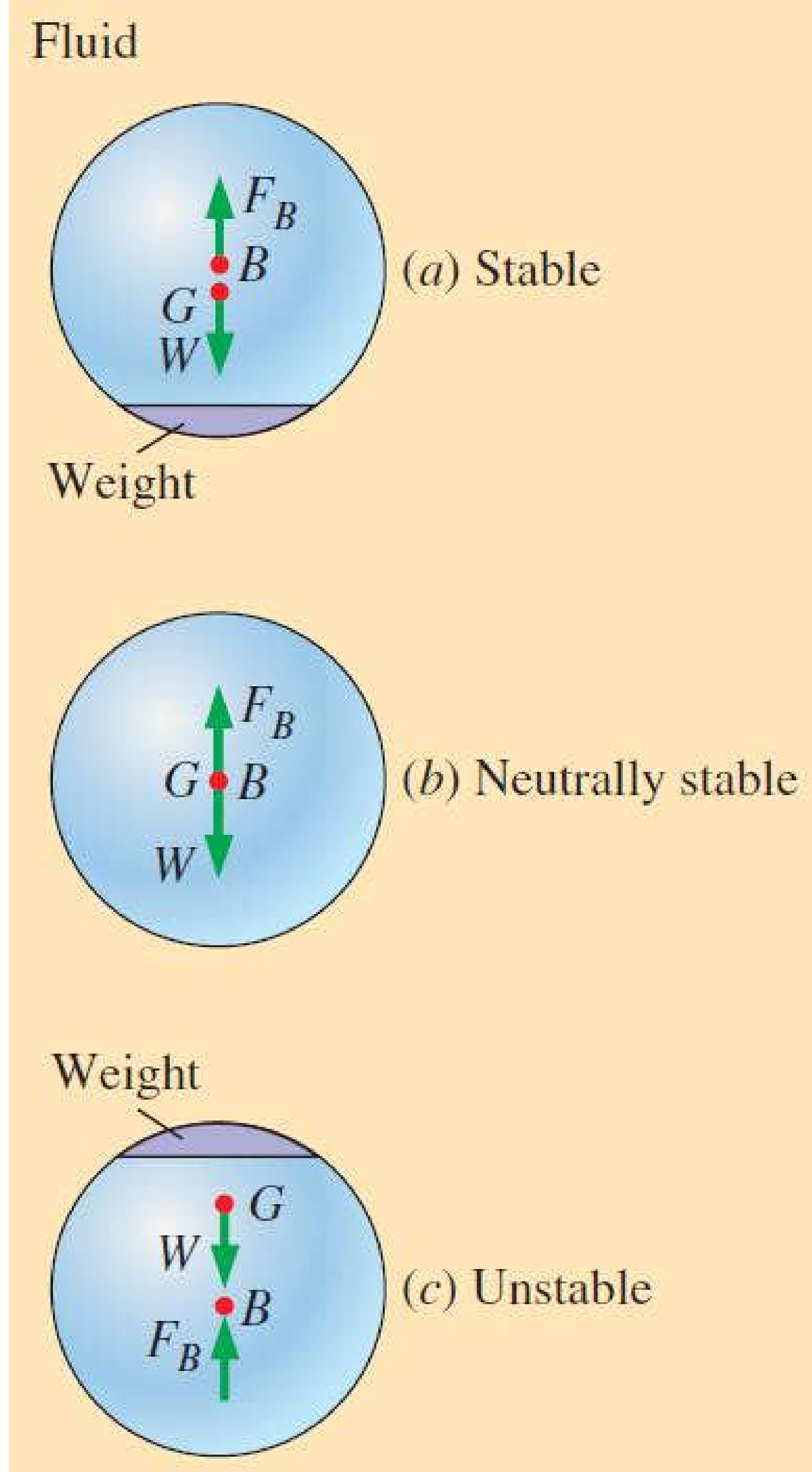


(a) Stable

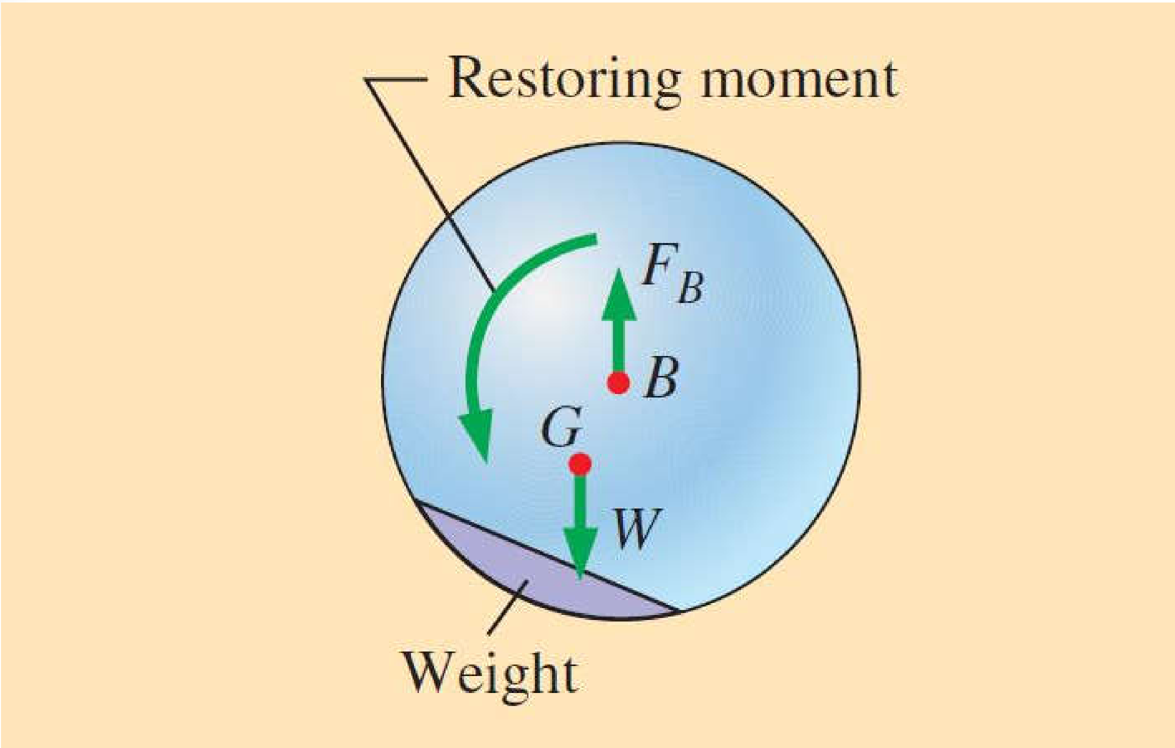
(b) Neutrally stable

(c) Unstable

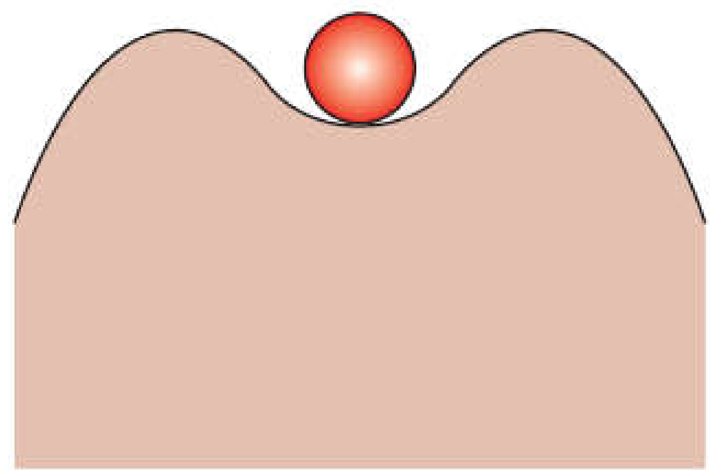
A floating body possesses vertical stability, while an immersed neutrally buoyant body is neutrally stable since it does not return to its original position after a disturbance.



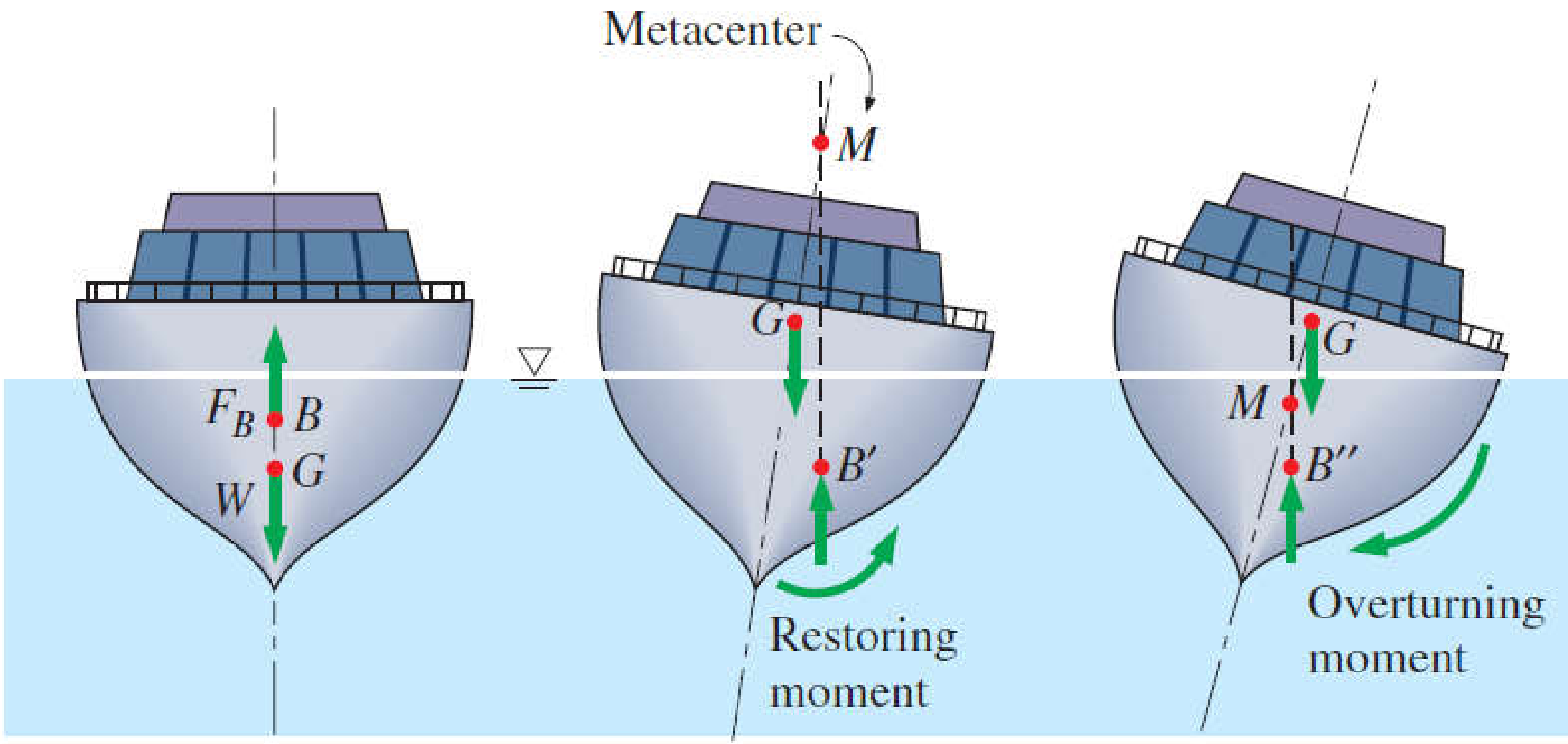
An immersed neutrally buoyant body is (*a*) stable if the center of gravity *G* is directly below the center of buoyancy *B* of the body, (*b*) neutrally stable if *G* and *B* are coincident, and (*c*) unstable if *G* is directly above *B*.



When the center of gravity *G* of an immersed neutrally buoyant body is not vertically aligned with the center of buoyancy *B* of the body, it is not in an equilibrium state and would rotate to its stable state, even without any disturbance.



A ball in a trough between two hills is stable for small disturbances, but unstable for large disturbances.



(*a*) Stable (*b*) Stable (*c*) Unstable

A floating body is *stable* if the body is bottom-heavy and thus the center of gravity *G* is below the centroid *B* of the body, or if the metacenter *M* is above point *G*. However, the body is *unstable* if point *M* is below point *G*.

Metacentric height *GM*: The distance between the center of gravity *G* and the metacenter *M*—the intersection point of the lines of action of the buoyant force through the body before and after rotation.

The length of the metacentric height *GM* above *G* is a measure of the stability: the larger it is, the more stable is the floating body.

Lecture 10

Fluid Kinematics

Part I

LAGRANGIAN AND EULERIAN DESCRIPTIONS

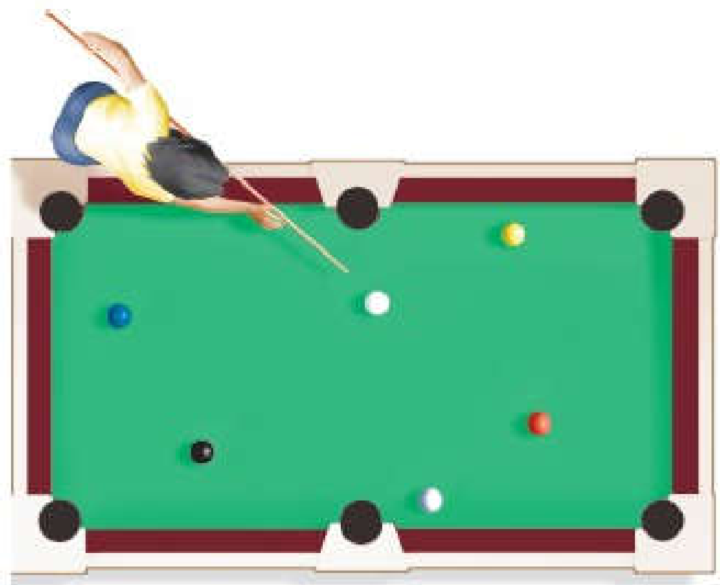
Kinematics: The study of motion.

Fluid kinematics: The study of how fluids flow and how to describe fluid motion.

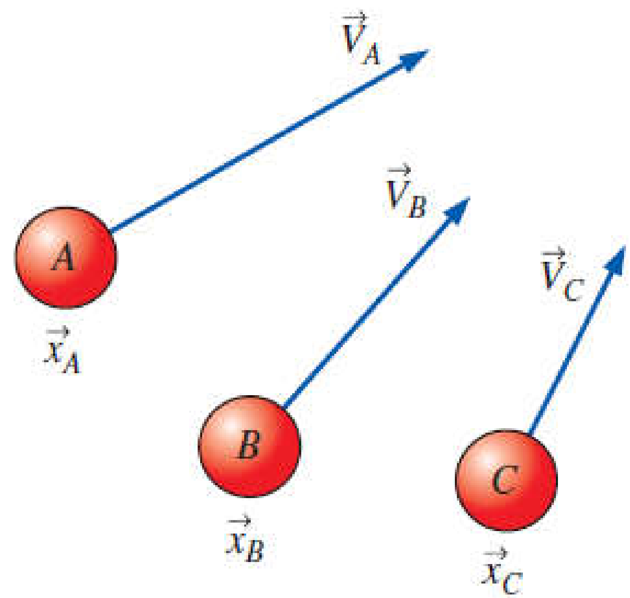
There are two distinct ways to describe motion: Lagrangian and Eulerian

Lagrangian description: To follow the path of individual objects.

This method requires us to track the position and velocity of each individual fluid parcel (**fluid particle**) and take to be a parcel of fixed identity.



With a small number of objects, such as billiard balls on a pool table, individual objects can be tracked.



In the Lagrangian description, one must keep track of the position and velocity of individual particles.

- A more common method is **Eulerian description** of fluid motion.
- In the Eulerian description of fluid flow, a finite volume called a **flow domain** or **control volume** is defined, through which fluid flows in and out.
- Instead of tracking individual fluid particles, we define **field variables**, functions of space and time, within the control volume.
- The field variable at a particular location at a particular time is the value of the variable for whichever fluid particle happens to occupy that location at that time.
- For example, the **pressure field** is a **scalar field variable**. We define the **velocity field** as a **vector field variable**.

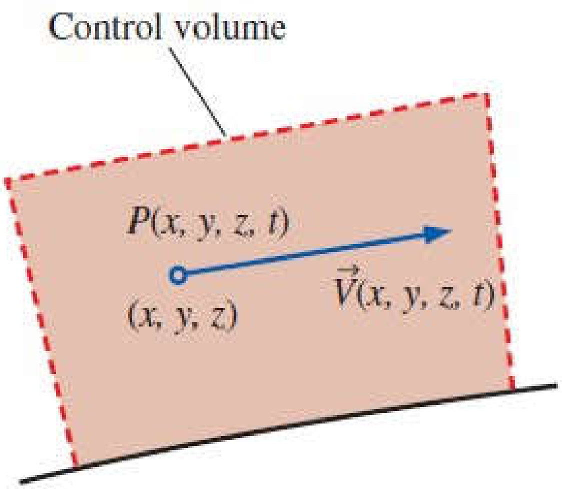
Pressure field: $P = P(x, y, z, t)$

Velocity field: $\vec{V} = \vec{V}(x, y, z, t)$

Acceleration field: $\vec{a} = \vec{a}(x, y, z, t)$

Collectively, these (and other) field variables define the **flow field**. The velocity field can be expanded in Cartesian coordinates as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$



(a)

(b)

- In the Eulerian description we don’t really care what happens to individual fluid particles; rather we are concerned with the pressure, velocity, acceleration, etc., of whichever fluid particle happens to be at the location of interest at the time of interest.
- While there are many occasions in which the Lagrangian description is useful, the Eulerian description is often more convenient for fluid mechanics applications.
- Experimental measurements are generally more suited to the Eulerian description.

(a) In the Eulerian description, we define field variables, such as the pressure field and the velocity field, at any location and instant in time. *(b) For example, the air speed probe* mounted under the wing of an airplane measures the air speed at that location.

EXAMPLE 4–1 A Steady Two-Dimensional Velocity Field

A steady, incompressible, two-dimensional velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j} \tag{1}$$

where the x- and y-coordinates are in meters and the magnitude of velocity is in m/s. A **stagnation point** is defined as *a point in the flow field where the velocity is zero.* (*a*) Determine if there are any stagnation points in this flow field and, if so, where? (*b*) Sketch velocity vectors at several locations in the domain between $x = -2$ m to 2 m and $y = 0$ m to 5 m; qualitatively describe the flow field.

Analysis (*a*) Since $\vec{V}$ is a vector, *all* its components must equal zero in order for $\vec{V}$ itself to be zero. Using Eq. 4–4 and setting Eq. 1 equal to zero,

Stagnation point:
$$u = 0.5 + 0.8x = 0 \quad \rightarrow \quad x = -0.625 \text{ m}$$
$$v = 1.5 - 0.8y = 0 \quad \rightarrow \quad y = 1.875 \text{ m}$$

Yes. There is one stagnation point located at **$x = -0.625$ m, $y = 1.875$ m.**

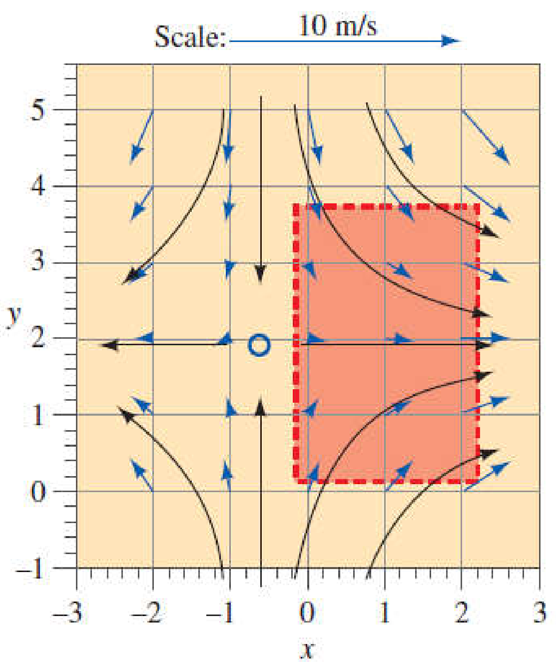
(*b*) The x- and y-components of velocity are calculated from Eq. 1 for several (x, y) locations in the specified range. For example, at the point ($x = 2$ m, $y = 3$ m), $u = 2.10$ m/s and $v = -0.900$ m/s. The magnitude of velocity (the *speed*) at that point is 2.28 m/s. At this and at an array of other locations, the velocity vector is constructed from its two components, the results of which are shown in Fig. 4–4. The flow can be described as stagnation point flow in which flow enters from the top and bottom and spreads out to the right and left about a horizontal line of symmetry at $y = 1.875$ m. The stagnation point of part (*a*) is indicated by the blue circle in Fig. 4–4.

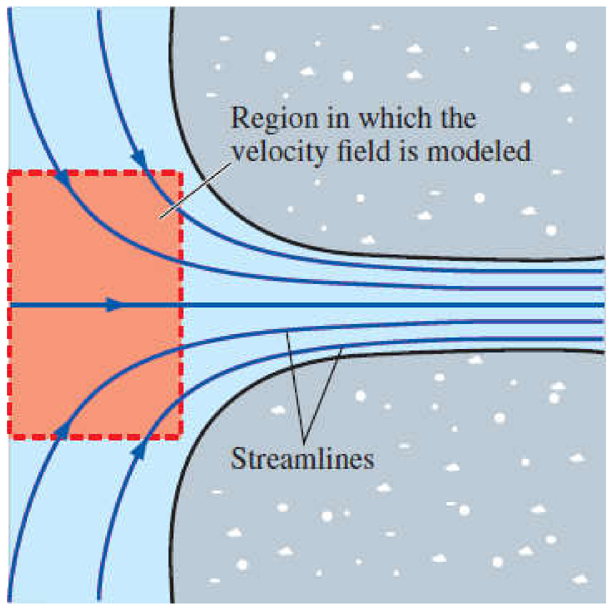
If we look only at the shaded portion of Fig. 4–4, this flow field models a converging, accelerating flow from the left to the right. Such a flow might be encountered, for example, near the submerged bell mouth inlet of a hydroelectric dam (Fig. 4–5). The useful portion of the given velocity field may be thought of as a first-order approximation of the shaded portion of the physical flow field of Fig. 4–5.

Discussion It can be verified from the material in Chap. 9 that this flow field is physically valid because it satisfies the differential equation for conservation of mass.

A Steady Two-Dimensional Velocity Field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$





Flow field near the bell mouth inlet of a hydroelectric dam; a portion of the velocity field of Example 4-1 may be used as a first-order approximation of this physical flow field.

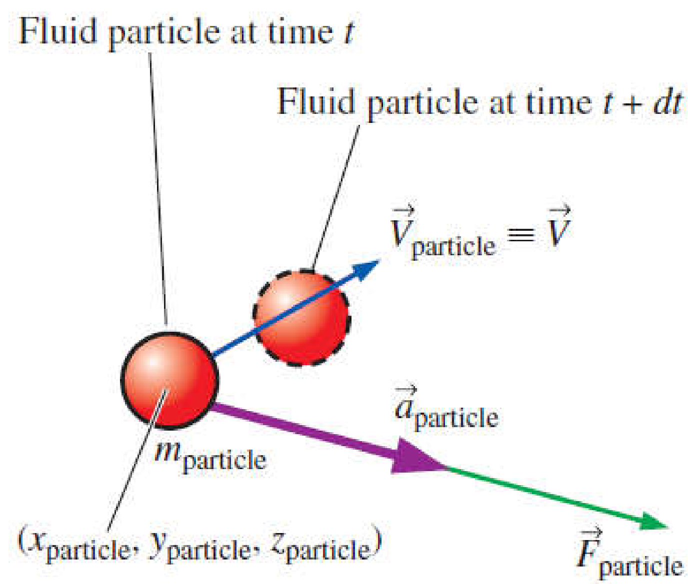
Velocity vectors for the velocity field of Example 4–1. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the blue circle. The shaded region represents a portion of the flow field that can approximate flow into an inlet.

Acceleration Field

The equations of motion for fluid flow (such as Newton's second law) are written for a fluid particle, which we also call a **material particle**.

If we were to follow a particular fluid particle as it moves around in the flow, we would be employing the Lagrangian description, and the equations of motion would be directly applicable.

For example, we would define the particle's location in space in terms of a **material position vector** ($x_{particle}(t)$, $y_{particle}(t)$, $z_{particle}(t)$).



Newton's second law applied to a fluid particle; the acceleration vector (purple arrow) is in the same direction as the force vector (green arrow), but the velocity vector (blue arrow) may act in a different direction.

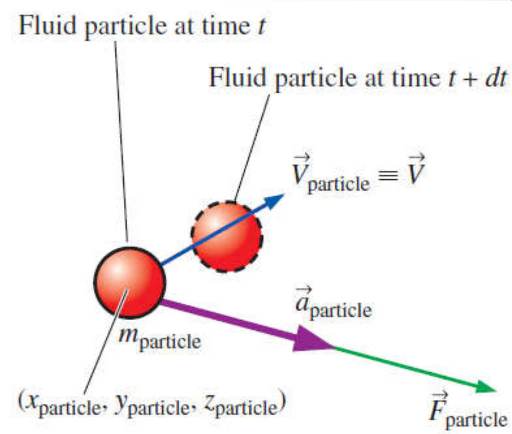
Newton's second law:

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

Acceleration of a fluid particle:

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

$$\vec{V}_{\text{particle}}(t) \equiv \vec{V}(x_{\text{particle}}(t),\ y_{\text{particle}}(t),\ z_{\text{particle}}(t),\ t)$$



$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt}$$

$$\boxed{\vec{a}_{\text{particle}}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}$$

∂: the Partial derivative operator
d: The total derivative operator

Acceleration of a fluid particle expressed as a field variable:

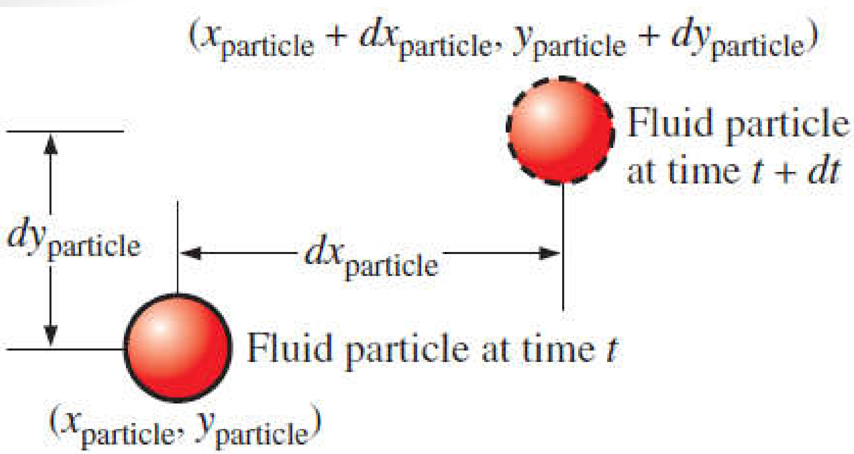
$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

$\frac{\partial \vec{V}}{\partial t}$ Local acceleration

$(\vec{V} \cdot \vec{\nabla})\vec{V}$ Advective (convective) acceleration

Gradient or del operator:

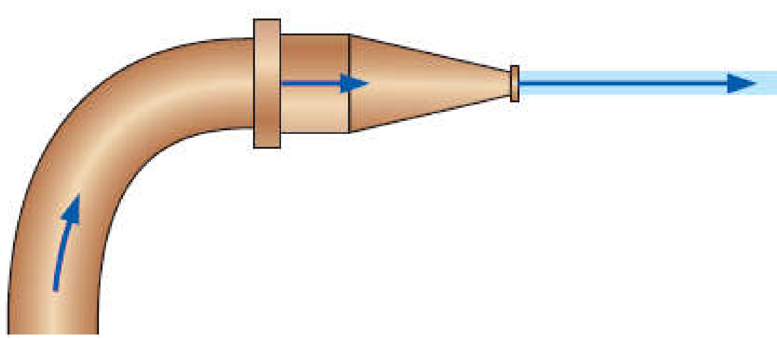
$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$



The components of the acceleration vector in cartesian coordinates:

$$\boxed{\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}}$$

When following a fluid particle, the *x*-component of velocity, *u*, is defined as $dx_{particle}/dt$. Similarly, $v=dy_{particle}/dt$ and $w=dz_{particle}/dt$. Movement is shown here only in two dimensions for simplicity.



Flow of water through the nozzle of a garden hose illustrates that fluid particles may accelerate, even in a steady flow. In this example, the exit speed of the water is much higher than the water speed in the hose, implying that fluid particles have accelerated even though the flow is steady.

EXAMPLE 4–2 Acceleration of a Fluid Particle through a Nozzle

Nadeen is washing her car, using a nozzle similar to the one sketched in Fig. 4–8. The nozzle is 3.90 in (0.325 ft) long, with an inlet diameter of 0.420 in (0.0350 ft) and an outlet diameter of 0.182 in (see Fig. 4–9). The volume flow rate through the garden hose (and through the nozzle) is $\dot{V}$ = 0.841 gal/min (0.00187 ft³/s), and the flow is steady. Estimate the magnitude of the acceleration of a fluid particle moving down the centerline of the nozzle.

Analysis The flow is steady, so you may be tempted to say that the acceleration is zero. However, even though the local acceleration $\partial\vec{V}/\partial t$ is identically zero for this steady flow field, the advective acceleration $(\vec{V}\cdot\vec{\nabla})\vec{V}$ is *not zero*. We first calculate the average *x*-component of velocity at the inlet and outlet of the nozzle by dividing volume flow rate by cross-sectional area:

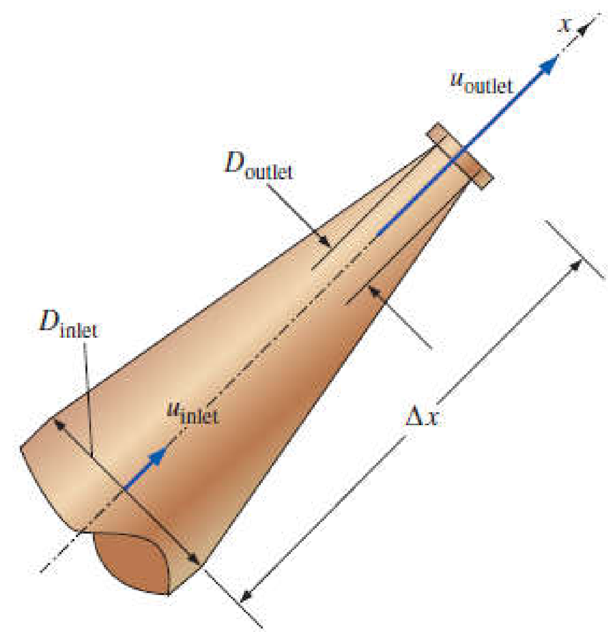
Inlet speed:

$$u_{\text{inlet}} \cong \frac{\dot{V}}{A_{\text{inlet}}} = \frac{4\dot{V}}{\pi D_{\text{inlet}}^2} = \frac{4(0.00187\ \text{ft}^3/\text{s})}{\pi(0.0350\ \text{ft})^2} = 1.95\ \text{ft/s}$$

Similarly, the average outlet speed is u_{outlet} = 10.4 ft/s. We now calculate the acceleration two ways, with equivalent results. First, a simple average value of acceleration in the *x*-direction is calculated based on the change in speed divided by an estimate of the **residence time** of a fluid particle in the nozzle, $\Delta t = \Delta x/u_{\text{avg}}$ (Fig. 4–10). By the fundamental definition of acceleration as the rate of change of velocity,

$$\textit{Method A:}\quad a_x \cong \frac{\Delta u}{\Delta t} = \frac{u_{\text{outlet}} - u_{\text{inlet}}}{\Delta x/u_{\text{avg}}} = \frac{u_{\text{outlet}} - u_{\text{inlet}}}{2\,\Delta x/(u_{\text{outlet}} + u_{\text{inlet}})} = \frac{u_{\text{outlet}}^2 - u_{\text{inlet}}^2}{2\,\Delta x}$$

The second method uses the equation for acceleration field components in Cartesian coordinates, Eq. 4–11,



Method B:

$$a_x = \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\substack{0\\ \text{Steady}}} + u\frac{\partial u}{\partial x} + \underbrace{v\cancel{\frac{\partial u}{\partial y}}}_{\substack{0\\ v\,=\,0\text{ along centerline}}} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{\substack{0\\ w\,=\,0\text{ along centerline}}} \cong u_{\text{avg}}\frac{\Delta u}{\Delta x}$$

Here we see that only one advective term is nonzero. We approximate the average speed through the nozzle as the average of the inlet and outlet speeds, and we use a **first-order finite difference approximation** (Fig. 4–11) for the average value of derivative $\partial u/\partial x$ through the centerline of the nozzle:

$$a_x \cong \frac{u_{\text{outlet}} + u_{\text{inlet}}}{2}\,\frac{u_{\text{outlet}} - u_{\text{inlet}}}{\Delta x} = \frac{u^2_{\text{outlet}} - u^2_{\text{inlet}}}{2\,\Delta x}$$

The result of method B is identical to that of method A. Substitution of the given values yields

Axial acceleration:

$$a_x \cong \frac{u^2_{\text{outlet}} - u^2_{\text{inlet}}}{2\,\Delta x} = \frac{(10.4\text{ ft/s})^2 - (1.95\text{ ft/s})^2}{2(0.325\text{ ft})} = \mathbf{160\ ft/s^2}$$

Discussion Fluid particles are accelerated through the nozzle at nearly five times the acceleration of gravity (almost five *g*'s)! This simple example clearly illustrates that the acceleration of a fluid particle can be nonzero, even in steady flow. Note that the acceleration is actually a **point function,** whereas we have estimated a simple average acceleration through the entire nozzle.

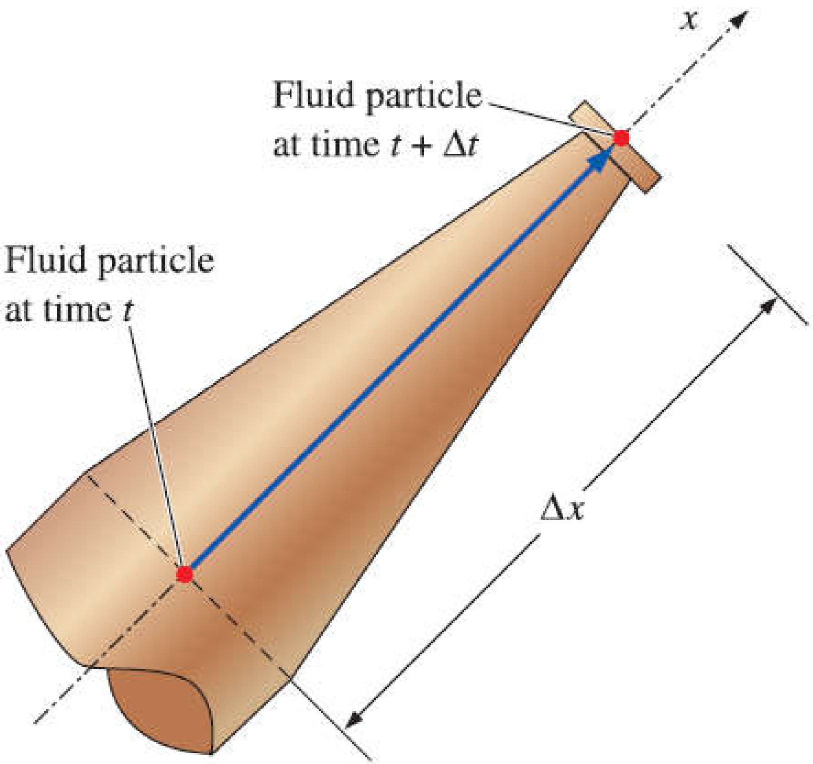


FIGURE 4–10
Residence time Δt is defined as the time it takes for a fluid particle to travel through the nozzle from inlet to outlet (distance Δx).

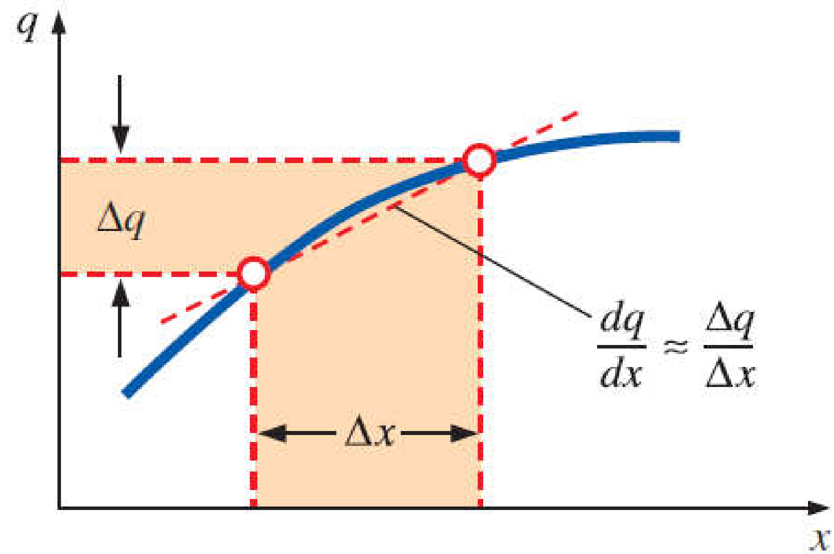


FIGURE 4–11
A *first-order finite difference approximation* for derivative dq/dx is simply the change in dependent variable (q) divided by the change in independent variable (x).

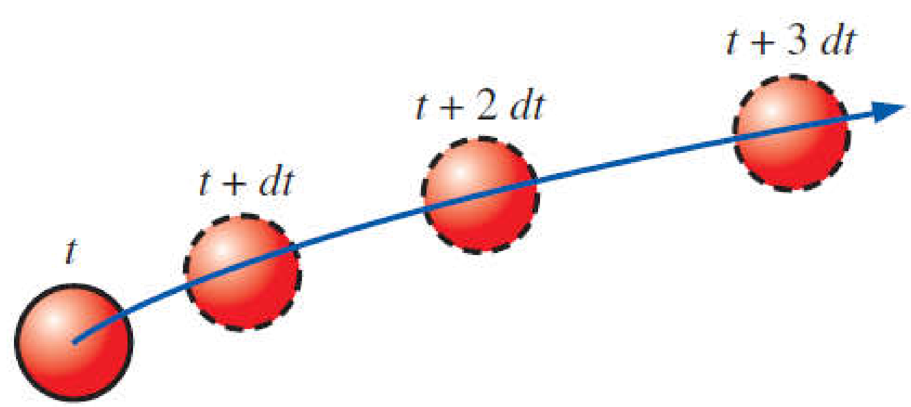
Material Derivative

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

The total derivative operator *d/dt* in this equation is given a special name, the **material derivative**;

it is assigned a special notation, *D/Dt*, in order to emphasize that it is formed by *following a fluid particle as it moves through the flow field*.

Other names for the material derivative include **total**, **particle**, **Lagrangian**, **Eulerian**, and **substantial derivative**.



The material derivative *D/Dt* is defined by following a fluid particle as it moves throughout the flow field. In this illustration, the fluid particle is accelerating to the right as it moves up and to the right.

Material derivative:

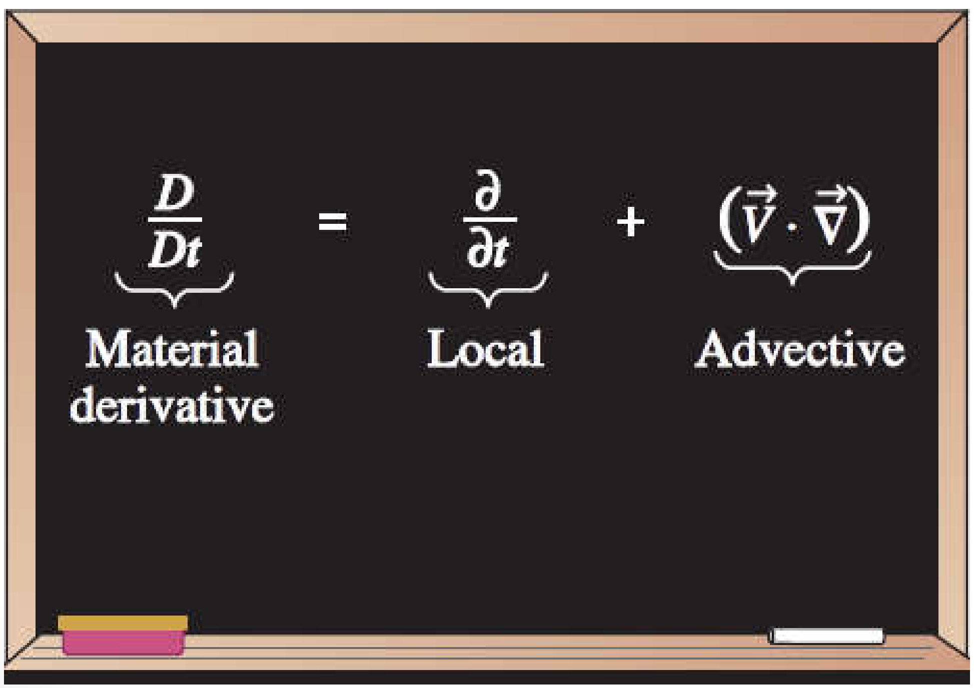
$$\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \vec{\nabla})$$

Material acceleration:

$$\vec{a}(x, y, z, t) = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

Material derivative of pressure:

$$\frac{DP}{Dt} = \frac{dP}{dt} = \frac{\partial P}{\partial t} + (\vec{V} \cdot \vec{\nabla})P$$



The material derivative *D/Dt* is composed of a *local* or *unsteady* part and a *convective* or *advective* part.

EXAMPLE 4–3 Material Acceleration of a Steady Velocity Field

Consider the steady, incompressible, two-dimensional velocity field of Example 4–1. (*a*) Calculate the material acceleration at the point (x = 2 m, y = 3 m). (*b*) Sketch the material acceleration vectors at the same array of x- and y-values as in Example 4–1.

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$

Analysis (*a*) Using the velocity field of Eq. 1 of Example 4–1 and the equation for material acceleration components in Cartesian coordinates (Eq. 4–11), we write expressions for the two nonzero components of the acceleration vector:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 0 + \overbrace{(0.5 + 0.8x)(0.8)} + \overbrace{(1.5 - 0.8y)(0)} + 0 = (0.4 + 0.64x)\ \text{m/s}^2$$

and

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

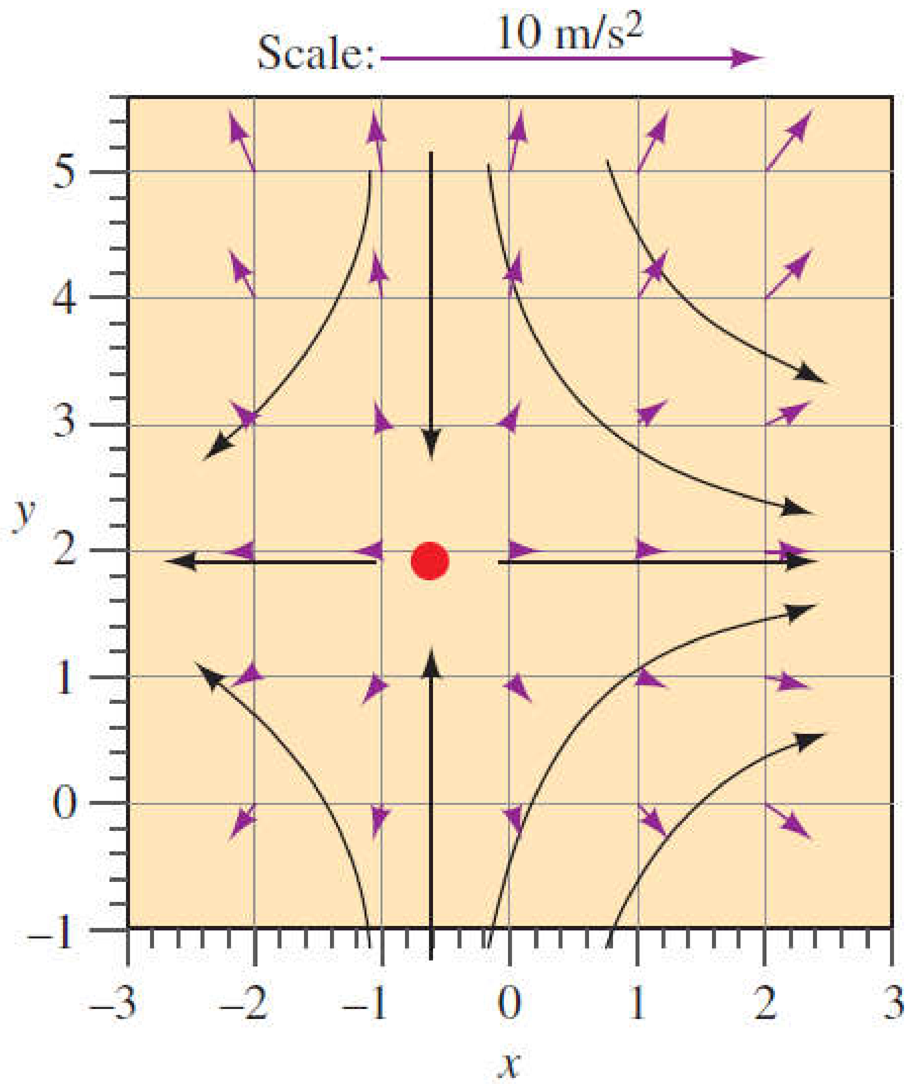
$$= 0 + \overbrace{(0.5 + 0.8x)(0)} + \overbrace{(1.5 - 0.8y)(-0.8)} + 0 = (-1.2 + 0.64y)\ \text{m/s}^2$$

At the point (x = 2 m, y = 3 m), $a_x = 1.68\ \text{m/s}^2$ and $a_y = 0.720\ \text{m/s}^2$.
(*b*) The equations in part (*a*) are applied to an array of x- and y-values in the flow domain within the given limits, and the acceleration vectors are plotted in Fig. 4–14.

Material acceleration: $\vec{a}(x, y, z, t) = \dfrac{D\vec{V}}{Dt} = \dfrac{d\vec{V}}{dt} = \dfrac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$

$$\boxed{\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}}$$

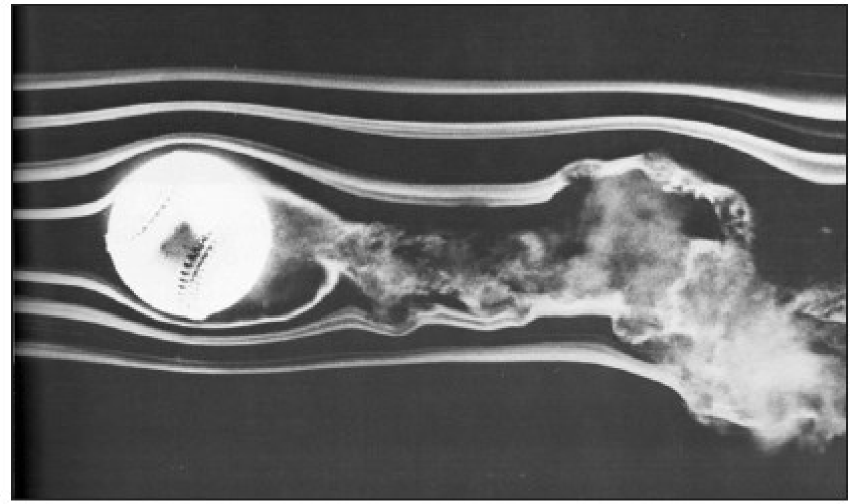
Material Acceleration of a Steady Velocity Field



Acceleration vectors for the velocity field of Examples 4–1 and 4–3. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the red circle.

FLOW PATTERNS AND FLOW VISUALIZATION

- **Flow visualization:** The visual examination of flow field features.
- While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from flow visualization.
- Flow visualization is useful not only in physical experiments but in *numerical* solutions as well [**computational fluid dynamics (CFD)**].
- In fact, the very first thing an engineer using CFD does after obtaining a numerical solution is simulate some form of flow visualization.

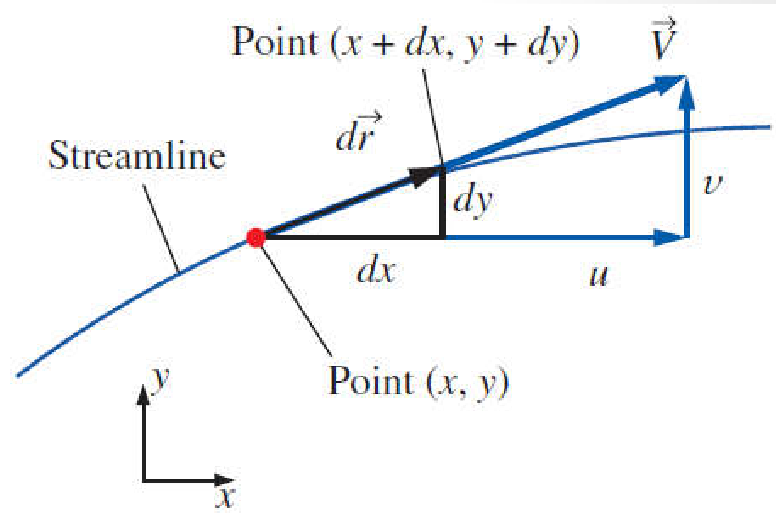


Spinning baseball. The late F. N. M. Brown devoted many years to developing and using smoke visualization in wind tunnels at the University of Notre Dame. Here the flow speed is about 23 m/s and the ball is rotated at 630 rpm.

Streamlines and Streamtubes

Streamline: A curve that is everywhere tangent to the instantaneous local velocity vector.

Streamlines are useful as indicators of the instantaneous direction of fluid motion throughout the flow field.



Equation for a streamline: $\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

Streamline in the xy-plane: $\left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}$

For example, regions of **recirculating flow and separation of a fluid off of a solid wall** are easily identified by the streamline pattern.

For two-dimensional flow in the xy-plane, arc length $d\vec{r} = (dx, dy)$ along a *streamline* is everywhere tangent to the local instantaneous velocity vector $\vec{V} = (u, v)$.

Streamlines cannot be directly observed experimentally except in steady flow fields.

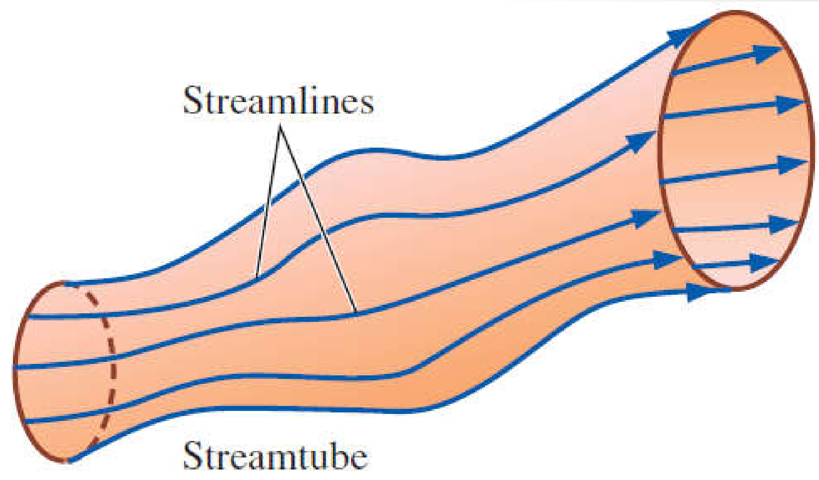
- A **streamtube** consists of a bundle of streamlines much like a communications cable consists of a bundle of fiber-optic cables.

Since streamlines are everywhere parallel to the local velocity, fluid cannot cross a streamline by definition.

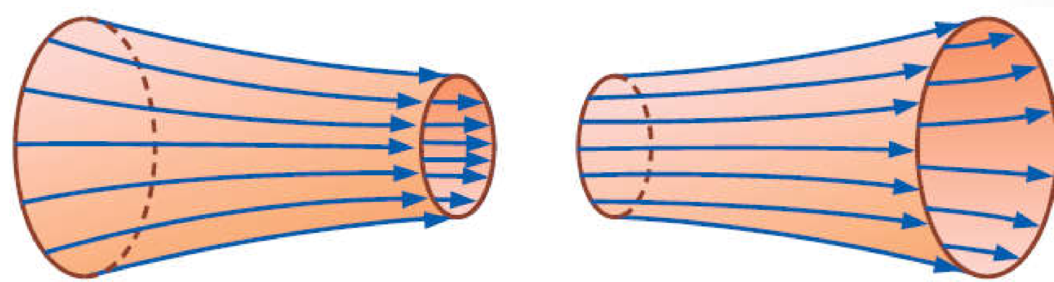
Fluid within a streamtube must remain there and cannot cross the boundary of the streamtube.

Both **streamlines and streamtubes are instantaneous quantities**,

defined at a particular instant in time according to the velocity field at that instant.



A *streamtube* consists of a bundle of individual streamlines.



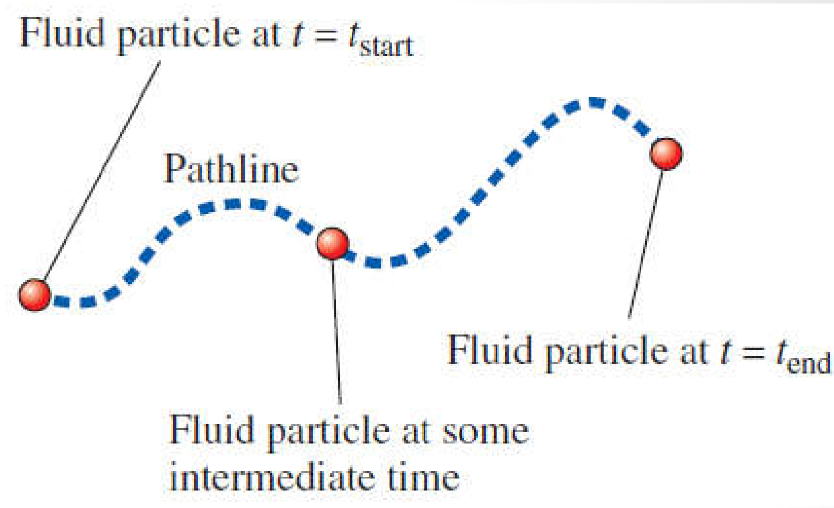
(*a*) (*b*)

In an incompressible flow field, a streamtube (*a*) decreases in diameter as the flow accelerates or converges and (*b*) increases in diameter as the flow decelerates or diverges.

Pathlines

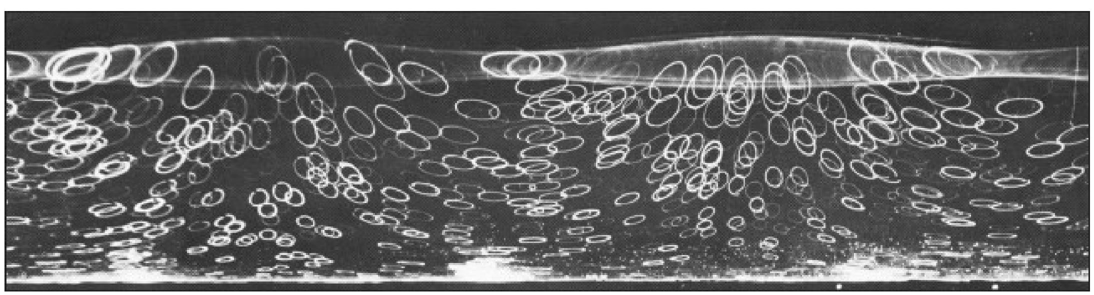
Pathline: The actual path traveled by an individual fluid particle over some time period.

- A pathline is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.
- Thus, a pathline is the same as the fluid particle's material position vector $(x_{particle}(t), y_{particle}(t), z_{particle}(t))$ traced out over some finite time interval.



A *pathline* is formed by following the actual path of a fluid particle.

Tracer particle location at time t: $\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V}\,dt$



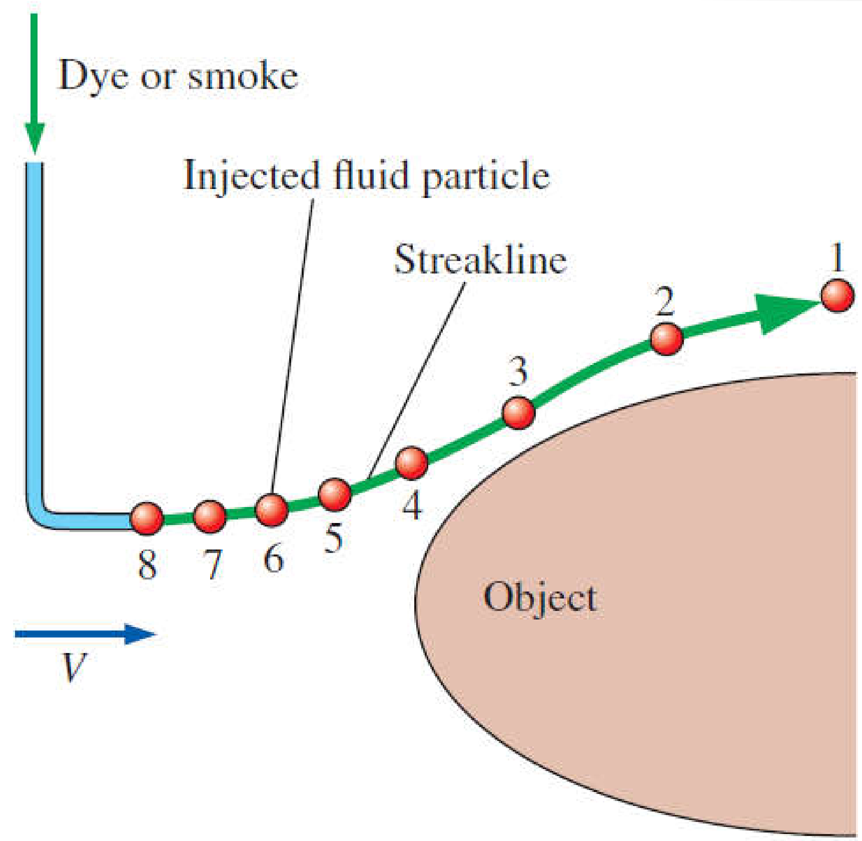
Pathlines produced by white tracer particles suspended in water and captured by time-exposure photography; as waves pass horizontally, each particle moves in an elliptical path during one wave period.

Streaklines

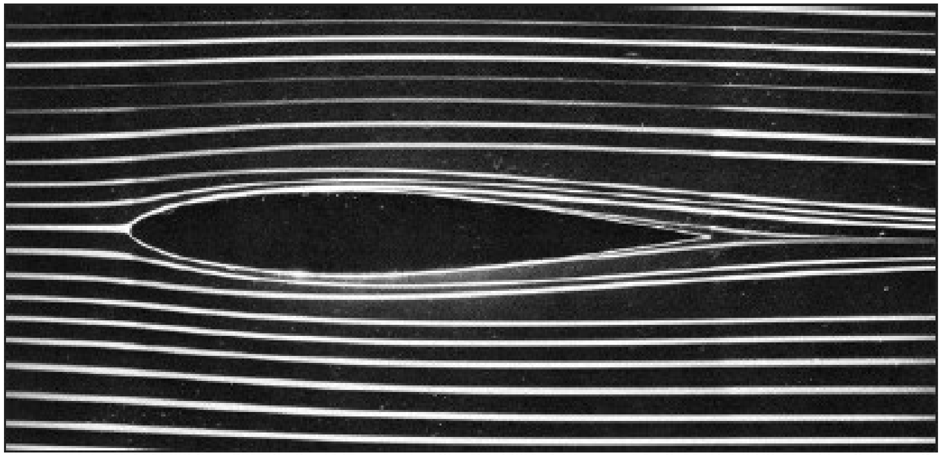
Streakline: The locus of fluid particles that have passed sequentially through a prescribed point in the flow.

Streaklines are the most common flow pattern generated in a physical experiment.

If you insert a small tube into a flow and introduce a continuous stream of tracer fluid (dye in a water flow or smoke in an air flow), the observed pattern is a streakline.



A *streakline* is formed by continuous introduction of dye or smoke from a point in the flow. Labeled tracer particles (1 through 8) were introduced sequentially.



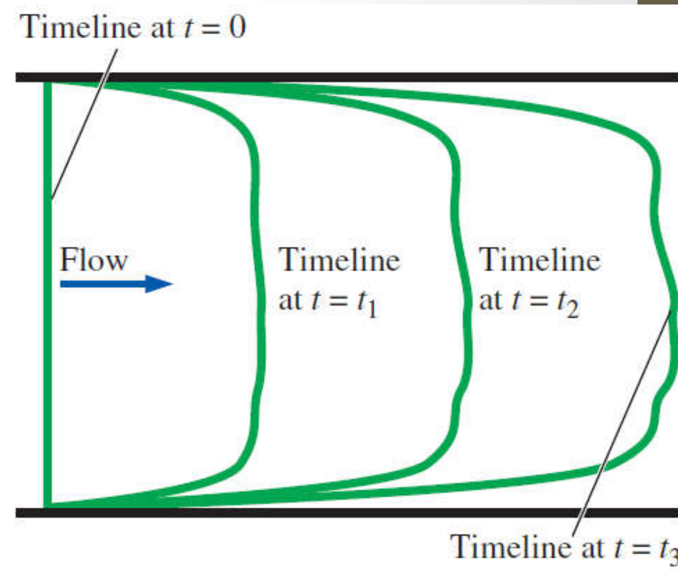
Streaklines produced by colored fluid introduced upstream; since the flow is steady, these streaklines are the same as streamlines and pathlines.

- Streaklines, streamlines, and pathlines are **identical in steady flow** but they can be **quite different in unsteady flow**.
- The main difference is that a **streamline** represents an ***instantaneous*** flow pattern at a given instant in time, while a **streakline** and a **pathline** are flow patterns that have some *age* and thus a ***time history*** associated with them.
- A streakline is an instantaneous snapshot of a *time-integrated* flow pattern.
- A pathline, on the other hand, is the *time-exposed* flow path of an individual particle over some time period.

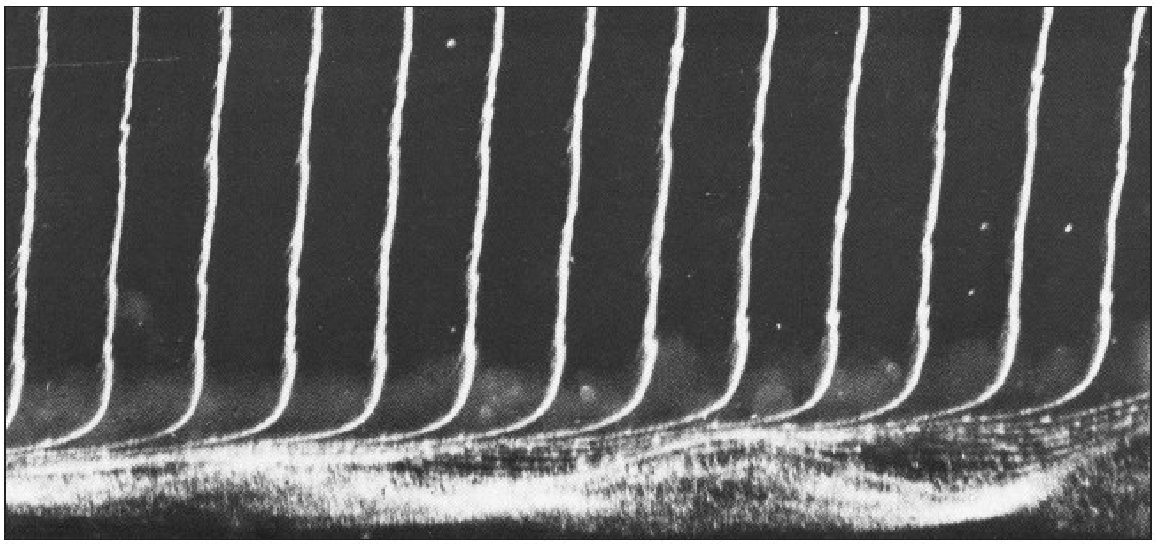
Timelines

Timeline: A set of adjacent fluid particles that were marked at the same (earlier) instant in time.

Timelines are particularly useful in situations where the uniformity of a flow (or lack thereof) is to be examined.



Timelines are formed by marking a line of fluid particles, and then watching that line move (and deform) through the flow field; timelines are shown at $t = 0$, t_1, t_2, and t_3.



Timelines produced by a hydrogen bubble wire are used to visualize the boundary layer velocity profile shape. Flow is from left to right, and the hydrogen bubble wire is located to the left of the field of view. Bubbles near the wall reveal a flow instability that leads to turbulence.

Particle image velocimetry (PIV): A modern experimental technique that utilizes short segments of particle pathlines to measure the velocity field over an entire plane in a flow.

Recent advances also extend the technique to three dimensions.

In PIV, tiny tracer particles are suspended in the fluid. However, the **flow is illuminated by two flashes of light** (usually a light sheet from a laser) to produce two bright spots (recorded by a camera) for each moving particle.

Then, both the magnitude and direction of the velocity vector at each particle location can be inferred, assuming that the tracer particles are small enough that they move with the fluid.

Lecture 11

OTHER KINEMATIC DESCRIPTIONS

Types of Motion or Deformation of Fluid Elements

In fluid mechanics, an element may undergo four fundamental types of motion or deformation:

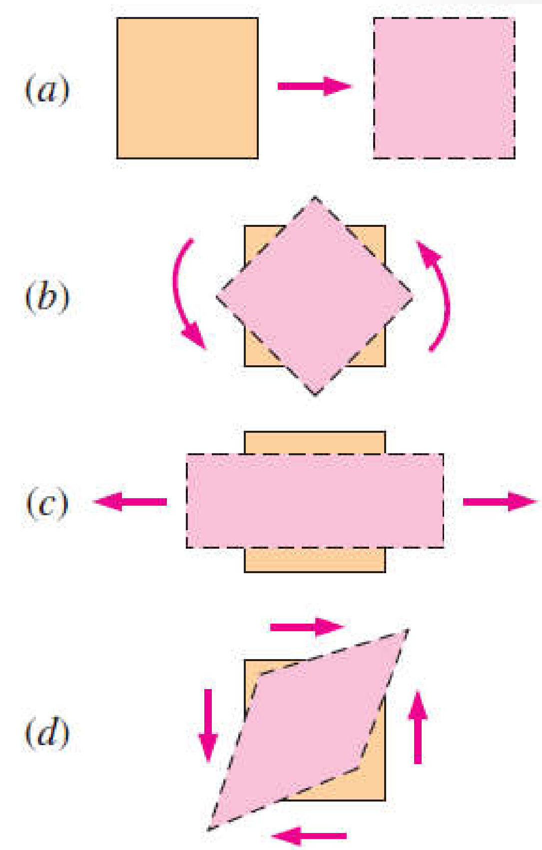
(*a*) **translation**, (*b*) **rotation**,
(*c*) **linear strain** (also called **extensional strain**), and
(*d*) **shear strain**.

All four types of motion or deformation usually occur simultaneously.

It is preferable in fluid dynamics to describe the motion and deformation of fluid elements in terms of *rates* such as

velocity (rate of translation),
angular velocity (rate of rotation),
linear strain rate (rate of linear strain), and
shear strain rate (rate of shear strain).

In order for these **deformation rates** to be useful in the calculation of fluid flows, we must express them in terms of velocity and derivatives of velocity.



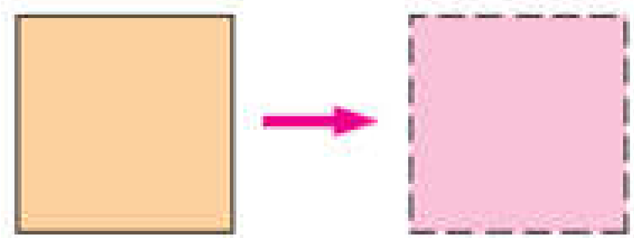
Fundamental types of fluid element motion or deformation: (*a*) translation, (*b*) rotation, (*c*) linear strain, and (*d*) shear strain.

Translation

The **rate of translation vector** is described mathematically as the **velocity vector**.

Rate of translation vector in Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

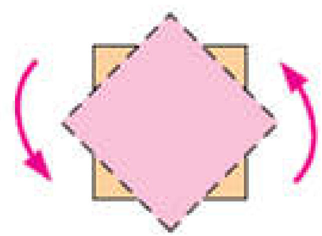


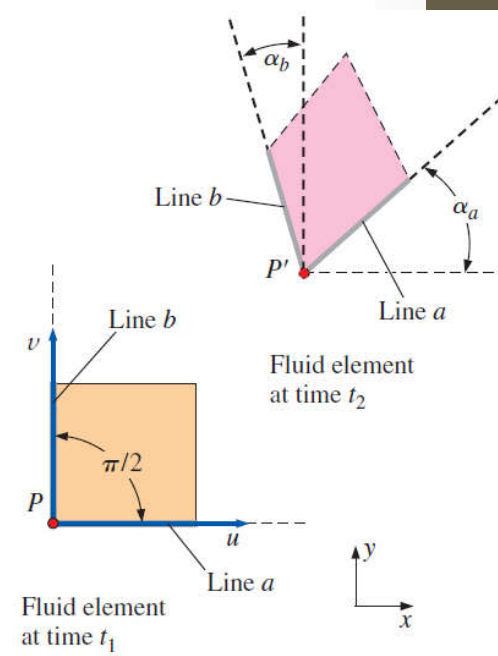
Rotation

Rate of rotation (angular velocity)

Rate of rotation vector in Cartesian coordinates:

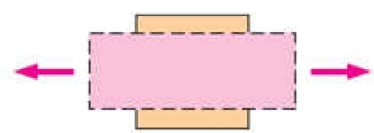
$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$





Linear strain rate

Linear strain rate: *The rate of increase in length per unit length.*



Linear strain rate in Cartesian coordinates:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Shear strain rate

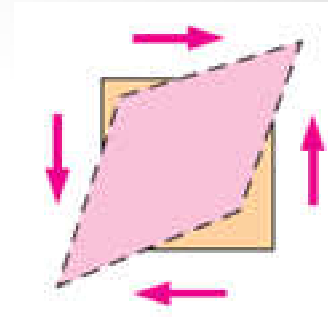
Shear strain rate

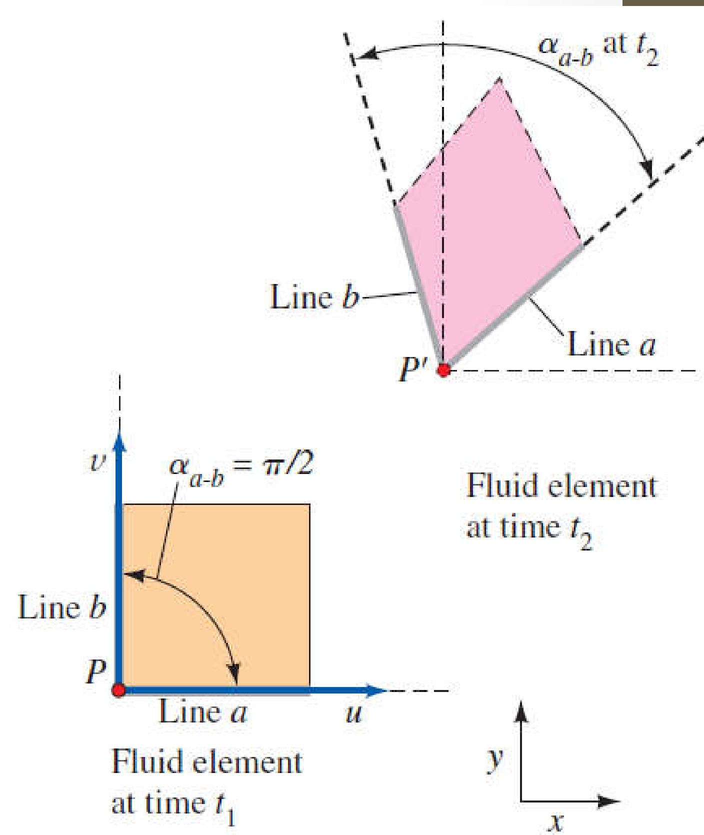
Shear strain rate in Cartesian coordinates:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Strain rate tensor in Cartesian coordinates:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$



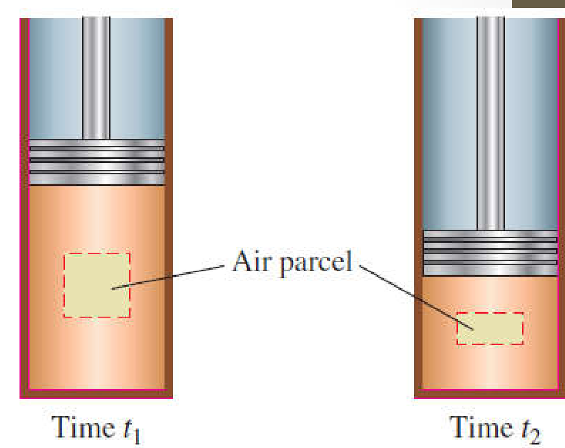


For a fluid element that translates and deforms as sketched, the *shear strain rate* at point *P* is defined as half of the rate of decrease of the angle between two initially perpendicular lines (lines *a* and *b*).

Linear strain rate

If the flow is ***incompressible***, the net **volume** of the fluid element must **remain constant**; thus if the element stretches in one direction, it must shrink by an appropriate amount in other direction(s) to compensate.

The volume of a ***compressible* fluid** element, however, may increase or decrease as its density decreases or increases, respectively.



Air being compressed by a piston in a cylinder; the volume of a fluid element in the cylinder decreases, corresponding to a negative rate of volumetric dilatation.

The rate of increase of volume of a fluid element per unit volume is called, **Volumetric strain rate** or **bulk strain rate** or **rate of volumetric dilatation.** This kinematic property is defined as *positive* when the volume *increases*.

The volumetric strain rate is the sum of the linear strain rates in three mutually orthogonal directions.

Volumetric strain rate in Cartesian coordinates:

$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The volumetric strain rate is zero in an incompressible flow.

EXAMPLE 4–6 Calculation of Kinematic Properties in a Two-Dimensional Flow

Consider the steady, two-dimensional velocity field of Example 4–1:

$$\vec{V} = (u, v) = (0.5 + 0.8\,x)\vec{i} + (1.5 - 0.8\,y)\vec{j} \tag{1}$$

where lengths are in units of m, time in s, and velocities in m/s. There is a stagnation point at (−0.625, 1.875) as shown in Fig. 4–41. Streamlines of the flow are also plotted in Fig. 4–41. Calculate the various kinematic properties, namely, the rate of translation, rate of rotation, linear strain rate, shear strain rate, and volumetric strain rate. Verify that this flow is incompressible.

SOLUTION We are to calculate several kinematic properties of a given velocity field and verify that the flow is incompressible.

Assumptions 1 The flow is steady. **2** The flow is two-dimensional, implying no z-component of velocity and no variation of u or v with z.

Analysis By Eq. 4–19, the rate of translation is simply the velocity vector itself, given by Eq. 1. Thus,

$$\textit{Rate of translation:}\quad u = \mathbf{0.5 + 0.8x} \qquad v = \mathbf{1.5 - 0.8y} \qquad w = \mathbf{0} \tag{2}$$

The rate of rotation is found from Eq. 4–21. In this case, since $w = 0$ everywhere, and since neither u nor v vary with z, the only nonzero component of rotation rate is in the z-direction. Thus,

$$\textit{Rate of rotation:}\qquad \vec{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \frac{1}{2}(0 - 0)\vec{k} = \mathbf{0} \tag{3}$$

In this case, we see that there is no net rotation of fluid particles as they move about. (This is a significant piece of information, to be discussed in more detail later in this chapter and also in Chap. 10.)

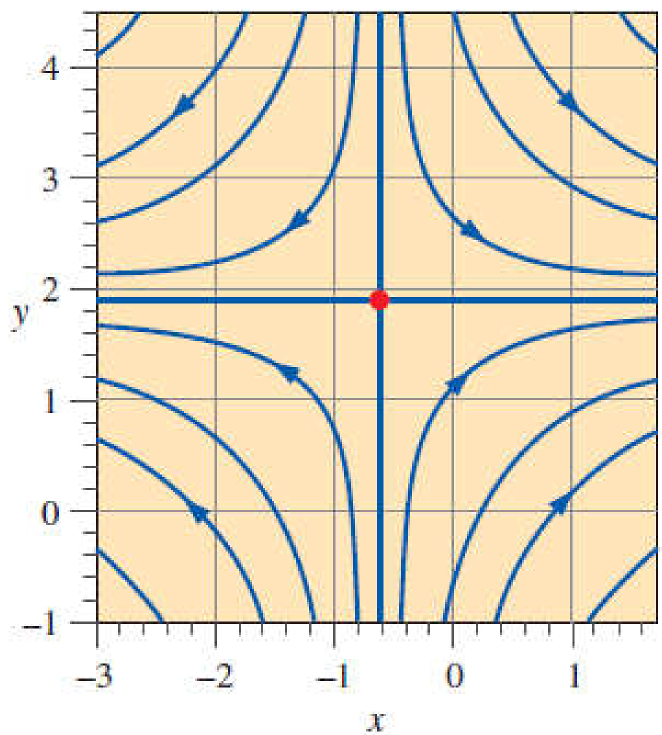


FIGURE 4–41
Streamlines for the velocity field of Example 4–6. The stagnation point is indicated by the red circle at $x = -0.625$ m and $y = 1.875$ m.

Linear strain rates can be calculated in any arbitrary direction using Eq. 4–23. In the x-, y-, and z-directions, the linear strain rates are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \mathbf{0.8\ s^{-1}} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \mathbf{-0.8\ s^{-1}} \qquad \varepsilon_{zz} = \mathbf{0} \tag{4}$$

Thus, we predict that fluid particles *stretch* in the x-direction (positive linear strain rate) and *shrink* in the y-direction (negative linear strain rate). This is illustrated in Fig. 4–42, where we have marked an initially square parcel of fluid centered at (0.25, 4.25). By integrating Eqs. 2 with time, we calculate the location of the four corners of the marked fluid after an elapsed time of 1.5 s. Indeed this fluid parcel has stretched in the x-direction and has shrunk in the y-direction as predicted.

Shear strain rate is determined from Eq. 4–26. Because of the two-dimensionality, nonzero shear strain rates can occur only in the xy-plane. Using lines parallel to the x- and y-axes as our initially perpendicular lines, we calculate ε_{xy},

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0 + 0) = \mathbf{0} \tag{5}$$

Thus, there is no shear strain in this flow, as also indicated by Fig. 4–42. Although the sample fluid particle deforms, it remains rectangular; its initially 90° corner angles remain at 90° throughout the time period of the calculation.

Finally, the volumetric strain rate is calculated from Eq. 4–24:

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.8 - 0.8 + 0)\ s^{-1} = \mathbf{0} \tag{6}$$

Since the volumetric strain rate is zero everywhere, we can say definitively that fluid particles are neither dilating (expanding) nor shrinking (compressing) in volume. Thus, **we verify that this flow is indeed incompressible.** In Fig. 4–42, the area of the shaded fluid particle (and thus its volume since it is a 2-D flow) remains constant as it moves and deforms in the flow field.

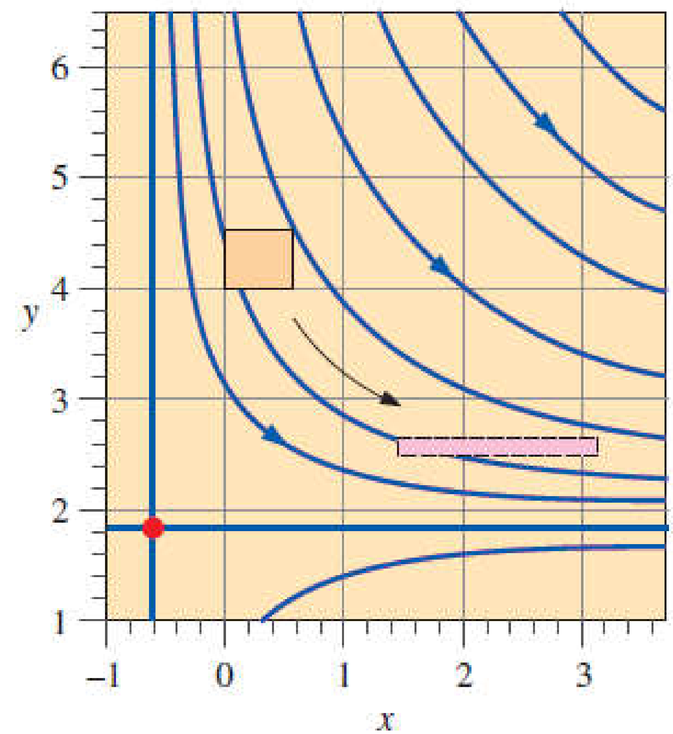


FIGURE 4–42
Deformation of an initially square parcel of marked fluid subjected to the velocity field of Example 4–6 for a time period of 1.5 s. The stagnation point is indicated by the red circle at $x = -0.625$ m and $y = 1.875$ m, and several streamlines are plotted.

VORTICITY AND ROTATIONALITY

Another kinematic property of great importance to the analysis of fluid flows is the **vorticity vector**, defined mathematically as the curl of the velocity vector

Vorticity vector: $$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \mathrm{curl}(\vec{V})$$

Rate of rotation vector: $$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\mathrm{curl}(\vec{V}) = \frac{\vec{\zeta}}{2}$$

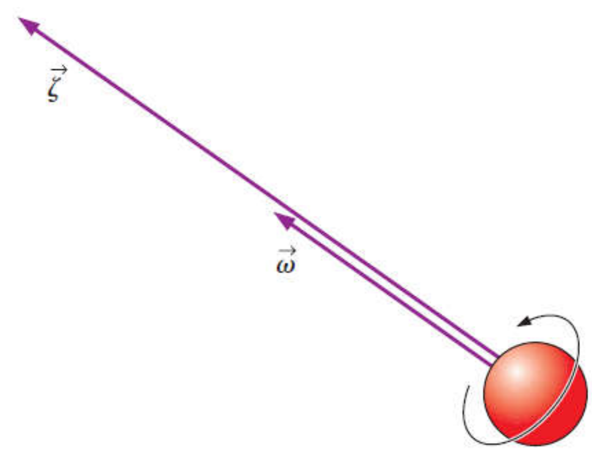
Vorticity is equal to twice the angular velocity of a fluid particle

Rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Vorticity vector in Cartesian coordinates:

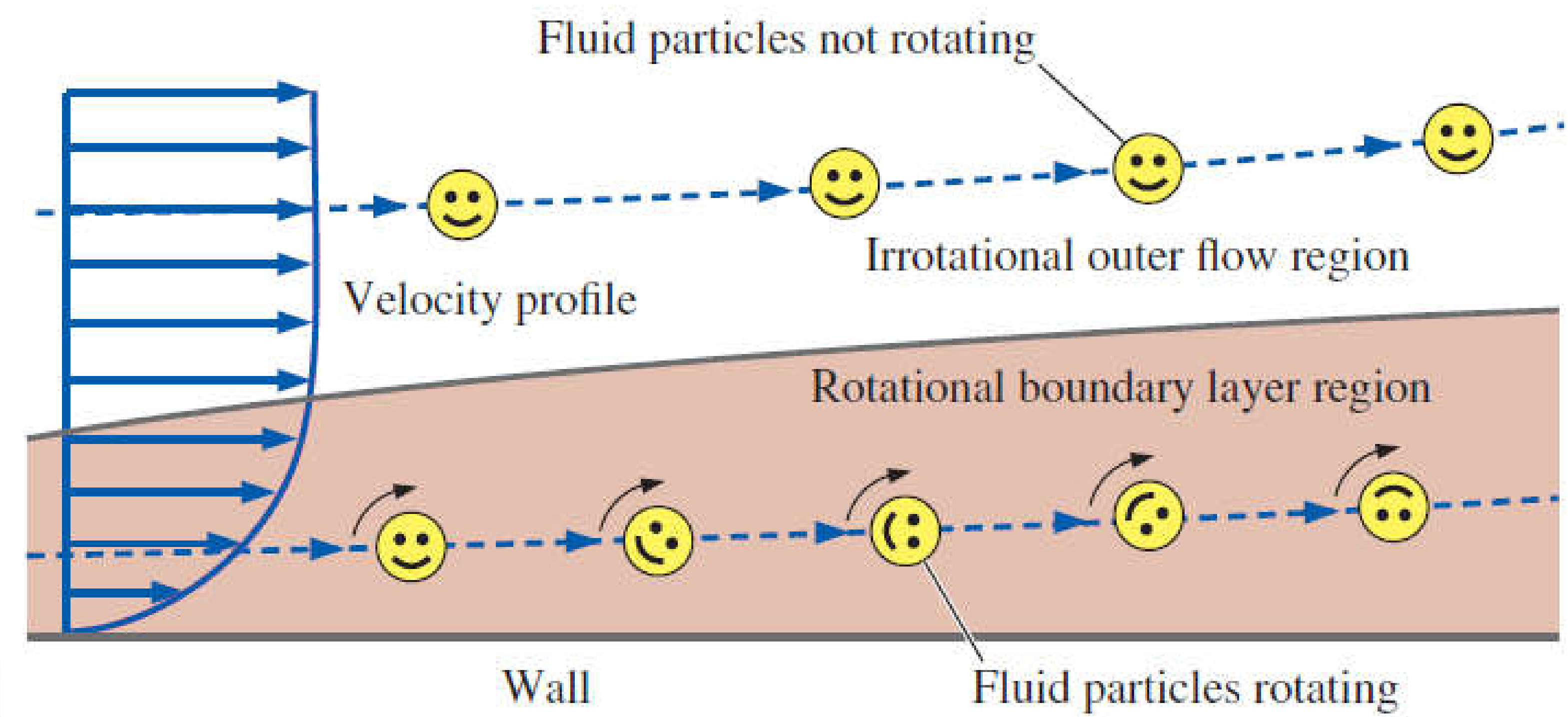
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$



The *vorticity vector* is equal to twice the angular velocity vector of a rotating fluid particle.

- If the vorticity at a point in a flow field is nonzero, the fluid particle that happens to occupy that point in space is rotating; the flow in that region is called **rotational**.
- Likewise, if the vorticity in a region of the flow is zero (or negligibly small), fluid particles there are not rotating; the flow in that region is called **irrotational**.
- Physically, fluid particles in a rotational region of flow rotate end over end as they move along in the flow.

The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.



Comparison of Two Circular Flows

Types of Vortices:-

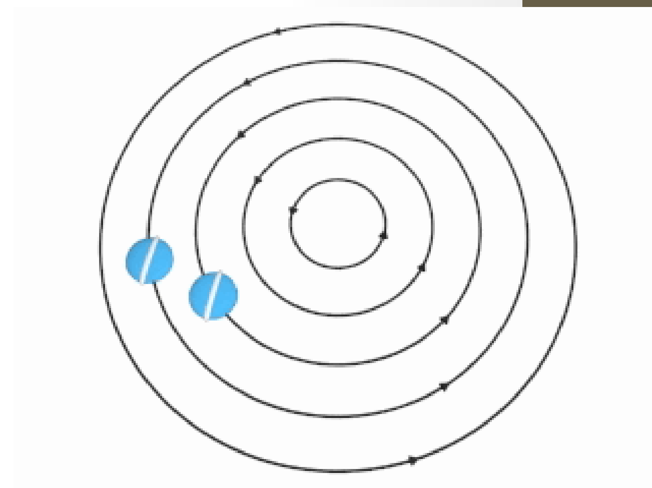
Forced Vortex

Forced Vortex is a type of motion for a flow rotating as a rigid body.

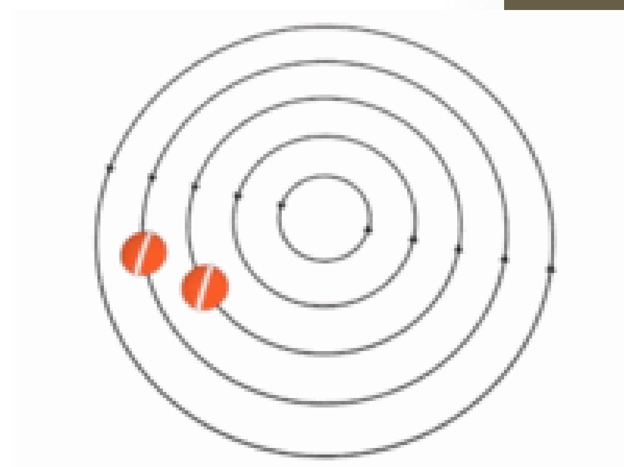
Free Vortex (an irrotational vortex)

If the fluid elements are going around in circles (evolves fairly quickly), but they are not rotating, this type of motion is known as free vortex.

Actual vortices (e.g. tornados and whirlpools) are approximated by a combination of forced and free vortices with the forced vortex at the center and the free vortex on the outside.



A rigid-body vortex



An irrotational vortex

Comparison of Two Circular Flows

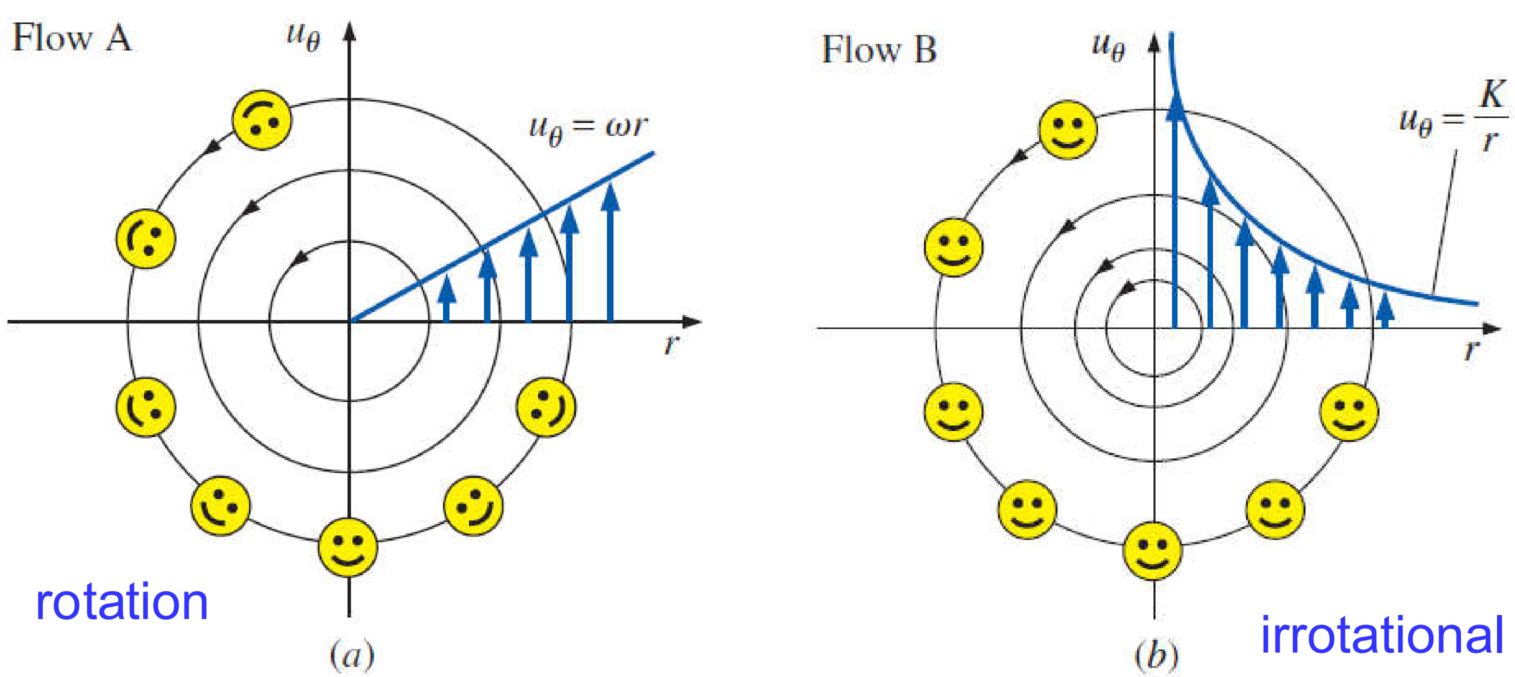
Flow A—solid-body rotation: $u_r = 0$ and $u_\theta = \omega r$

Flow B—line vortex: $u_r = 0$ and $u_\theta = \dfrac{K}{r}$

Flow A—solid-body rotation: $\vec{\zeta} = \dfrac{1}{r}\left(\dfrac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{k} = 2\omega\vec{k}$

Flow B—line vortex: $\vec{\zeta} = \dfrac{1}{r}\left(\dfrac{\partial(K)}{\partial r} - 0\right)\vec{k} = 0$

Streamlines and velocity profiles for (*a*) flow A, solid-body rotation and (*b*) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.



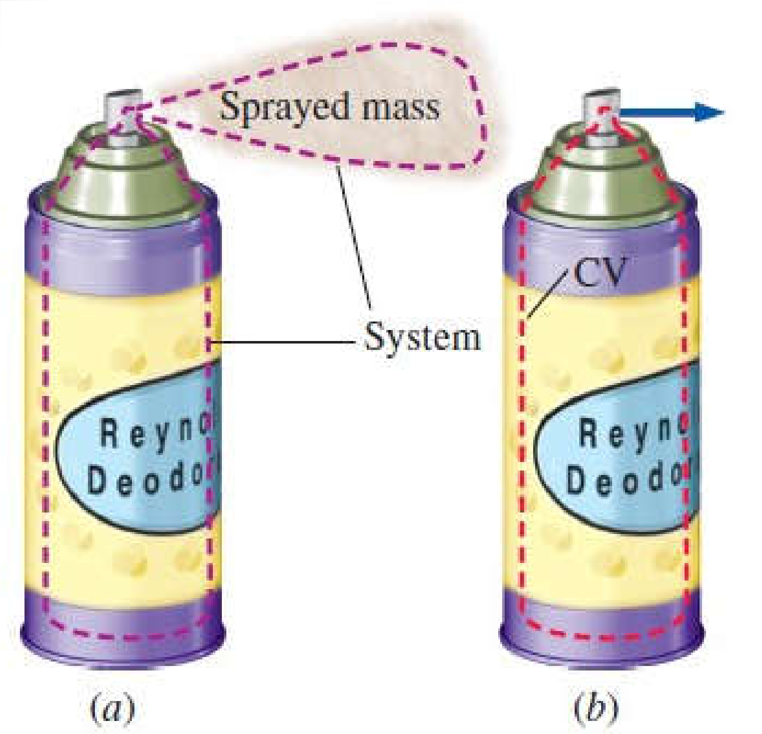
A simple analogy can be made between flow A and a merry-go-round or roundabout, and flow B and a Ferris wheel.

As children revolve around a roundabout, they also rotate at the same angular velocity as that of the ride itself. This is analogous to a rotational flow.

In contrast, children on a Ferris wheel always remain oriented in an upright position as they trace out their circular path. This is analogous to an irrotational flow.

A simple analogy: (*a*) *rotational* circular flow is analogous to a roundabout, while (*b*) *irrotational* circular flow is analogous to a Ferris wheel.

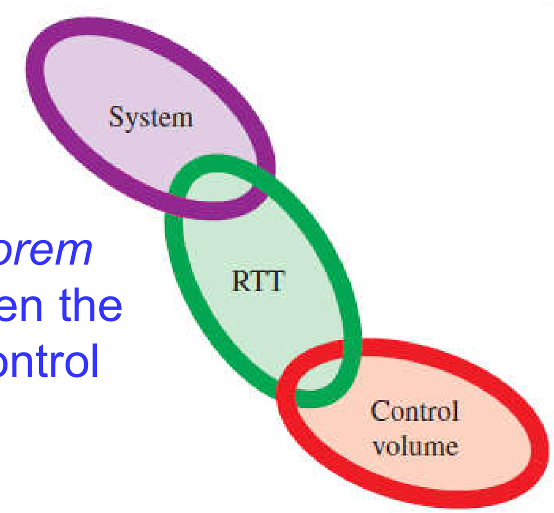
THE REYNOLDS TRANSPORT THEOREM

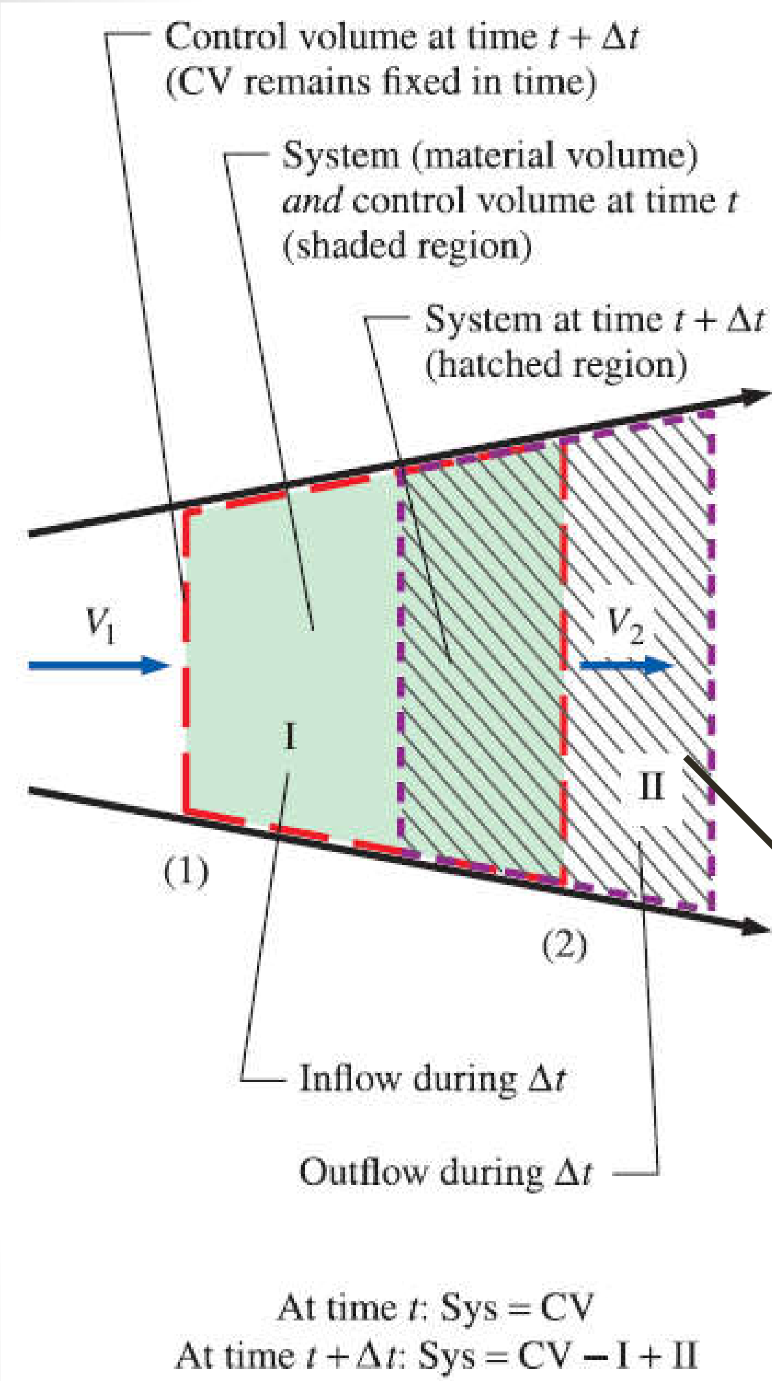


Two methods of analyzing the spraying of deodorant from a spray can:
(***a***) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed.
(***b***) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

The relationship between the time rates of change of an extensive property for a system and for a control volume is expressed by the **Reynolds Transport Theorem (RTT)**.

The *Reynolds transport theorem* (RTT) provides a link between the system approach and the control volume approach.





- Consider flow from left to right through a diverging portion of a flow field.

- The control volume chosen to be fixed between sections (1) and (2) of the flow field.

- At some initial time t, the system coincides with the control volume (**Identical system and control volume** **(greenish shaded region)**).

- During time interval Δt, the system moves in the flow direction at uniform speeds V_1 at section (1) and V_2 at section (2) (system hatched region).

At time $t + \Delta t$, the system consists of the same fluid, but it occupies the region.

CV – I +II.

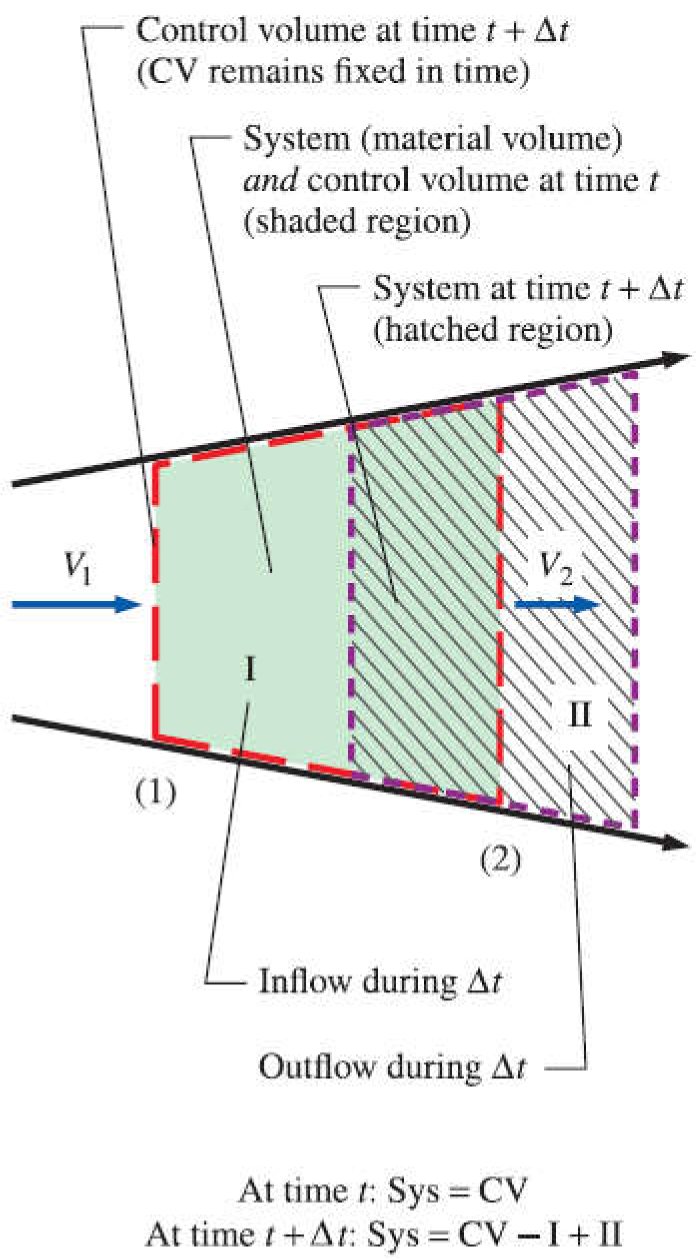
A moving *system* (hatched region) and a fixed *control volume* (shaded region) in a diverging portion of a flow field at times t and $t+\Delta t$. The upper and lower bounds are streamlines of the flow.

At time $t+\Delta t$, the system consists of the same fluid, but it occupies the region.

CV – I + II.

b=B/m

B **=** any **extensive property** (energy, momentum)
b **=** ***B/m*** the corresponding **intensive property**



$$B_{sys,t} = B_{CV,t} \qquad \text{(the system and CV concide at time t)}$$

$$B_{sys,t+\Delta t} = B_{CV,t+\Delta t} - B_{I,t+\Delta t} + B_{II,t+\Delta t}$$

$$\frac{B_{sys,t+\Delta t} - B_{sys,t}}{\Delta t} = \frac{B_{CV,t+\Delta t} - B_{CV,t}}{\Delta t} - \frac{B_{I,t+\Delta t}}{\Delta t} + \frac{B_{II,t+\Delta t}}{\Delta t}$$

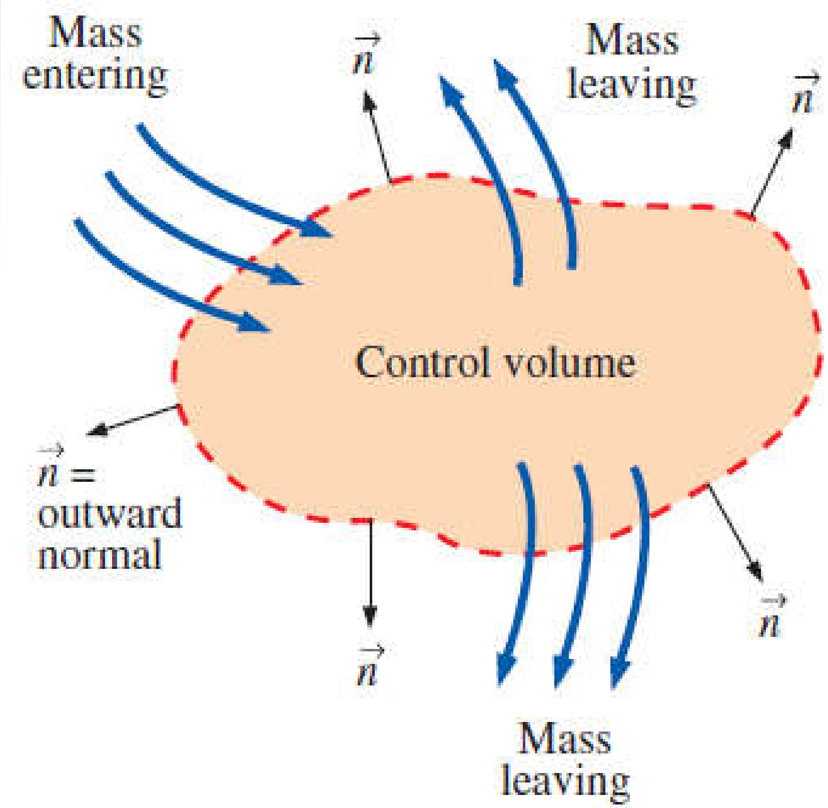
$$\boxed{\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out}}$$

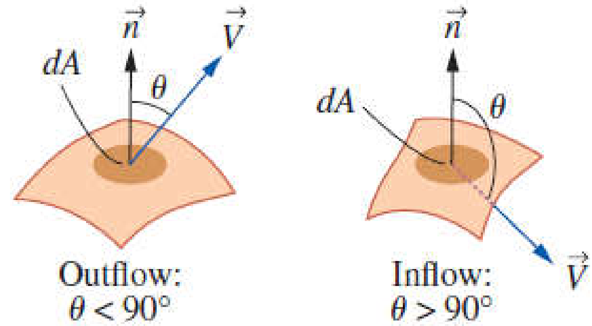
$\boxed{B = bm}$

Or

$$\boxed{\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - b_1\rho_1 V_1 A_1 + b_2\rho_2 V_2 A_2}$$

The **time rate of change of the property *B*** of the system is equal to the time rate of **change of *B* of the control volume** plus the **net flux of *B*** out of the control volume by mass crossing the control surface.





b=B/m

$\vec{V}\cdot\vec{n} = |\vec{V}||\vec{n}|\cos\theta = V\cos\theta$
If $\theta < 90°$, then $\cos\theta > 0$ (outflow).
If $\theta > 90°$, then $\cos\theta < 0$ (inflow).
If $\theta = 90°$, then $\cos\theta = 0$ (no flow).

Outflow and inflow of mass across the differential area of a control surface.

$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V}\cdot\vec{n}\, dA$$

$$\boxed{\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out}}$$

$\boxed{B = bm}$

The integral of $b\rho\vec{V}\cdot\vec{n}\,dA$ over the control surface gives the net amount of the property B flowing out of the control volume (into the control volume if it is negative) per unit time.

RTT, fixed CV:

$$\boxed{\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b\vec{V}\cdot\vec{n}\, dA}$$

Alternate RTT, fixed CV:

$$\boxed{\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b\vec{V}\cdot\vec{n}\, dA}$$

$$B_{CV} = \int_{CV} \rho b\, dV$$

Reynolds transport theorem applied to a control volume moving at constant velocity.

RTT, fixed CV:

$$\boxed{\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA}$$

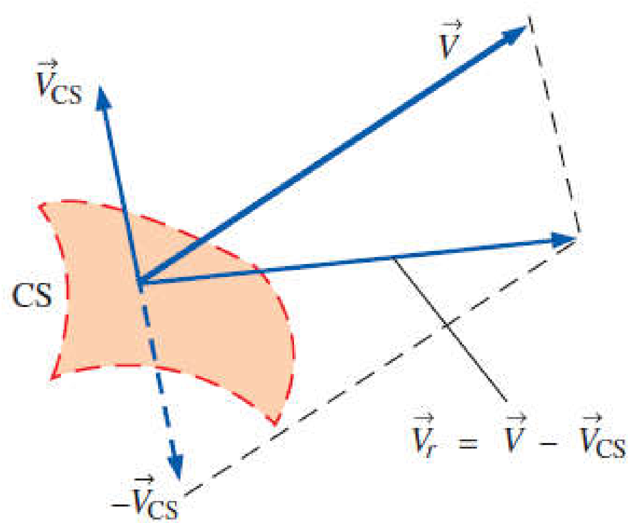
Relative velocity: $\vec{V}_r = \vec{V} - \vec{V}_{CS}$

RTT, nonfixed CV:

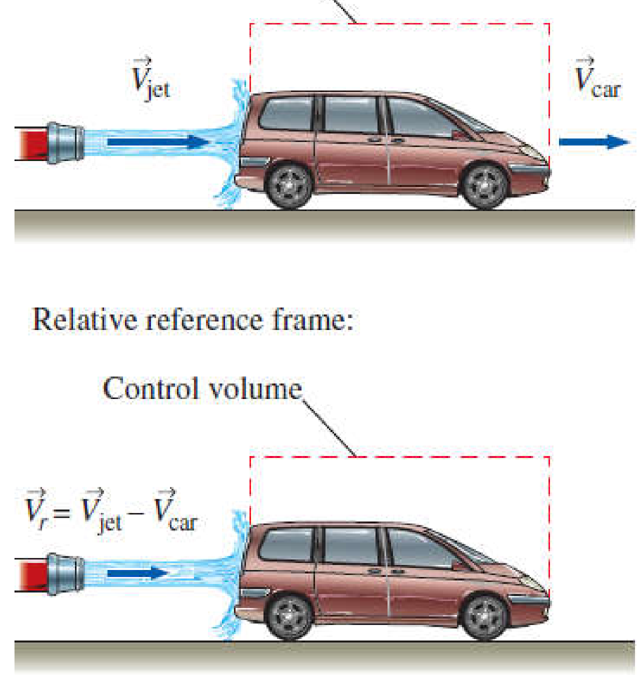
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

RTT, steady flow:

$$\frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$$



Relative velocity crossing a control surface is found by vector addition of the absolute velocity of the fluid and the negative of the local velocity of the control surface.



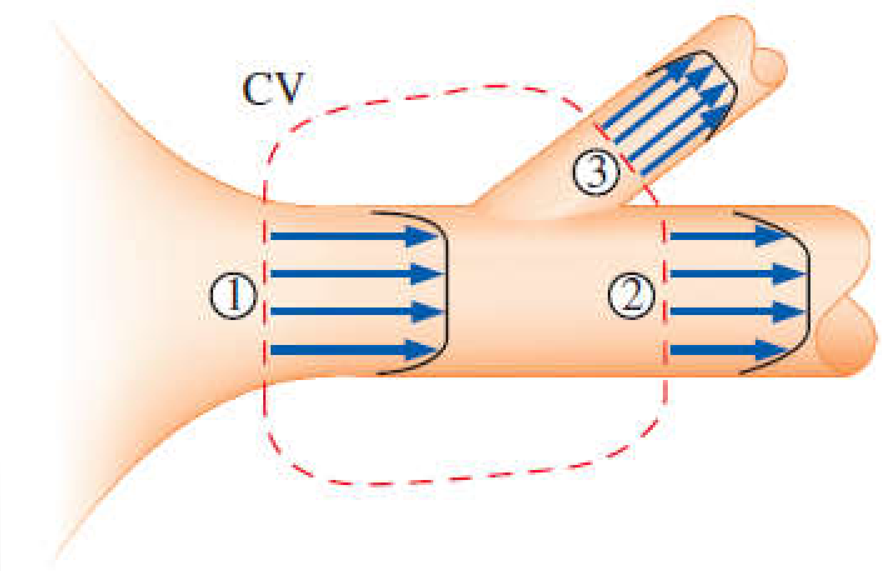
$$\int_A \rho b \vec{V}_r \cdot \vec{n}\, dA \cong b_{avg} \int_A \rho \vec{V}_r \cdot \vec{n}\, dA = b_{avg}\dot{m}_r$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \sum_{out} \underbrace{\dot{m}_r b_{avg}}_{\text{for each outlet}} - \sum_{in} \underbrace{\dot{m}_r b_{avg}}_{\text{for each inlet}}$$

Approximate RTT for well-defined inlets and outlets:

$$\dot{m}_r \approx \rho_{avg}\dot{V}_r = \rho_{avg} V_{r,\,avg} A$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \sum_{out} \underbrace{\rho_{avg} b_{avg} V_{r,\,avg} A}_{\text{for each outlet}} - \sum_{in} \underbrace{\rho_{avg} b_{avg} V_{r,avg} A}_{\text{for each inlet}}$$



An example control volume in which there is one well-defined inlet (1) and two well-defined outlets (2 and 3). In such cases, the control surface integral in the RTT can be more conveniently written in terms of the average values of fluid properties crossing each inlet and outlet.

Conservation laws such as the laws of conservation of mass, conservation of energy, and conservation of momentum.

Historically, the conservation laws are first applied to a fixed quantity of matter called a *closed system* or just a *system*, and then extended to regions in space called *control volumes*.

The conservation relations are also called *balance equations* since any conserved quantity must balance during a process.

Many fluid flow devices such as this Pelton wheel hydraulic turbine are analyzed by applying the conservation of mass and energy principles, along with the linear momentum equation.

Conservation of Mass

The conservation of mass relation for a closed system undergoing a change is expressed as m_{sys} = constant or $dm_{sys}/dt = 0$, which is the statement that the mass of the system remains constant during a process.

Mass balance for a control volume (CV) in rate form:

Conservation of mass: $$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{dm_{\text{CV}}}{dt}$$

$\dot{m}_{\text{in}}$ and $\dot{m}_{\text{out}}$ the total rates of mass flow into and out of the control volume

dm_{CV}/dt the rate of change of mass within the control volume boundaries.

Continuity equation: In fluid mechanics, the conservation of mass relation written for a differential control volume is usually called the *continuity equation*.

The Linear Momentum Equation

Linear momentum: The product of the **mass and the velocity** of a body is called the *linear momentum* or just the *momentum* of the body.

The momentum of a rigid body of mass m moving with a velocity V is mV.

Newton's second law: The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass, and that **the rate of change of the momentum of a body is equal to the net force acting on the body**.

Conservation of momentum principle: The momentum of a system remains constant only when the net force acting on it is zero, and thus the momentum of such systems is conserved.

Linear momentum equation: In fluid mechanics, Newton's second law is usually referred to as the *linear momentum equation.*

Conservation of Energy

The conservation of energy principle (the energy balance): The net energy transfer to or from a system during a process be equal to the change in the energy content of the system.

Energy can be transferred to or from a closed system by heat or work.

Control volumes also involve energy transfer via mass flow.

Conservation of energy: $$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \frac{dE_{\text{CV}}}{dt}$$

$\dot{E}_{\text{in}}$ and $\dot{E}_{\text{out}}$ — the total rates of energy transfer into and out of the control volume

dE_{CV}/dt — the rate of change of energy within the control volume boundaries

In fluid mechanics, we usually limit our consideration to mechanical forms of energy only.

CONSERVATION OF MASS

Conservation of mass: Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process.

Closed systems: The mass of the system remain constant during a process.

Control volumes: Mass can cross the boundaries, and so we must keep track of the amount of mass entering and leaving the control volume.

$$\boxed{\begin{matrix}2\text{ kg}\\ H_2\end{matrix}} + \boxed{\begin{matrix}16\text{ kg}\\ O_2\end{matrix}} \longrightarrow \boxed{\begin{matrix}18\text{ kg}\\ H_2O\end{matrix}}$$

Mass is conserved even during chemical reactions.

Mass m and energy E can be converted to each other:

$$E = mc^2$$

c is the speed of light in a vacuum, $c = 2.9979\times10^8$ m/s

The mass change due to energy change is negligible.

Mass and Volume Flow Rates

Mass flow rate: The amount of mass flowing through a cross section per unit time.

The differential mass flow rate

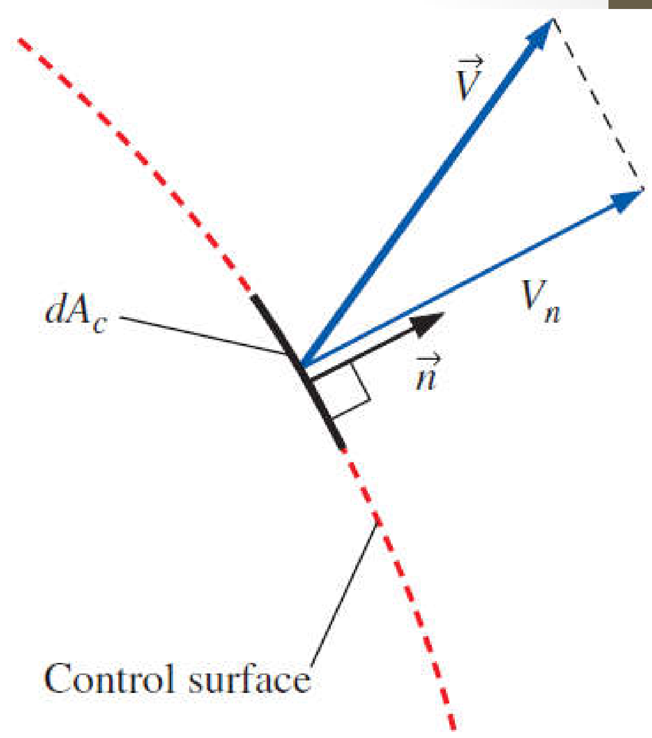
$$\delta \dot{m} = \rho V_n \, dA_c$$

Point functions have ***exact differentials***

$$\int_1^2 dA_c = A_{c2} - A_{c1} = \pi(r_2^2 - r_1^2)$$

Path functions have ***inexact differentials***

$$\int_1^2 \delta \dot{m} = \dot{m}_{\text{total}} \qquad \text{not } \dot{m}_2 - \dot{m}_1$$



The normal velocity V_n for a surface is the component of velocity perpendicular to the surface.

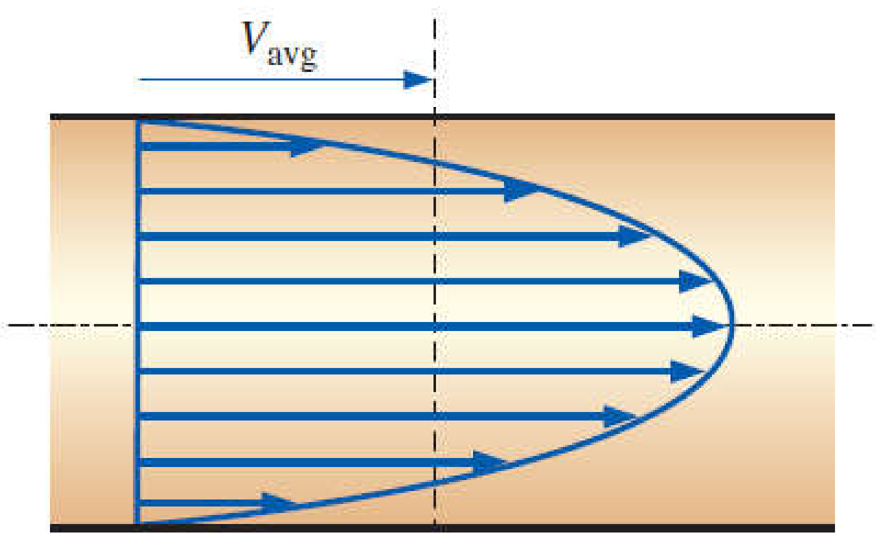
$$\delta \dot{m} = \rho V_n \, dA_c$$

$$\dot{m} = \int_{A_c} \delta \dot{m} = \int_{A_c} \rho V_n \, dA_c$$

Mass flow rate

$$\boxed{\dot{m} = \rho V_{avg} A_c \qquad (\mathrm{kg/s})}$$

$$\boxed{\dot{m} = \rho \dot{V} = \frac{\dot{V}}{v}}$$



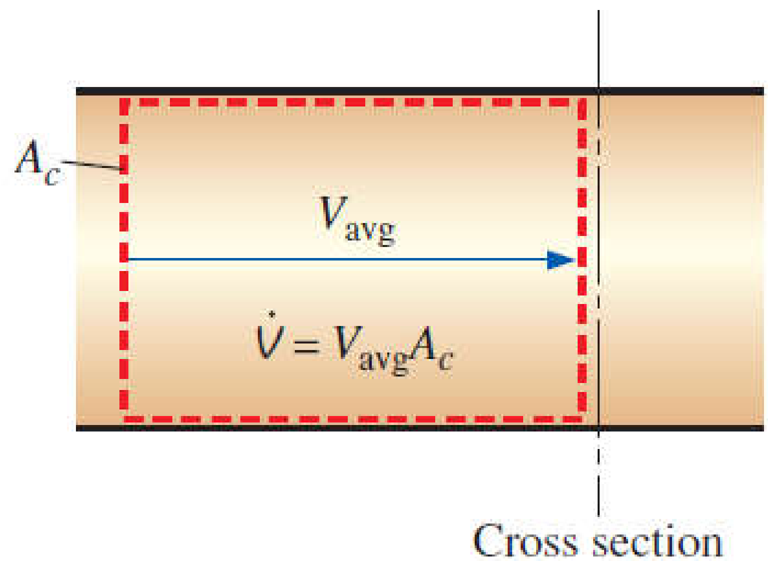
The average velocity V_{avg} is defined as the average speed through a cross section.

Average velocity

$$V_{avg} = \frac{1}{A_c} \int_{A_c} V_n \, dA_c$$

Volume flow rate

$$\boxed{\dot{V} = \int_{A_c} V_n \, dA_c = V_{avg} A_c = V A_c \qquad (\mathrm{m^3/s})}$$



The volume flow rate is the volume of fluid flowing through a cross section per unit time.

Conservation of Mass Principle

The conservation of mass principle for a control volume: The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the control volume during Δt.

$$\begin{pmatrix}\text{Total mass entering}\\ \text{the CV during } \Delta t\end{pmatrix} - \begin{pmatrix}\text{Total mass leaving}\\ \text{the CV during } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Net change of mass}\\ \text{within the CV during } \Delta t\end{pmatrix}$$

$$\boxed{m_{\text{in}} - m_{\text{out}} = \Delta m_{\text{CV}} \qquad (\text{kg})}$$

$$\boxed{\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = dm_{\text{CV}}/dt \qquad (\text{kg/s})}$$

$\dot{m}_{\text{in}}$ and $\dot{m}_{\text{out}}$ — the total rates of mass flow into and out of the control volume

dm_{CV}/dt — the rate of change of mass within the control volume boundaries.

Mass balance is applicable to any control volume undergoing any kind of process.

Conservation of mass principle for an ordinary bathtub.

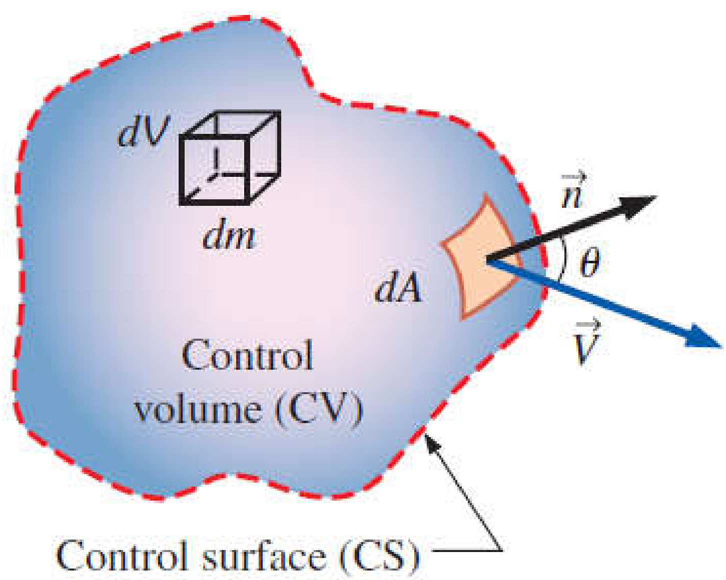
$dm = \rho \, dV$. *Total mass within the CV:* $\quad m_{CV} = \int_{CV} \rho \, dV$

Rate of change of mass within the CV: $\quad \frac{dm_{CV}}{dt} = \frac{d}{dt} \int_{CV} \rho \, dV$

Normal component of velocity: $\quad V_n = V \cos \theta = \vec{V} \cdot \vec{n}$

Differential mass flow rate: $\quad \delta \dot{m} = \rho V_n \, dA = \rho (V \cos \theta) \, dA = \rho (\vec{V} \cdot \vec{n}) \, dA$

Net mass flow rate: $\quad \dot{m}_{net} = \int_{CS} \delta \dot{m} = \int_{CS} \rho V_n \, dA = \int_{CS} \rho (\vec{V} \cdot \vec{n}) \, dA$



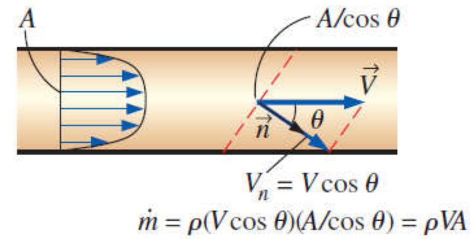
The differential control volume *dV* and the differential control surface *dA* used in the derivation of the conservation of mass relation.

General conservation of mass: $$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho(\vec{V}\cdot\vec{n})\, dA = 0$$

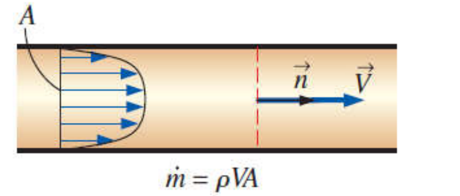
The time rate of change of mass within the control volume plus the net mass flow rate through the control surface is equal to zero.

$$\frac{d}{dt}\int_{CV} \rho \, dV + \sum_{\text{out}} \rho |V_n| A - \sum_{\text{in}} \rho |V_n| A = 0$$

$$\frac{d}{dt}\int_{CV} \rho \, dV = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \qquad \frac{dm_{CV}}{dt} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$$



(*a*) Control surface *at an angle* to the flow



(*b*) Control surface *normal* to the flow

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b(\vec{V}\cdot\vec{n})\, dA$$

$$B = m \qquad b = 1 \qquad b = 1$$

$$\frac{dm_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho(\vec{V}\cdot\vec{n})\, dA$$

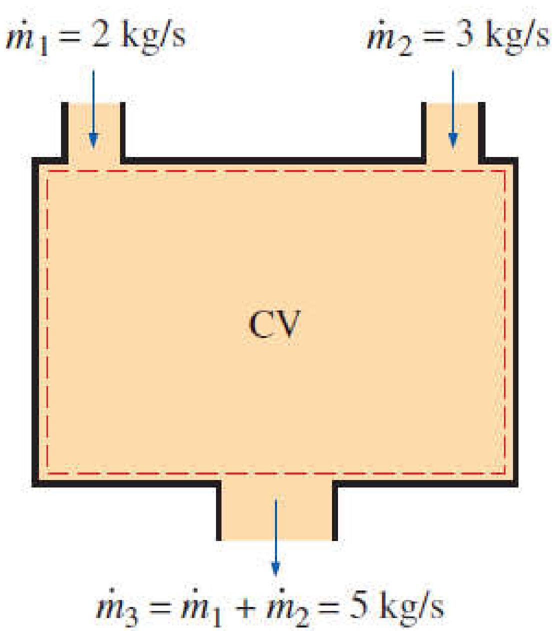
The conservation of mass equation is obtained by replacing *B* in the Reynolds transport theorem by mass *m*, and *b* by 1 (*m* per unit mass = *m*/*m* = 1).

A control surface should always be selected *normal to the flow* at all locations where it crosses the fluid flow to avoid complications, even though the result is the same.

Mass Balance for Steady-Flow Processes

During a steady-flow process, the total amount of mass contained within a control volume does not change with time (m_{CV} = constant).

Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it.



Conservation of mass principle for a two-inlet–one-outlet steady-flow system.

For steady-flow processes, we are interested in the amount of mass flowing per unit time, that is, *the mass flow rate*.

$$\sum_{\text{in}} \dot{m} = \sum_{\text{out}} \dot{m} \qquad (\text{kg/s})$$

Multiple inlets and exits

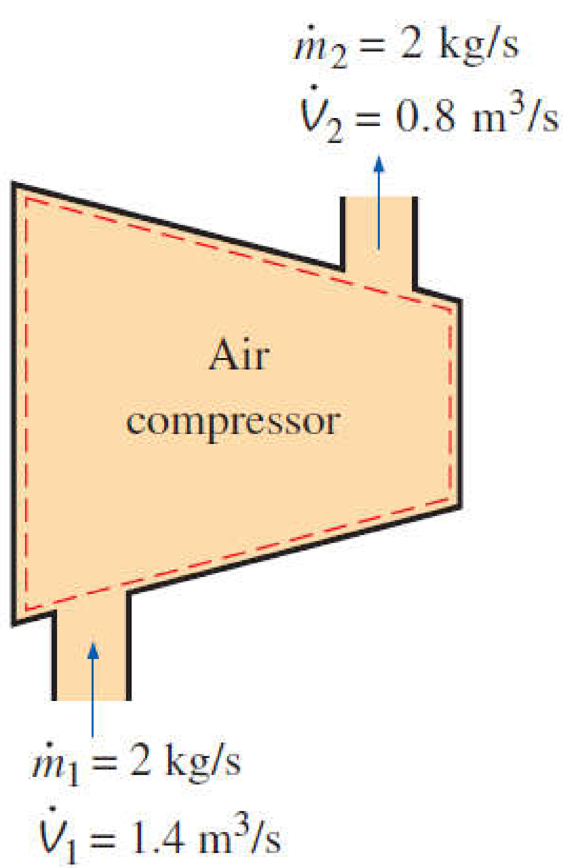
$$\dot{m}_1 = \dot{m}_2 \quad \rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Single stream

Many engineering devices such as nozzles, diffusers, turbines, compressors, and pumps involve a single stream (only one inlet and one outlet).

Special Case: Incompressible Flow

The conservation of mass relations can be simplified even further when the fluid is incompressible, which is usually the case for liquids.



$$\sum_{\text{in}} \dot{V} = \sum_{\text{out}} \dot{V} \qquad (\text{m}^3/\text{s})$$

Steady, incompressible

$$\dot{V}_1 = \dot{V}_2 \rightarrow V_1 A_1 = V_2 A_2$$

Steady, incompressible flow (single stream)

There is no such thing as a "conservation of volume" principle.

However, for steady flow of liquids, the volume flow rates, as well as the mass flow rates, remain constant since liquids are essentially incompressible substances.

During a steady-flow process, volume flow rates are not necessarily conserved although mass flow rates are.

EXAMPLE 5–1 Water Flow through a Garden Hose Nozzle

A garden hose attached with a nozzle is used to fill a 10-gal bucket. The inner diameter of the hose is 2 cm, and it reduces to 0.8 cm at the nozzle exit (Fig. 5–12). If it takes 50 s to fill the bucket with water, determine (*a*) the volume and mass flow rates of water through the hose, and (*b*) the average velocity of water at the nozzle exit.

Properties We take the density of water to be 1000 kg/m^3 = 1 kg/L.

$$\dot{V} = \frac{V}{\Delta t} = \frac{10\ \text{gal}}{50\ \text{s}}\left(\frac{3.7854\ \text{L}}{1\ \text{gal}}\right) = \mathbf{0.757\ L/s}$$

$$\dot{m} = \rho\dot{V} = (1\ \text{kg/L})(0.757\ \text{L/s}) = \mathbf{0.757\ kg/s}$$

(*b*) The cross-sectional area of the nozzle exit is

$$A_e = \pi r_e^2 = \pi(0.4\ \text{cm})^2 = 0.5027\ \text{cm}^2 = 0.5027 \times 10^{-4}\ \text{m}^2$$

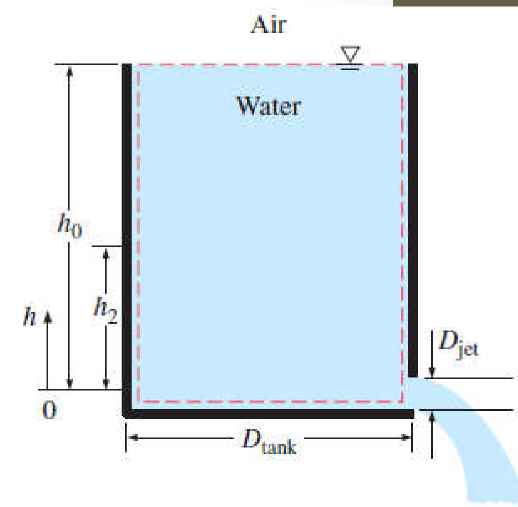
The volume flow rate through the hose and the nozzle is constant. Then the average velocity of water at the nozzle exit becomes

$$V_e = \frac{\dot{V}}{A_e} = \frac{0.757\ \text{L/s}}{0.5027 \times 10^{-4}\ \text{m}^2}\left(\frac{1\ \text{m}^3}{1000\ \text{L}}\right) = \mathbf{15.1\ m/s}$$

Discussion It can be shown that the average velocity in the hose is 2.4 m/s. Therefore, the nozzle increases the water velocity by over six times.

EXAMPLE 5–2 Discharge of Water from a Tank

A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out (Fig. 5–13). The average velocity of the jet is approximated as $V = \sqrt{2gh}$, where h is the height of water in the tank measured from the center of the hole (a variable) and g is the gravitational acceleration. Determine how long it takes for the water level in the tank to drop to 2 ft from the bottom.



Analysis We take the volume occupied by water as the control volume. The size of the control volume decreases in this case as the water level drops, and thus this is a variable control volume. (We could also treat this as a fixed control volume that consists of the interior volume of the tank by disregarding the air that replaces the space vacated by the water.) This is obviously an unsteady-flow problem since the properties (such as the amount of mass) within the control volume change with time.

The conservation of mass relation for a control volume undergoing any process is given in rate form as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt} \tag{1}$$

During this process no mass enters the control volume ($\dot{m}_{in} = 0$), and the mass flow rate of discharged water is

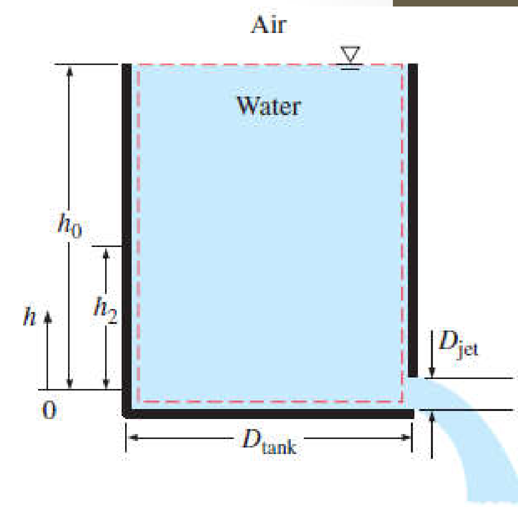
$$\dot{m}_{out} = (\rho V A)_{out} = \rho\sqrt{2gh}A_{jet} \tag{2}$$

where $A_{jet} = \pi D_{jet}^2/4$ is the cross-sectional area of the jet, which is constant. Noting that the density of water is constant, the mass of water in the tank at any time is

$$m_{CV} = \rho V = \rho A_{tank} h \tag{3}$$

where $A_{tank} = \pi D_{tank}^2/4$ is the base area of the cylindrical tank. Substituting Eqs. 2 and 3 into the mass balance relation (Eq. 1) gives

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank} h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)dh}{dt}$$



Canceling the densities and other common terms and separating the variables give

$$dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{dh}{\sqrt{2gh}}$$

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

Substituting, the time of discharge is determined to be

$$t = \frac{\sqrt{4\text{ ft}} - \sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}}\left(\frac{3 \times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \mathbf{12.6\text{ min}}$$

Therefore, it takes 12.6 min after the discharge hole is unplugged for half of the tank to be emptied.

Discussion Using the same relation with $h_2 = 0$ gives $t = 43.1$ min for the discharge of the entire amount of water in the tank. Therefore, emptying the bottom half of the tank takes much longer than emptying the top half. This is due to the decrease in the average discharge velocity of water with decreasing h.

MECHANICAL ENERGY AND EFFICIENCY

Mechanical energy: The form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine.

Mechanical energy of a flowing fluid per unit mass:

$$e_{\text{mech}} = \frac{P}{\rho} + \frac{V^2}{2} + gz$$

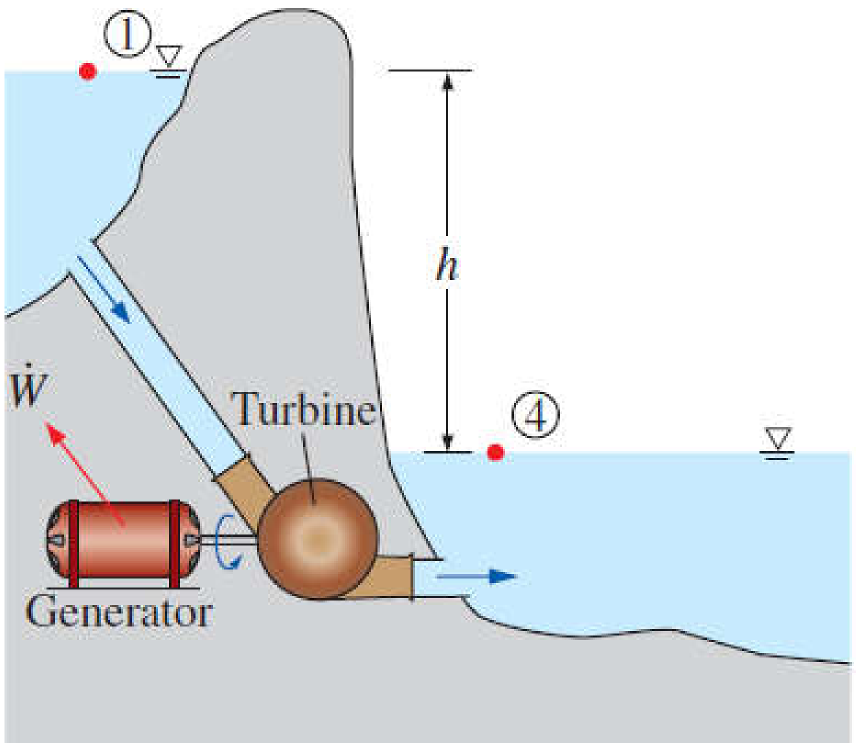
Flow energy + kinetic energy + potential energy

Mechanical energy change:

$$\Delta e_{\text{mech}} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) \qquad (\text{kJ/kg})$$

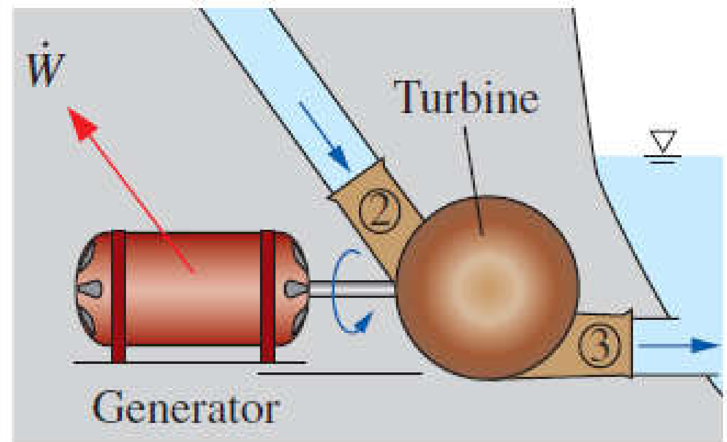
- The mechanical energy of a fluid does not change during flow if its pressure, density, velocity, and elevation remain constant.
- In the absence of any irreversible losses, the mechanical energy change represents the mechanical work supplied to the fluid (if $\Delta e_{\text{mech}} > 0$) or extracted from the fluid (if $\Delta e_{\text{mech}} < 0$).

Mechanical energy is a useful concept for flows that do not involve significant heat transfer or energy conversion, such as the flow of gasoline from an underground tank into a car.



$$\dot{W}_{max} = \dot{m}\Delta e_{mech} = \dot{m}g(z_1 - z_4) = \dot{m}gh$$
since $P_1 \approx P_4 = P_{atm}$ and $V_1 = V_4 \approx 0$

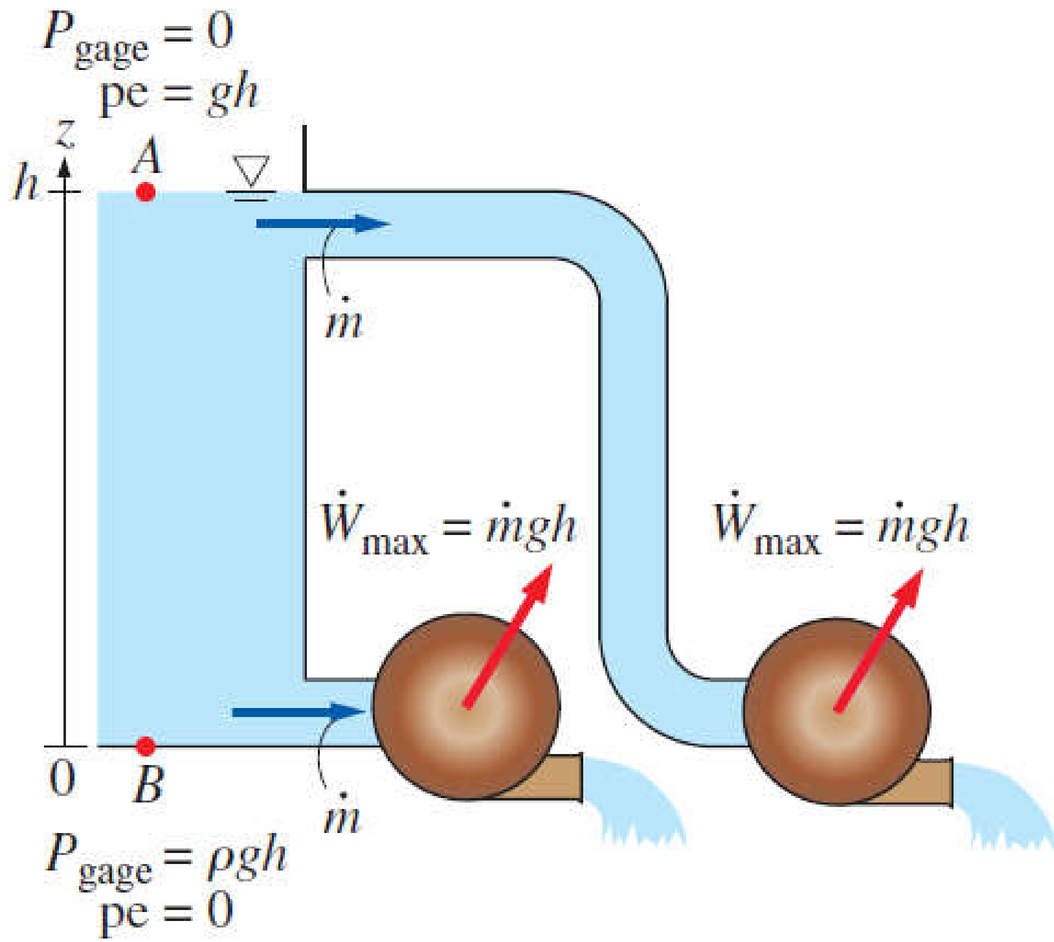
(*a*)



$$\dot{W}_{max} = \dot{m}\Delta e_{mech} = \dot{m}\frac{P_2 - P_3}{\rho} = \dot{m}\frac{\Delta P}{\rho}$$
since $V_2 \approx V_3$ and $z_2 \approx z_3$

(*b*)

Mechanical energy is illustrated by an ideal hydraulic turbine coupled with an ideal generator. In the absence of irreversible losses, the maximum produced power is proportional to (*a*) the change in water surface elevation from the upstream to the downstream reservoir or (*b*) (close-up view) the drop in water pressure from just upstream to just downstream of the turbine.



The available mechanical energy of water at the bottom of a container is equal to the avaiable mechanical energy at any depth including the free surface of the container.

Transfer of mechanical energy is accomplished rotating shaft

Shaft work: The transfer of mechanical energy is usually accomplished by a rotating shaft, and thus mechanical work is often referred to as *shaft work*.

A pump or a fan receives shaft work and transfers it to the fluid as mechanical energy.

A turbine converts the mechanical energy of a fluid to shaft work.

$$\eta_{mech} = \frac{\text{Mechanical energy output}}{\text{Mechanical energy input}} = \frac{E_{mech, out}}{E_{mech, in}} = 1 - \frac{E_{mech, loss}}{E_{mech, in}}$$

Mechanical efficiency of a device or process

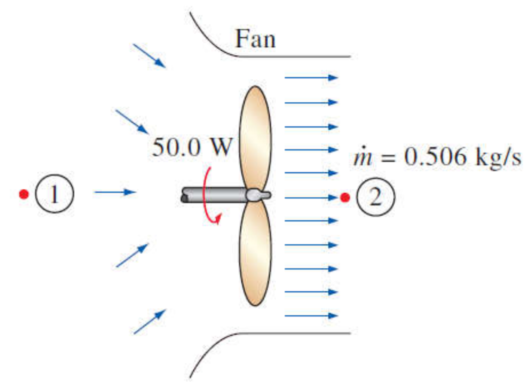
The effectiveness of the conversion process between the mechanical work supplied or extracted and the mechanical energy of the fluid is expressed by the **pump efficiency** and **turbine efficiency**,

$$\eta_{pump} = \frac{\text{Mechanical power increase of the fluid}}{\text{Mechanical power input}} = \frac{\Delta \dot{E}_{mech, fluid}}{\dot{W}_{shaft, in}} = \frac{\dot{W}_{pump, u}}{\dot{W}_{pump}}$$

$$\Delta \dot{E}_{mech, fluid} = \dot{E}_{mech, out} - \dot{E}_{mech, in}$$

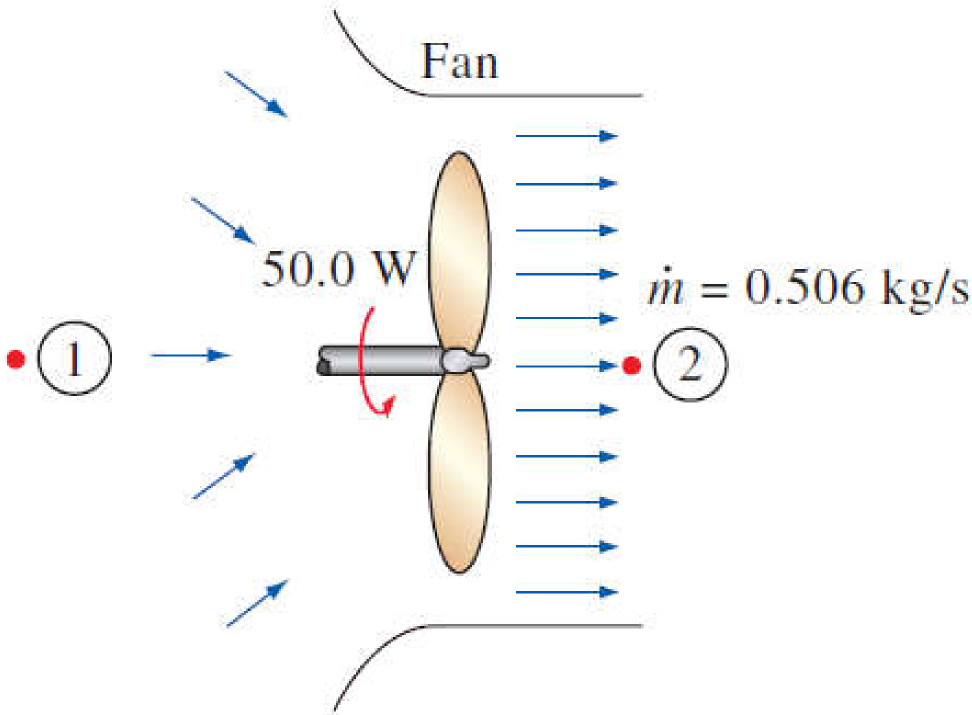
$$\eta_{turbine} = \frac{\text{Mechanical power output}}{\text{Mechanical power decrease of the fluid}} = \frac{\dot{W}_{shaft, out}}{|\Delta \dot{E}_{mech, fluid}|} = \frac{\dot{W}_{turbine}}{\dot{W}_{turbine, e}}$$

$$|\Delta \dot{E}_{mech, fluid}| = \dot{E}_{mech, in} - \dot{E}_{mech, out}$$



$V_1 \approx 0$, $V_2 = 12.1$ m/s
$z_1 = z_2$
$P_1 \approx P_{atm}$ and $P_2 \approx P_{atm}$

$$\eta_{mech, fan} = \frac{\Delta \dot{E}_{mech, fluid}}{\dot{W}_{shaft, in}} = \frac{\dot{m} V_2^2/2}{\dot{W}_{shaft, in}} = \frac{(0.506 \text{ kg/s})(12.1 \text{ m/s})^2/2}{50.0 \text{ W}} = 0.741$$



$$V_1 \approx 0,\ V_2 = 12.1\ \text{m/s}$$
$$z_1 = z_2$$
$$P_1 \approx P_{atm} \text{ and } P_2 \approx P_{atm}$$
$$\eta_{\text{mech, fan}} = \frac{\Delta \dot{E}_{\text{mech, fluid}}}{\dot{W}_{\text{shaft, in}}} = \frac{\dot{m}V_2^2/2}{\dot{W}_{\text{shaft, in}}} = \frac{(0.506\ \text{kg/s})(12.1\ \text{m/s})^2/2}{50.0\ \text{W}} = 0.741$$

The mechanical efficiency of a fan is the ratio of the kinetic energy of air at the fan exit to the mechanical power input.

The mechanical efficiency should not be confused with the **motor efficiency** and the **generator efficiency,** which are defined as

$$\eta_{motor} = \frac{\text{Mechanical power output}}{\text{Electric power input}} = \frac{\dot{W}_{shaft,out}}{\dot{W}_{elect,in}}$$ Motor efficiency

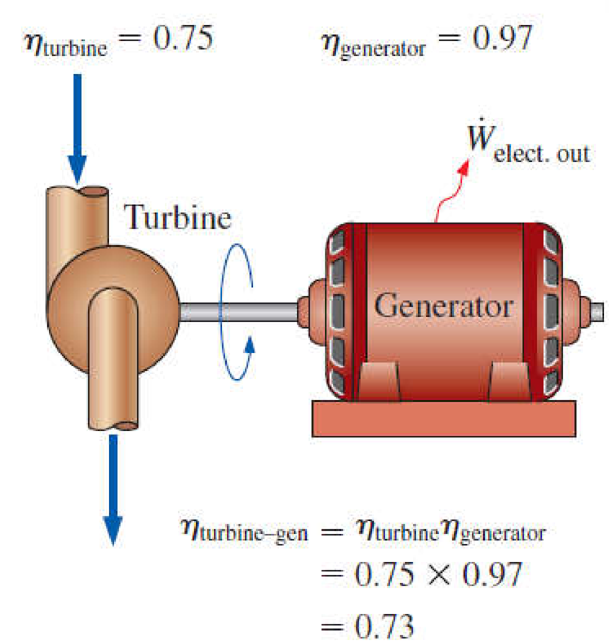
$$\eta_{generator} = \frac{\text{Electric power output}}{\text{Mechanical power input}} = \frac{\dot{W}_{elect,out}}{\dot{W}_{shaft,in}}$$ Generator efficiency

$$\eta_{pump-motor} = \eta_{pump}\eta_{motor} = \frac{\dot{W}_{pump,u}}{\dot{W}_{elect,in}} = \frac{\Delta\dot{E}_{mech,fluid}}{\dot{W}_{elect,in}}$$ Pump-Motor overall efficiency

Turbine-Generator overall efficiency:

$$\eta_{turbine-gen} = \eta_{turbine}\eta_{generator} = \frac{\dot{W}_{elect,out}}{\dot{W}_{turbine,e}} = \frac{\dot{W}_{elect,out}}{|\Delta\dot{E}_{mech,fluid}|}$$

The overall efficiency of a turbine–generator is the product of the efficiency of the turbine and the efficiency of the generator, and represents the fraction of the mechanical energy of the fluid converted to electric energy.



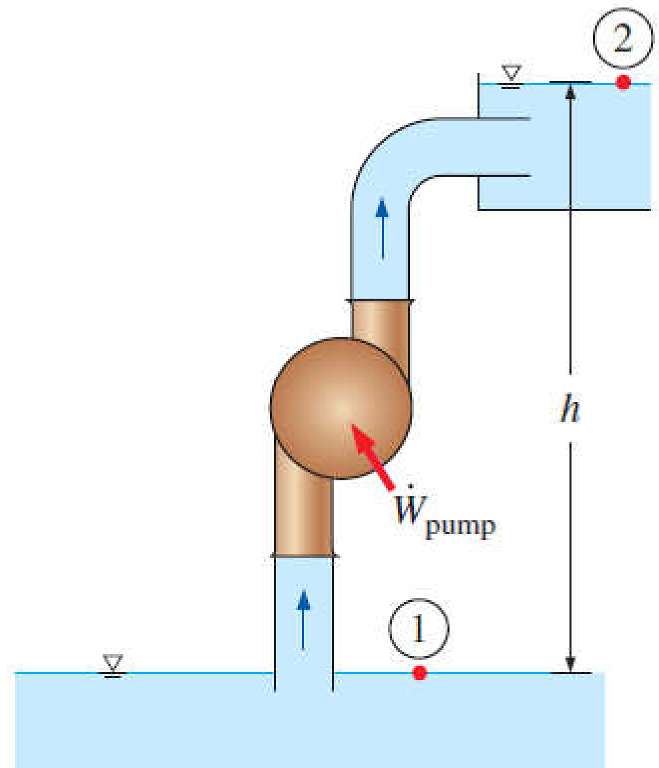
The efficiencies defined range between 0 and 100%.

For systems that involve only *mechanical forms of energy* and its transfer as *shaft work*, **the conservation of energy is**

$$E_{\text{mech, in}} - E_{\text{mech, out}} = \Delta E_{\text{mech, system}} + E_{\text{mech, loss}}$$

***E*mech, loss** : The conversion of mechanical energy to thermal energy due to irreversibilities such as friction.

Many fluid flow problems involve mechanical forms of energy only, and such problems are conveniently solved by using a *mechanical energy* balance.



Steady flow

$$V_1 = V_2 \approx 0$$
$$z_2 = z_1 + h$$
$$P_1 = P_2 = P_{\text{atm}}$$

$$\dot{E}_{\text{mech, in}} = \dot{E}_{\text{mech, out}} + \dot{E}_{\text{mech, loss}}$$
$$\dot{W}_{\text{pump}} + \dot{m}gz_1 = \dot{m}gz_2 + \dot{E}_{\text{mech, loss}}$$
$$\dot{W}_{\text{pump}} = \dot{m}gh + \dot{E}_{\text{mech, loss}}$$

EXAMPLE 5–3 Performance of a Hydraulic Turbine–Generator

The water in a large lake is to be used to generate electricity by the installation of a hydraulic turbine–generator. The elevation difference between the free surfaces upstream and downstream of the dam is 50 m (Fig. 5–19). Water is to be supplied at a rate of 5000 kg/s. If the electric power generated is measured to be 1862 kW and the generator efficiency is 95 percent, determine (*a*) the overall efficiency of the turbine–generator, (*b*) the mechanical efficiency of the turbine, and (*c*) the shaft power supplied by the turbine to the generator.

SOLUTION A hydraulic turbine–generator is to generate electricity from the water of a lake. The overall efficiency, the turbine efficiency, and the shaft power are to be determined.

Assumptions **1** The elevation of the lake and that of the discharge site remain constant. **2** Irreversible losses in the pipes are negligible.

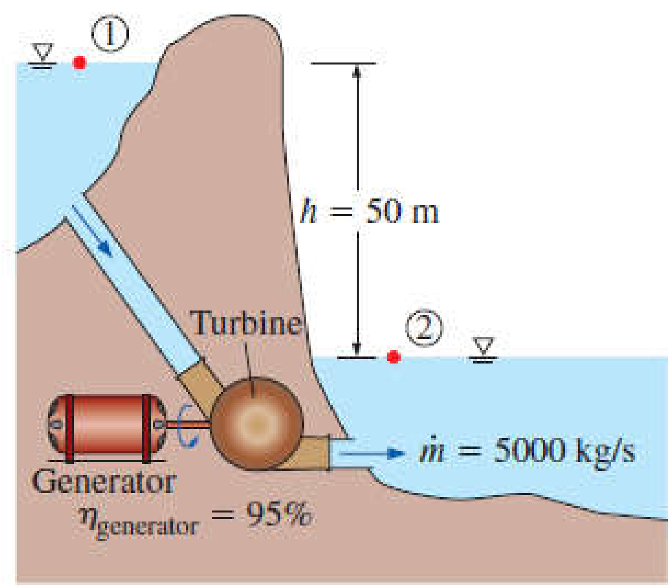
Properties The density of water is taken to be $\rho = 1000\ \text{kg/m}^3$.

Analysis (*a*) We perform our analysis from inlet (1) at the free surface of the lake to outlet (2) at the free surface of the downstream discharge site. At both free surfaces the pressure is atmospheric and the velocity is negligibly small. The change in the water's mechanical energy per unit mass is then

$$e_{\text{mech, in}} - e_{\text{mech, out}} = \underbrace{\frac{P_{\text{in}} - \cancel{P_{\text{out}}}}{\rho}}_{0} + \underbrace{\frac{V_{\text{in}}^2 - \cancel{V_{\text{out}}^2}}{2}}_{0} + g(z_{\text{in}} - z_{\text{out}})$$

$$= gh$$

$$= (9.81\ \text{m/s}^2)(50\ \text{m})\left(\frac{1\ \text{kJ/kg}}{1000\ \text{m}^2/\text{s}^2}\right) = 0.491\frac{\text{kJ}}{\text{kg}}$$



$$\text{ke} = m^2/s^2 \, x \, \frac{1J/kg}{1m^2/s^2} = (\,)\text{J/kg}$$

Then the rate at which mechanical energy is supplied to the turbine by the fluid and the overall efficiency become

$$|\Delta \dot{E}_{\text{mech, fluid}}| = \dot{m}(e_{\text{mech, in}} - e_{\text{mech, out}}) = (5000 \text{ kg/s})(0.491 \text{ kJ/kg}) = 2455 \text{ kW}$$

$$\eta_{\text{overall}} = \eta_{\text{turbine-gen}} = \frac{\dot{W}_{\text{elect, out}}}{|\Delta \dot{E}_{\text{mech, fluid}}|} = \frac{1862 \text{ kW}}{2455 \text{ kW}} = \mathbf{0.760}$$

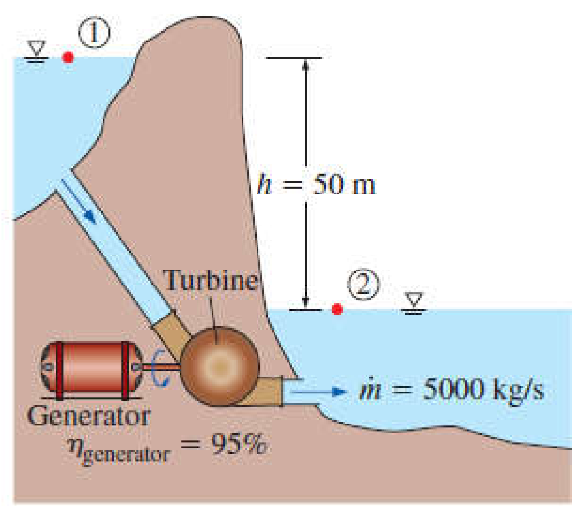
(*b*) Knowing the overall and generator efficiencies, the mechanical efficiency of the turbine is determined from

$$\eta_{\text{turbine-gen}} = \eta_{\text{turbine}}\,\eta_{\text{generator}} \rightarrow \eta_{\text{turbine}} = \frac{\eta_{\text{turbine-gen}}}{\eta_{\text{generator}}} = \frac{0.76}{0.95} = \mathbf{0.800}$$

(*c*) The shaft power output is determined from the definition of mechanical efficiency,

$$\dot{W}_{\text{shaft, out}} = \eta_{\text{turbine}}|\Delta \dot{E}_{\text{mech, fluid}}| = (0.800)(2455 \text{ kW}) = 1964 \text{ kW} \approx \mathbf{1960 \text{ kW}}$$

Discussion Note that the lake supplies 2455 kW of mechanical power to the turbine, which converts 1964 kW of it to shaft power that drives the generator, which generates 1862 kW of electric power. There are irreversible losses through each component. Irreversible losses in the pipes are ignored here; you will learn how to account for these in Chap. 8.

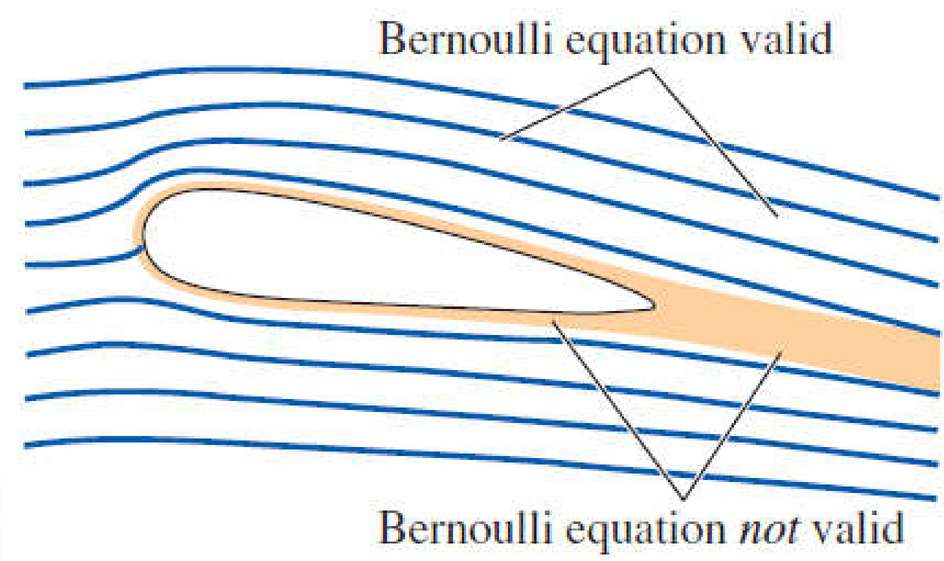


THE BERNOULLI EQUATION

Bernoulli equation: An approximate relation between pressure, velocity, and elevation, and is valid in regions of steady, incompressible flow where net frictional forces are negligible.

Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics.

The Bernoulli approximation is typically useful in flow regions outside of boundary layers and wakes, where the fluid motion is governed by the combined effects of pressure and gravity forces.

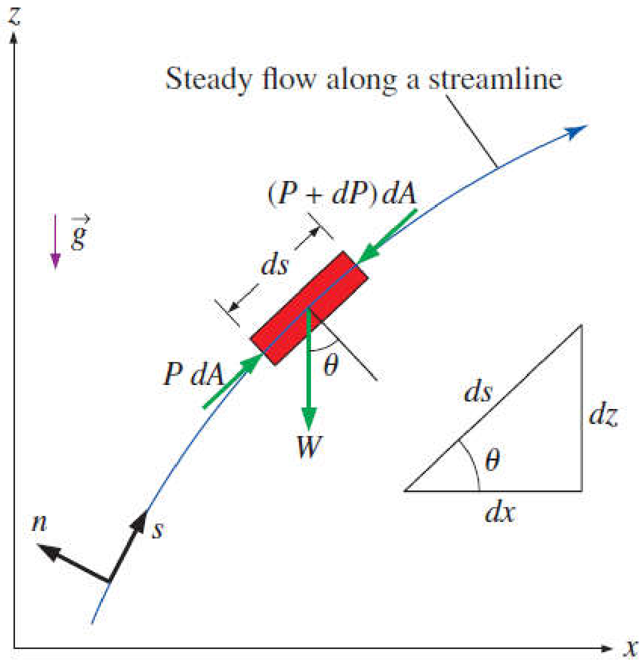


The *Bernoulli equation* is an *approximate* equation that is valid only in *inviscid regions of flow* where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of *boundary layers* and *wakes*.

Derivation of the Bernoulli Equation

$\boxed{V = ds/dt}$

Acceleration in steady flow is due to the change of velocity with position.



The forces acting on a fluid particle along a streamline.

$$\sum F_s = ma_s \qquad P\,dA - (P + dP)\,dA - W\sin\theta = mV\frac{dV}{ds}$$

$$m = \rho V = \rho\,dA\,ds \qquad W = mg = \rho g\,dA\,ds$$

$$\sin\theta = dz/ds. \qquad -dP\,dA - \rho g\,dA\,ds\frac{dz}{ds} = \rho\,dA\,ds\,V\frac{dV}{ds}$$

$$-dP - \rho g\,dz = \rho V\,dV \qquad V\,dV = \tfrac{1}{2}\,d(V^2)$$

$$\frac{dP}{\rho} + \tfrac{1}{2}\,d(V^2) + g\,dz = 0$$

***Steady flow*:**

$$\int\frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}$$

***Steady, incompressible flow*:** **Bernoulli equation**

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}$$

The sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow when compressibility and frictional effects are negligible.

The Bernoulli equation between any two points on the same streamline:

$$\boxed{\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2}$$

(Steady flow along a streamline)

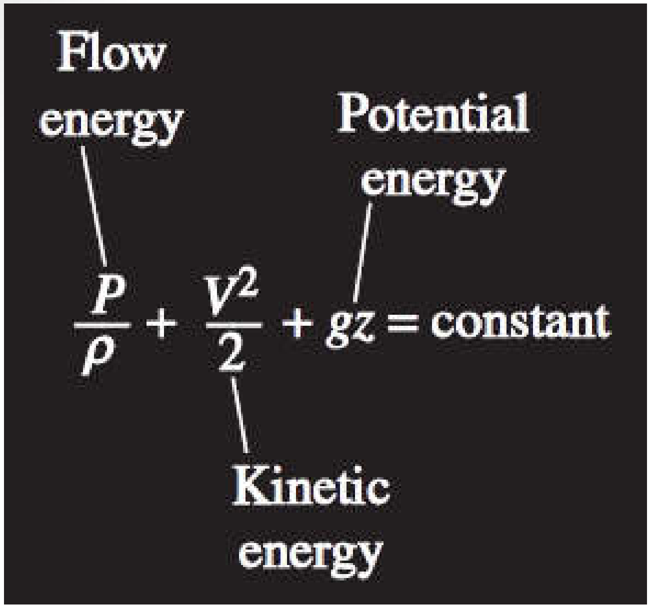
General:

$$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Incompressible flow ($\rho = \text{constant}$):

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

The incompressible Bernoulli equation is derived assuming incompressible flow, and thus it should not be used for flows with significant compressibility effects.



The Bernoulli equation states that the sum of the kinetic, potential, and flow energies of a fluid particle is constant along a streamline during steady flow.

- The Bernoulli equation can be viewed as the "*conservation of mechanical energy principle*."
- This is equivalent to the general conservation of energy principle for systems that do not involve any conversion of mechanical energy and thermal energy to each other, and thus the mechanical energy and thermal energy are conserved separately.
- The Bernoulli equation states that during steady, incompressible flow with negligible friction, the various forms of mechanical energy are converted to each other, but their sum remains constant.
- There is no dissipation of mechanical energy during such flows since there is no friction that converts mechanical energy to sensible thermal (internal) energy.
- The Bernoulli equation is commonly used in practice since a variety of practical fluid flow problems can be analyzed to reasonable accuracy with it.

Unsteady, Compressible Flow

The Bernoulli equation for *unsteady, compressible flow:*

Unsteady, compressible flow: $$\int \frac{dP}{\rho} + \int \frac{\partial V}{\partial t}\,ds + \frac{V^2}{2} + gz = \text{constant}$$

Static, Dynamic, and Stagnation Pressures

The kinetic and potential energies of the fluid can be converted to flow energy (and vice versa) during flow, causing the pressure to change. Multiplying the Bernoulli equation by the density gives

$$P + \rho \frac{V^2}{2} + \rho g z = \text{constant (along a streamline)}$$

***P* is the static pressure:** It does not incorporate any dynamic effects; it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.

$\rho V^2/2$ is the dynamic pressure: It represents the pressure rise when the fluid in motion is brought to a stop isentropically.

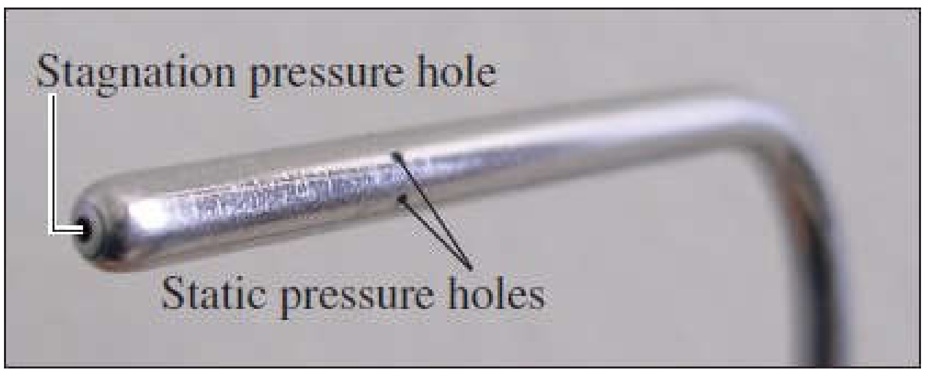
ρgz is the hydrostatic pressure: It is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects, i.e., fluid weight on pressure. (Be careful of the sign—unlike hydrostatic pressure ρgh which *increases* with fluid depth h, the hydrostatic pressure term ρgz *decreases* with fluid depth.)

Total pressure: The sum of the static, dynamic, and hydrostatic pressures. Therefore, the Bernoulli equation states that *the total pressure along a streamline is constant*.

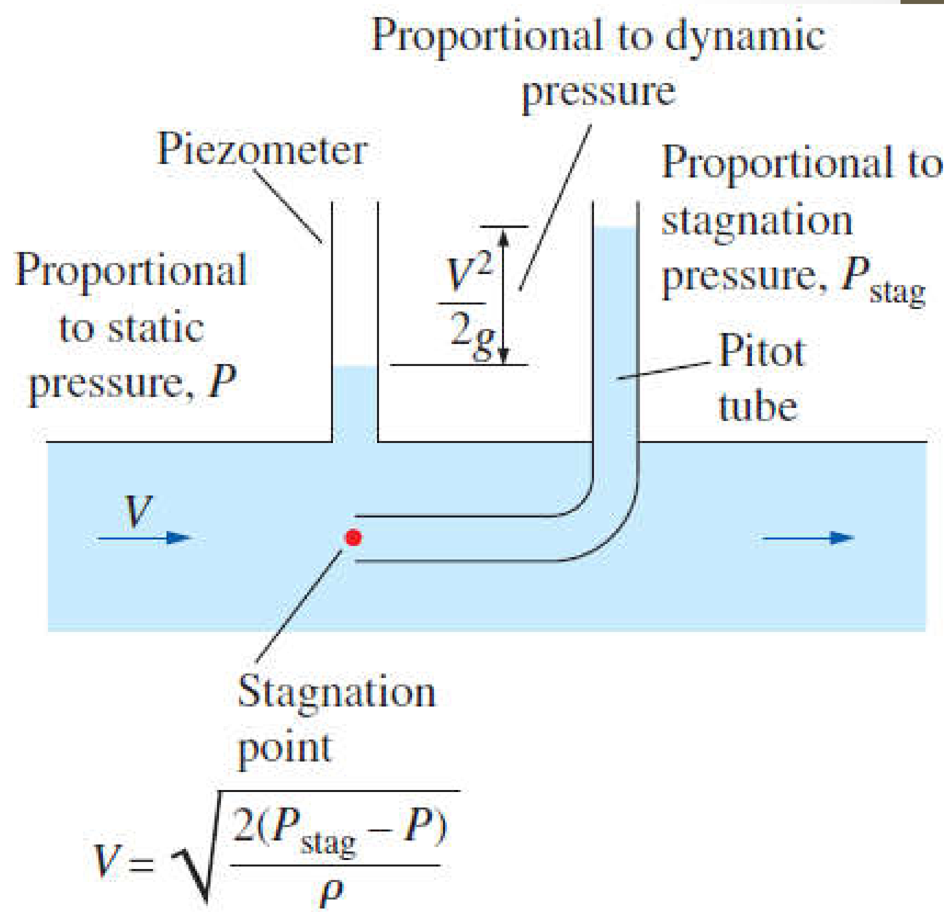
Stagnation pressure: The sum of the static and dynamic pressures. It represents the pressure at a point where the fluid is brought to a complete stop isentropically.

$$P_{stag} = P + \rho \frac{V^2}{2} \qquad \text{(kPa)}$$

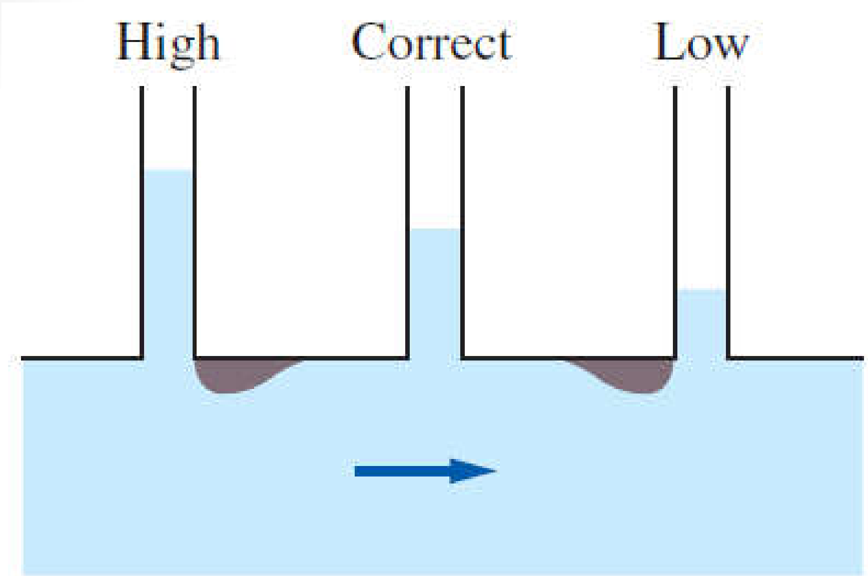
$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$



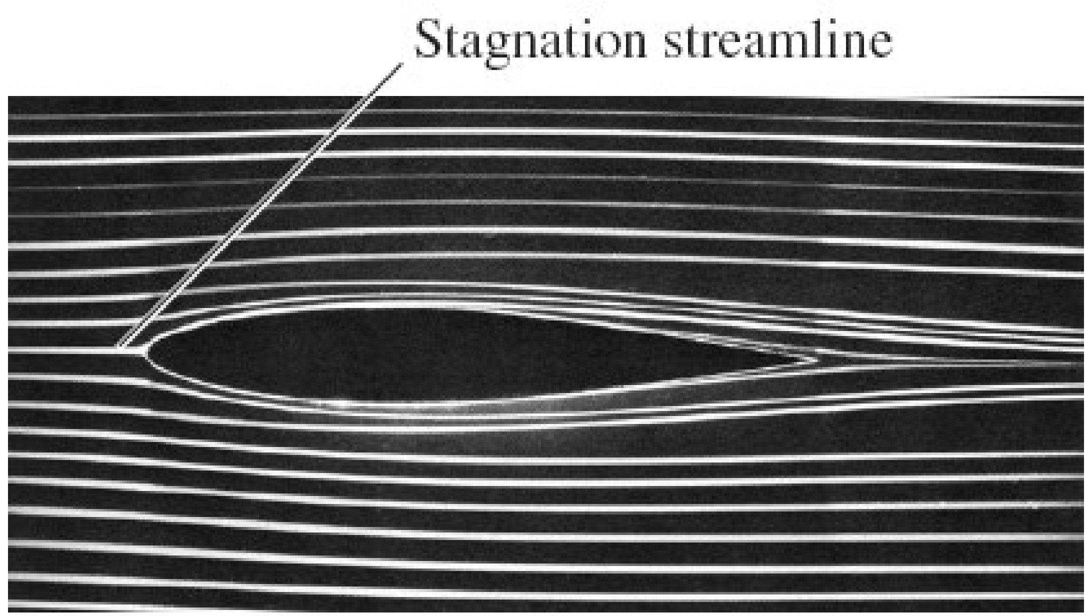
Close-up of a **Pitot-static probe**, showing the stagnation pressure hole and two of the five static circumferential pressure holes.



The static, dynamic, and stagnation pressures measured using **piezometer tubes**.



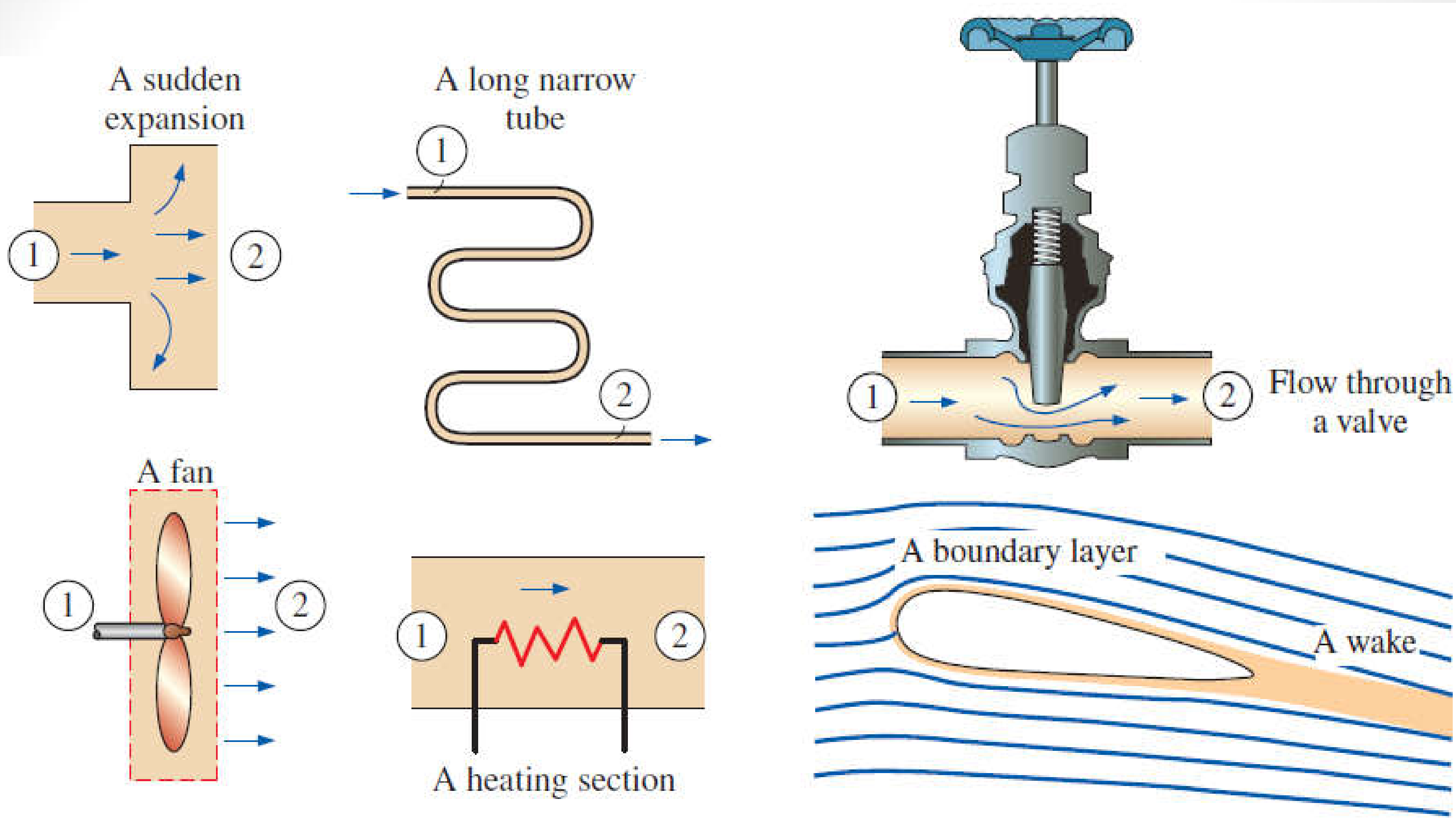
Careless drilling of the static pressure tap may result in an erroneous reading of the static pressure head.



Streaklines produced by colored fluid introduced upstream of an airfoil; since the flow is steady, the streaklines are the same as streamlines and pathlines. The **stagnation streamline** is marked.

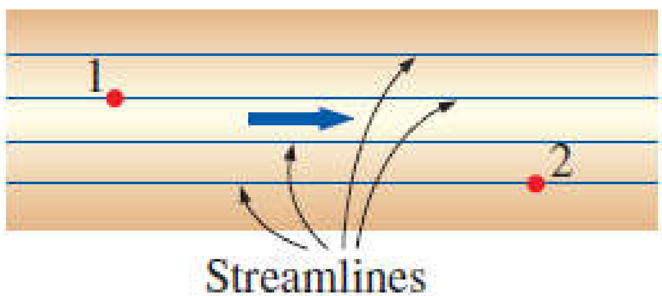
Limitations on the Use of the Bernoulli Equation

1. **Steady flow** The Bernoulli equation is applicable to *steady flow*.
2. **Frictionless flow** Every flow involves some friction, no matter how small, and *frictional effects* may or may not be negligible.
3. **No shaft work** The Bernoulli equation is not applicable in a flow section that involves a pump, turbine, fan, or any other machine or impeller since such devices destroy the streamlines and carry out energy interactions with the fluid particles. When these devices exist, the energy equation should be used instead.
4. **Incompressible flow** Density is taken constant in the derivation of the Bernoulli equation. The flow is incompressible for liquids and also by gases at Mach numbers less than about 0.3.
5. **No heat transfer** The density of a gas is inversely proportional to temperature, and thus the Bernoulli equation should not be used for flow sections that involve significant temperature change such as heating or cooling sections.
6. **Flow along a streamline** Strictly speaking, the Bernoulli equation is applicable along a streamline. However, when a region of the flow is *irrotational* and there is negligibly small *vorticity* in the flow field, the Bernoulli equation becomes applicable *across* streamlines as well.

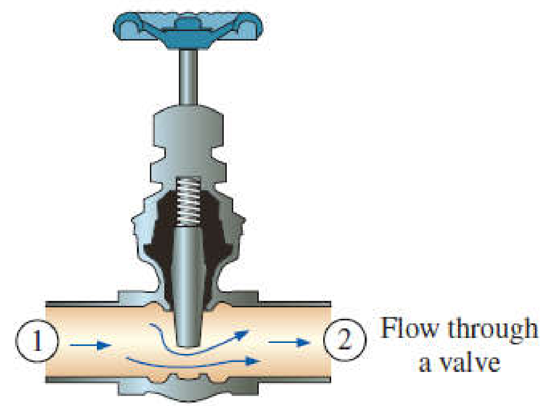


Frictional effects, heat transfer, and components that disturb the streamlined structure of flow make the Bernoulli equation invalid. It should *not* be used in any of the flows shown here.

Flow along a streamline Strictly speaking, the Bernoulli equation is applicable along a streamline. However, when a region of the flow is *irrotational* and there is negligibly small *vorticity* in the flow field, the Bernoulli equation becomes applicable *across* streamlines as well.



$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$



When the flow is irrotational, the Bernoulli equation becomes applicable between any two points along the flow (not just on the same streamline).

Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)

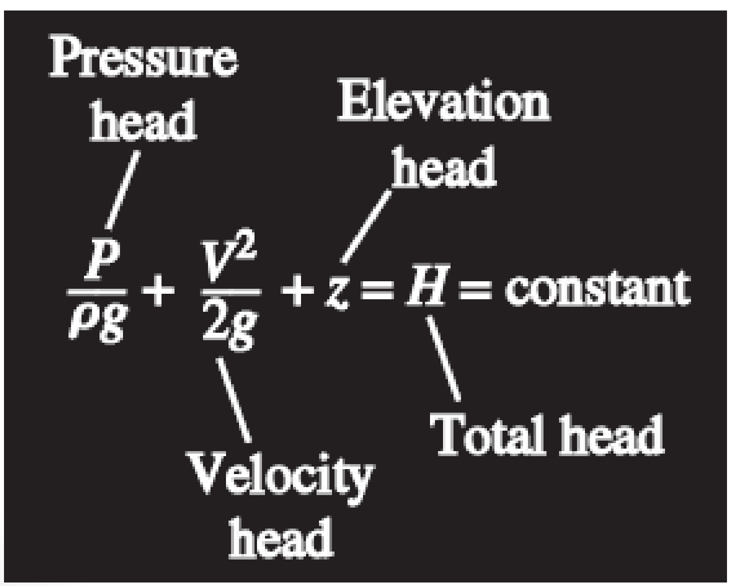
It is often convenient to represent the level of mechanical energy graphically using *heights* to facilitate visualization of the various terms of the Bernoulli equation. Dividing each term of the Bernoulli equation by *g* gives

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant} \qquad \text{(along a streamline)}$$

$P/\rho g$ is the pressure head; it represents the height of a fluid column that produces the static pressure *P*.

$V^2/2g$ is the velocity head; it represents the elevation needed for a fluid to reach the velocity *V* during frictionless free fall.

z is the elevation head; it represents the potential energy of the fluid.

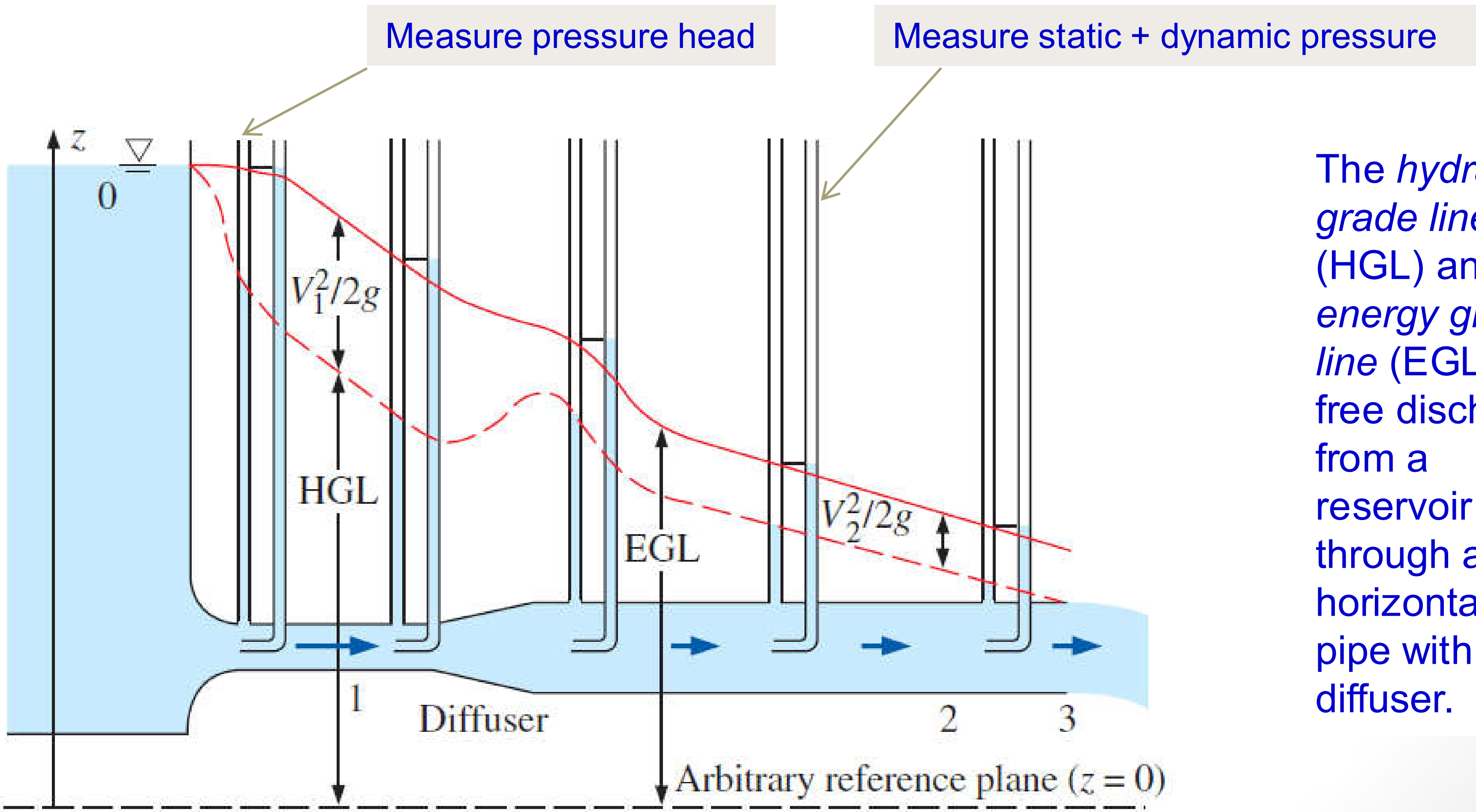


An alternative form of the Bernoulli equation is expressed in terms of heads as: ***The sum of the pressure, velocity, and elevation heads is constant along a streamline.***

Hydraulic grade line (HGL), $P/\rho g + z$ The line that represents the sum of the **static pressure** and the **elevation heads**.

Energy grade line (EGL), $P/\rho g + V^2/2g + z$ The line that represents **the total head of the fluid**.

Dynamic head, $V^2/2g$ The difference between the heights of EGL and HGL.



The *hydraulic grade line* (HGL) and the *energy grade line* (EGL) for free discharge from a reservoir through a horizontal pipe with a diffuser.

Notes on HGL and EGL

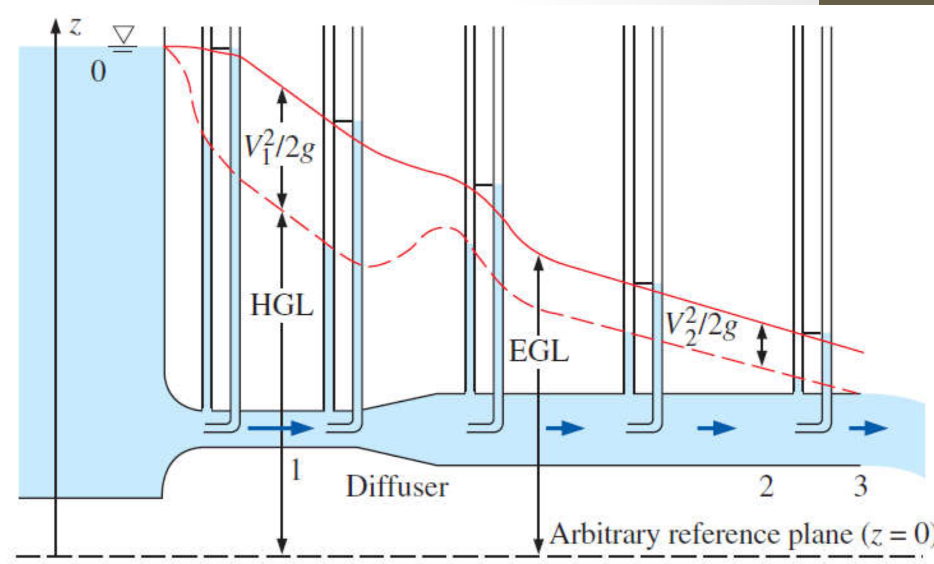
- For *stationary bodies* such as reservoirs or lakes, the EGL and HGL coincide with the free surface of the liquid.

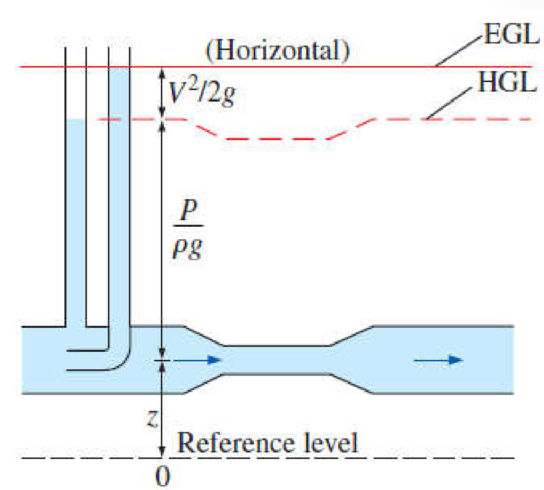
- The EGL is always a distance $V^2/2g$ above the HGL. These two curves approach each other as the velocity decreases, and they diverge as the velocity increases.

- At a *pipe exit*, the pressure head is zero (atmospheric pressure) and thus the HGL coincides with the pipe outlet.

- In an *idealized Bernoulli-type flow*, EGL is horizontal and its height remains constant.

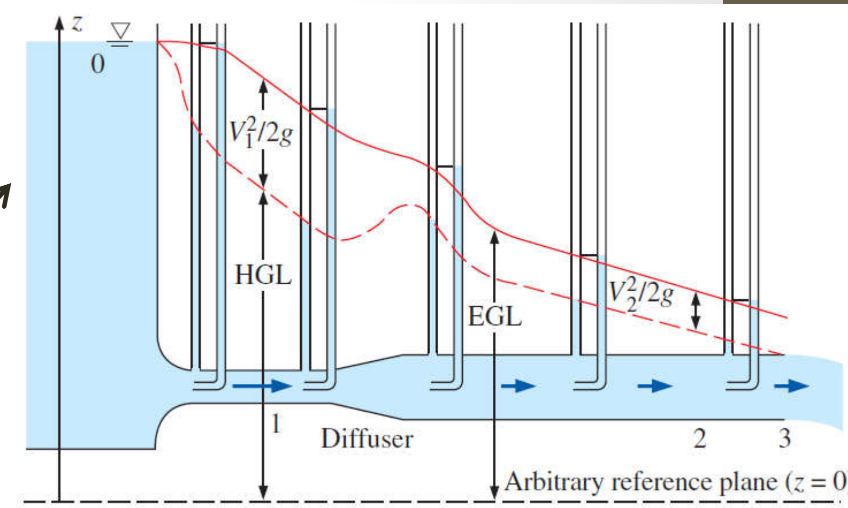
- For *open-channel flow*, the HGL coincides with the free surface of the liquid, and the EGL is a distance $V^2/2g$ above the free surface.



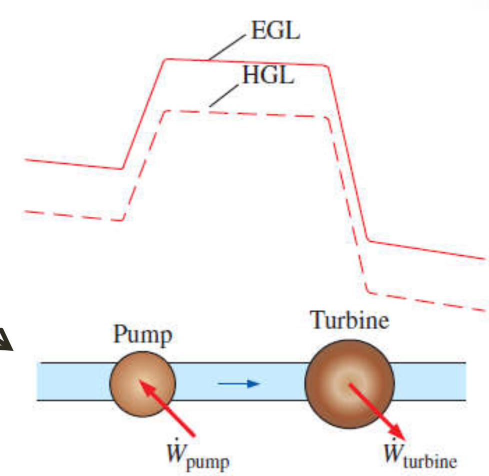


Notes on HGL and EGL

- The *mechanical energy loss* due to frictional effects (conversion to thermal energy) causes the EGL and HGL to slope downward in the direction of flow. The slope is a measure of the head loss in the pipe.



- A *steep jump/drop* occurs in EGL and HGL **whenever mechanical energy is added or removed** to or from the fluid (pump, turbine).



Example

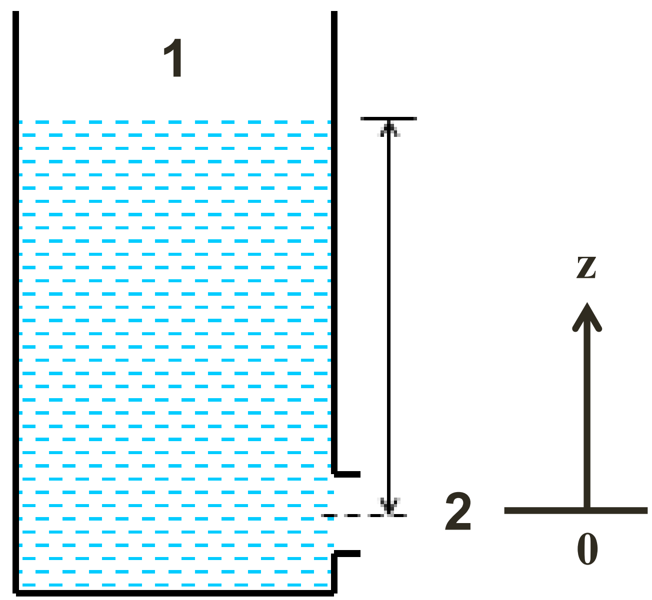
A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap. A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

Solution:

- **Assumption:**
  - **No work interaction.**
  - **The rounded outlet is frictionless**
  - **Tank is very large that the velocity of water in tank is relatively small, $u_1 \cong 0$**
  - **Steady flow**

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$z_1 = 5$ m, $P_1 = 0$ kPa, $u_1 \cong 0$



$z_2 = 0$ m , $P_2 = 0$ kPa, $u_2 = ?$

Solution:

$$\boxed{\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2}$$

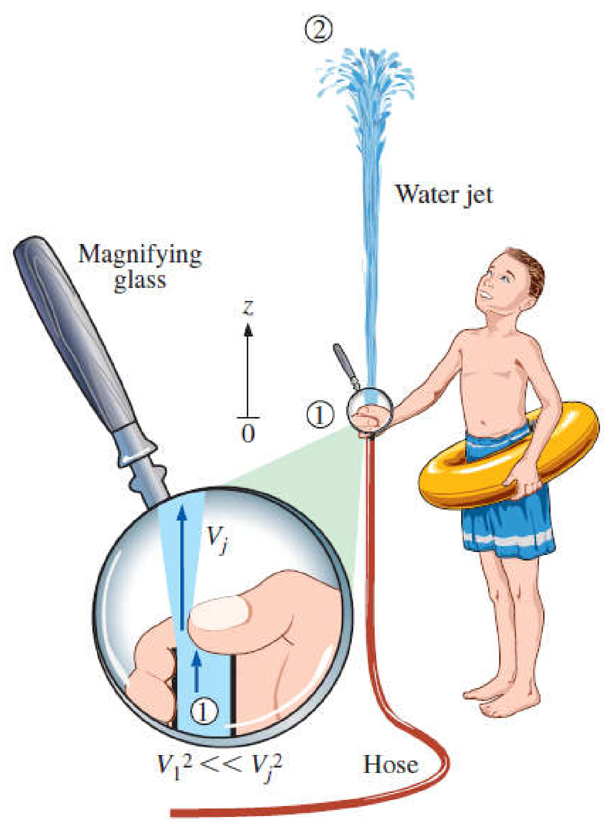
$$\frac{\Delta P}{\rho} + g\Delta z + \frac{\Delta \mathrm{v}^2}{2} = 0$$

$$g(z_2 - z_1) + \frac{\mathrm{v}_2{}^2 - \mathrm{v}_1{}^2}{2} = 0$$

$$\begin{aligned} \mathrm{v}_2 &= \sqrt{2g(z_1 - z_2)} \\ &= \sqrt{2(9.81)(5-0)} \\ &= 9.9\,\frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

Example

Water is flowing from a hose attached to a water main at 400 kPa gage. A child places his thumb to cover most of the hose outlet, causing a thin jet of high-speed water to emerge. If the hose is held upward, what is the maximum height that the jet could achieve?



Solution:

$$\boxed{\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2}$$

$$\frac{P_1}{\rho g} + \overbrace{\frac{V_1^2}{2g}}^{\approx 0} + \overbrace{z_1}^{0} = \frac{P_2}{\rho g} + \overbrace{\frac{V_2^2}{2g}}^{0} + z_2 \quad \rightarrow \quad \frac{P_1}{\rho g} = \frac{P_{\text{atm}}}{\rho g} + z_2$$

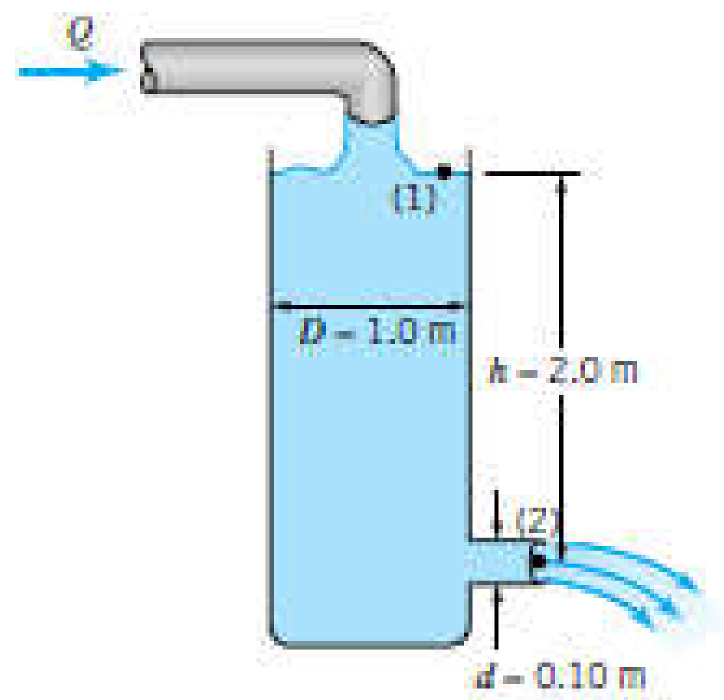
$$z_2 = \frac{P_1 - P_{atm}}{\rho g} = \frac{P_{1,gage}}{\rho g} = \frac{400x1000}{(1000)x(9.81)} = 40.8m$$

Example

A stream of water of diameter d = 0.1 m flows steadily from a tank of diameter D = 1 m.
Determine:

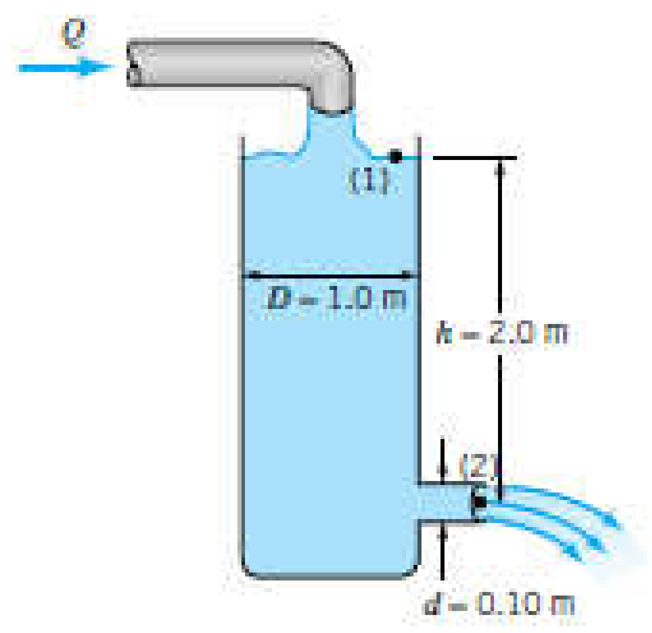
- The flowrate, Q, needed from the inflow pipe if the water depth remains constant, h = 2.0 m.

Assume that the pressure at Point (1 and 2) are equal.



Solution

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$



$$p_1 = p_2 = 0,\ z_1 = h,\ \text{and}\ z_2 = 0,$$

$$\boxed{\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2} \longrightarrow \text{Eq. (1)}$$

From the conservation of mass equation $Q_1 = Q_2$, Thus

$$A_1V_1 = A_2V_2 \longrightarrow \frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

$$\boxed{V_1 = \left(\frac{d}{D}\right)^2 V_2} \longrightarrow \text{Eq. (2)}$$

Equation (1) and (2) can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}} = \sqrt{\frac{2(9.81\ \text{m/s}^2)(2.0\ \text{m})}{1-(0.1\text{m}/1\text{m})^4}} = 6.26\ \text{m/s}$$

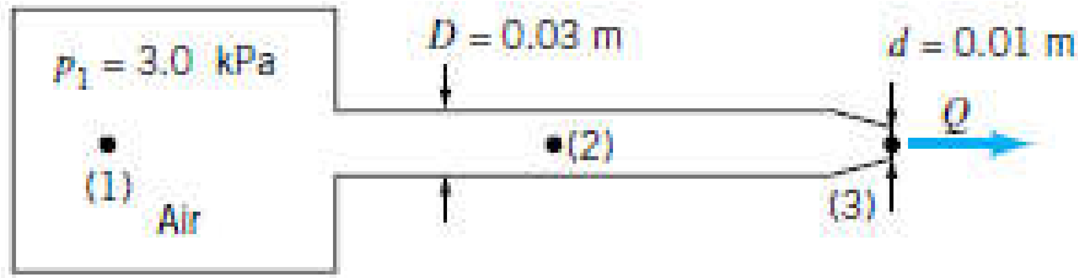
$$\boxed{Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.1\ \text{m})^2(6.26\ \text{m/s}) = 0.0492\ \text{m}^3/\text{s}}$$

Example

Air at 15°C flows steadily from a tank, through a hose of diameter D = 0.03 m and exits to the atmosphere from a nozzle of diameter d = 0.01m. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure. Determine:

- The flowrate
- The pressure in the hose.

Assume the air as a perfect gas

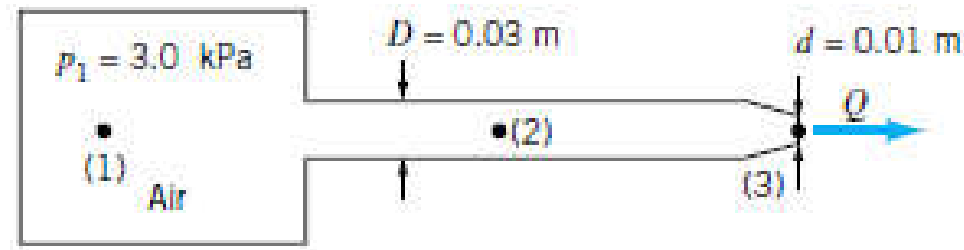


Given: $P_1 = 3$ kPa, $V_1 = 0$ (Large tank), $z_1 = z_2 = z_3$ (horizontal hose), $P_3 = 0$ (free jet).

Unknown: P_2, V_2, V_3

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

Solution



For steady, inviscid, and incompressible, applying Bernoulli equation along the streamline

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

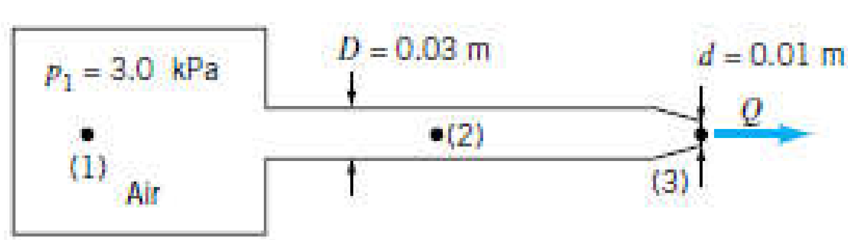
Applying Bernoulli equation between (1 and 2) → $p_2 = p_1 - \tfrac{1}{2}\rho V_2^2$

Applying Bernoulli equation between (1 and 3) → $V_3 = \sqrt{\dfrac{2p_1}{\rho}}$

Assume the air as a perfect gas

$$\rho = \frac{p_1}{RT_1}$$
$$= [(3.0 + 101)\ \text{kN/m}^2] \times \frac{10^3\ \text{N/kN}}{(286.9\ \text{N}\cdot\text{m/kg}\cdot\text{K})(15 + 273)\text{K}}$$
$$= 1.26\ \text{kg/m}^3$$

Solution



Applying Bernoulli equation between (1 and 3)

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

$$Q = A_3 V_3 = \frac{\pi}{4} d^2 V_3 = \frac{\pi}{4}(0.01\ \text{m})^2 (69.0\ \text{m/s})$$

$$= 0.00542\ \text{m}^3/\text{s}$$

Applying Bernoulli equation between (1 and 2)

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2$$

Applying continuity equation to find V_2

$$A_2 V_2 = A_3 V_3$$

$$V_2 = A_3 V_3 / A_2 = \left(\frac{d}{D}\right)^2 V_3 = \left(\frac{0.01\ \text{m}}{0.03\ \text{m}}\right)^2 (69.0\ \text{m/s}) = 7.67\ \text{m/s}$$

$$p_2 = 3.0 \times 10^3\ \text{N/m}^2 - \tfrac{1}{2}(1.26\ \text{kg/m}^3)(7.67\ \text{m/s})^2$$

$$= (3000 - 37.1)\text{N/m}^2 = 2963\ \text{N/m}^2$$

GENERAL ENERGY EQUATION

The energy content of a fixed quantity of mass can be changed by two mechanisms:

- *heat transfer Q*
- *work transfer W*.

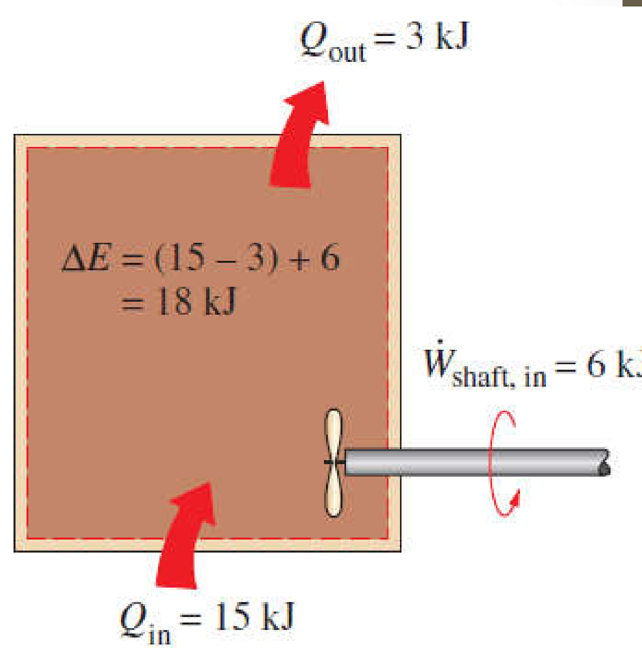
Then the conservation of energy for a fixed quantity of mass can be expressed in rate form as

$$E_{in} - E_{out} = \Delta E.$$

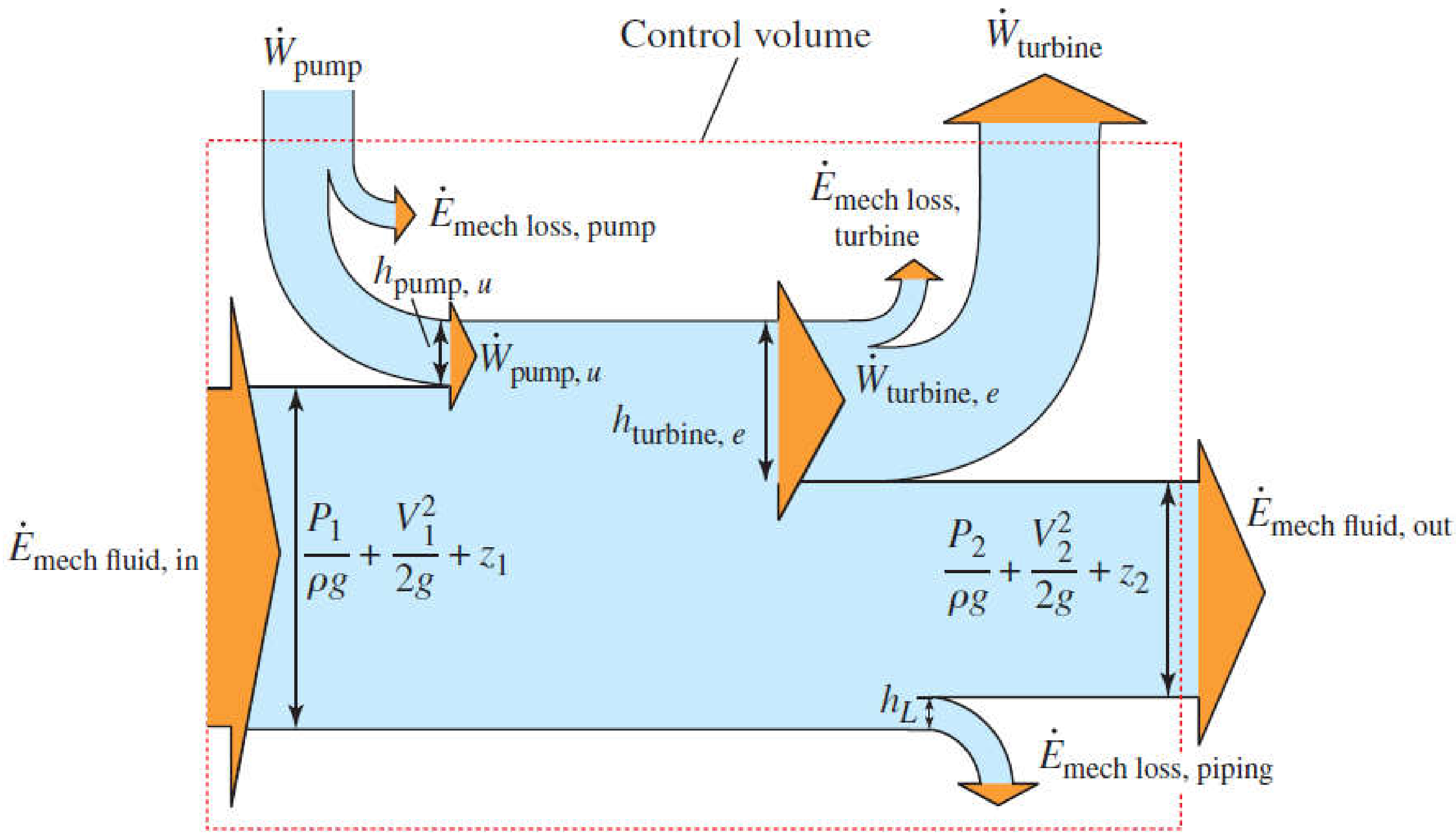
$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{net in}} = \frac{dE_{sys}}{dt} \qquad \dot{Q}_{\text{net in}} + \dot{W}_{\text{net in}} = \frac{d}{dt}\int_{sys} \rho e \, dV$$

$$\dot{Q}_{\text{net in}} = \dot{Q}_{in} - \dot{Q}_{out} \qquad \dot{W}_{\text{net in}} = \dot{W}_{in} - \dot{W}_{out}$$

$$e = u + \text{ke} + \text{pe} = u + \frac{V^2}{2} + gz$$



$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$



Mechanical energy flow chart for a fluid flow system that involves a pump and a turbine. Vertical dimensions show each energy term expressed as an equivalent column height of fluid, i.e., *head*.

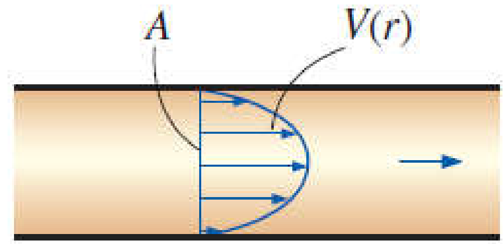
$$\dot{m}\left(\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right) + \dot{W}_{pump} = \dot{m}\left(\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) + \dot{W}_{turbine} + \dot{E}_{mech,\ loss}$$

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{pump,\ u} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{turbine,\ e} + h_L$$

Kinetic Energy Correction Factor, α

The kinetic energy of a fluid stream obtained from $V^2/2$ is not the same as the actual kinetic energy of the fluid stream since the square of a sum is not equal to the sum of the squares of its components.

This error can be corrected by replacing the kinetic energy terms $V^2/2$ in the energy equation by $\alpha V_{avg}^2/2$, where α is the **kinetic energy correction factor**.



$$\dot{m} = \rho V_{avg} A, \qquad \rho = \text{constant}$$

$$\dot{KE}_{act} = \int ke\,\delta\dot{m} = \int_A \frac{1}{2}[V(r)]^2 [\rho V(r)\,dA]$$

$$= \frac{1}{2}\rho \int_A [V(r)]^3\,dA$$

$$\dot{KE}_{avg} = \frac{1}{2}\dot{m}V_{avg}^2 = \frac{1}{2}\rho A V_{avg}^3$$

$$\alpha = \frac{\dot{KE}_{act}}{\dot{KE}_{avg}} = \frac{1}{A}\int_A \left(\frac{V(r)}{V_{avg}}\right)^3 dA$$

The correction factor is 2.0 for fully developed laminar pipe flow, and it ranges between 1.04 and 1.11 for fully developed turbulent flow in a round pipe.

The determination of the *kinetic energy correction factor* using the actual velocity distribution $V(r)$ and the average velocity V_{avg} at a cross section.

$$\dot{m}\left(\frac{P_1}{\rho} + \alpha_1 \frac{V_1^2}{2} + gz_1\right) + \dot{W}_{pump} = \dot{m}\left(\frac{P_2}{\rho} + \alpha_2 \frac{V_2^2}{2} + gz_2\right) + \dot{W}_{turbine} + \dot{E}_{mech,\,loss}$$

$$\frac{P_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_{pump,\,u} = \frac{P_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_{turbine,\,e} + h_L \qquad (5\text{–}77)$$

Kinetic Energy Correction Factor, α

The **kinetic energy correction factor** are often ignored for fully developed flow.(α =1)

- most flows are turbulent (almost = unity)
- Kinetic energy term are often small to the other terms

For laminar flow it is significant.