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Chapter One

Introduction to Operation Research

1.1. Origin of Operation Research (OR)

The term "Operations Research" (OR) was first coined by MC Closky and Trefthen in 1940 in a small town, Bowdsey of UK. The main origin of OR was during the second world war – The military commands of UK and USA engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals, to arrive at the decision on optimal utilization of scarce military resources, and to implement the decisions effectively. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it had gained popularity and was called "An art of winning the war without actually fighting it"

The name Operations Research (OR) was invented because the team was dealing with research on military operations. Now OR activities has become universally applicable to any area such as transportation, hospital management, agriculture, libraries, city planning, financial institutions, construction management and so forth.

Given the importance indicated for the methods of operations research in resolving many the problems facing business facilities. The engineer industrial must have sufficient knowledge. In these methods and how to apply them and benefit from them in the analysis and interpretation of accounting data or financial and provide them to management in a way that helps them in rationalizing the various administrative decisions.

1.2. Concept and definition of OR

Operations research is a branch of mathematics that deals with the application of scientific methods and techniques to obtain optimal solutions for decision-making problems. With the growth of the global market and thus the competition, operations research has gained much importance. In this era of competitiveness, every business aims at providing good value products and services to its customers. **Good value results from a fine combination of low cost, high quality, rapid availability, and real-time information about goods and services.** Operations research with its techniques provides an optimal solution to achieve business targets. The purpose is to provide the management with explicit quantitative understanding and assessment of complex situations to have sound basics for arriving at best decisions.

Definition: OR is a scientific methodology (analytical, experimental and quantitative) which by assessing the overall implications of various alternative courses of action in a management system provides an improved basis for management decisions.

1.3. Characterize & Application of OR

In recent years, service organizations such as hospitals, airlines, banks, railways, and libraries have started recognizing the usefulness of operations research for improving efficiency. The importance of applications of operations research in solving various problems in the engineering field is now accepted universally. **Operations research techniques are used in solving problems in the following fields**:

1. Optimal inventory cost.
2. Optimal designing of a control system.
3. Optimal planning, scheduling, and controlling.
4. Optimal designing of chemical processing equipment and plants 4.
5. Optimal selection of a new site for an industry.
6. Optimal designing of pipeline networks for process industries.
7. Optimal designing of computer structure at minimum cost.
8. Optimal designing of plastic structures.
9. Optimal planning to obtain maximum profit t in the presence of one or more competitors.
10. Optimal designing of the electrical networks.

1.4. Phases of OR

In general, various phases are involved in solving a problem through the operations research techniques. These phases are considered most important and relevant in operations research. Some of the main phases involved are as follows (refer to Fig. 1.1):

Decision phase: In this phase, real-life problems comprising the selection of an appropriate aim, values of various variables with appropriate measurement, and formulation of an appropriate model for the problem are identified.

Research phase: This phase is the longest phase and is extremely important. It involves observation and data collection, formulation of hypothesis, analysis of available information, and prediction of various results.

Planning and action phase: This phase includes the implementation of the recommendations and the results predicted in the research phase.

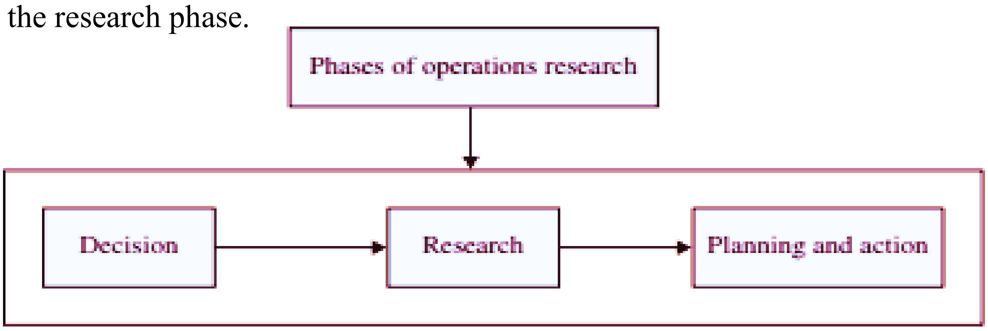

Figure 1.1 Main phases of operations research

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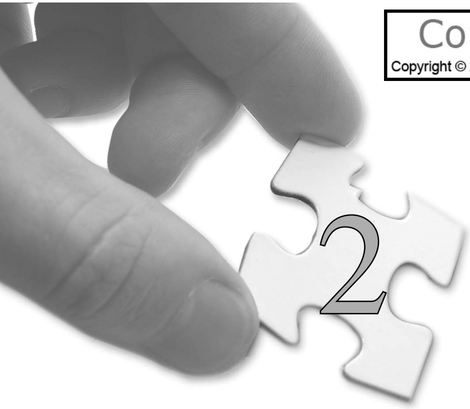

Linear Programming Problem I—Formulation

After studying this chapter, the reader will be able to

- understand the formulation of a linear programming problem
- choose the appropriate decision variables for a real-life problem
- identify the objective of the given problem based on the chosen decision variables to be optimized
- understand the restrictions to be satisfied by the decision variables
- convert a real-life problem into a mathematical form consolidating the concepts learnt from the chapter

2.1 INTRODUCTION

Linear programming is an important optimization technique used in operations research. It was developed during the Second World War (1939–1945) by a group of British scientists for efficient utilization of depleting resources. The success of experiments with this technique attracted the attention of industries that were seeking methods to minimize their total costs and maximize their total profits. However, it was only in 1947 that the efforts of G.B. Dantzig led to the invention of the simplex method of linear programming. Later, new techniques in linear programming were developed and applied to various real-life problems.

Let us consider an example of a manufacturing company that produces personal computers and laptops. It follows the steps given here for the production of these goods:

1. The production department receives an order to produce a certain number of laptops and computers.
2. The material department ensures availability of the necessary raw materials for servicing this order.
3. After the production process is completed, the production department hands over the manufactured product to the sales department.

In each of these departments, various parameters can be optimized for gaining maximum profit—availability of machines and labour hours in the production department; availability of materials and storage capacity in the materials department; and the capacity of distributions and marketing in the sales department. The solution to such real-life problems can be effectively found using mathematical modelling.

2.2 LINEAR PROGRAMMING PROBLEM

Linear programming problem (LPP) is very useful for decision-making in business and other fields of day-to-day life for obtaining the optimum values of linear expressions subject to certain restrictions that are linear in nature. It is a powerful tool for dealing with problems of allocating limited resources among the activities necessary to attain the goals. In general, the problem is to maximize the net profit or minimize the total cost, taking into consideration the various restrictions imposed on the decision variables involved in the problem.

A general LPP can be described as follows:

$$\text{Optimize (maximize or minimize) } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{2.1}$$

Subject to conditions

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le \text{ or } = \text{ or } \ge b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le \text{ or } = \text{ or } \ge b_2 \\ &\qquad . \qquad\qquad . \qquad\qquad . \\ &a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le \text{ or } = \text{ or } \ge b_m \end{aligned}\right\} \tag{2.2}$$

$$\text{and} \quad x_1, x_2, \ldots, x_n \ge 0 \tag{2.3}$$

The function Z in Eq. (2.1) is called the *objective function*. The conditions (restrictions) given in Eq. (2.2) are called *constraints* and those given in Eq. (2.3) are called *non-negative restrictions*. The variables $x_1, x_2, \ldots, x_n$ are known as *decision variables*. The coefficients $c_1, c_2, \ldots, c_n$ in objective function are known as *costs*. The coefficients a_{ij} $(i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n)$ are known as *technological coefficients* and $b_1, b_2, \ldots, b_m$ are known as *availabilities*.

The aforementioned LPP can also be written in *matrix* notation as follows:

$$\text{Optimize (maximize or minimize) } Z = (c_1, c_2, \ldots, c_n)\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$$

Subject to

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \le \text{ or} = \text{or } \ge \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$$

$$\text{and} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \geq 0 \quad \text{i.e., } (x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0)$$

or Optimize (maximize or minimize) $Z = C^{T}X$

Subject to $AX \leq$ or $=$ or $\geq b$

and $X \geq 0$

where $C = [c_1, c_2, \ldots, c_n]^{T}$, $b = [b_1, b_2, \ldots, b_m]^{T}$, $X = [x_1, x_2, \ldots, x_n]^{T}$, and $A = [a_{ij}]_{m \times n}$.

It has to be noted that C^{T} is a row matrix and both X and b are column matrices.

2.3 BASIC ASSUMPTIONS OF LINEAR PROGRAMMING PROBLEM

For an LPP, some basic assumptions are made. None of these assumptions should be violated; otherwise finding the correct solution is not possible. These assumptions are as follows:

1. The objective function of an LPP must have a linear function in decision variables.
2. The objective function must be to maximize or minimize under certain restrictions, which are again of '≤' or '≥' types.
3. The restrictions should not involve strict inequalities like '<' or '>'.
4. The decision variables must be non-negative.

2.4 FORMULATION OF LINEAR PROGRAMMING MODEL

The formulation of a linear programming model (LP model) means conversion of a descriptive problem into a mathematical model. Further, if some linear function is to be maximized or minimized under certain linear restrictions and the variables involved are required to be non-negative, then it is known as formulation of the problem in LP model. The following steps are required to formulate a problem in an LP model:

1. Identify the decision variables $x_1, x_2, \ldots, x_n$.
2. Identify the objective function and express it as a linear function of $x_1, x_2, \ldots, x_n$ and check whether it is to be maximized or minimized.
3. Identify the constraints as a linear function of $x_1, x_2, \ldots, x_n$ and express them in the form of inequalities.
4. Include proper restrictions for decision variables (i.e., x_i's ≥ 0; $i = 1, 2, \ldots, n$).

2.5 LIMITATIONS OF LINEAR PROGRAMMING PROBLEM

The following are some of the limitations of LPPs:

1. In linear programming, there is only one objective function. However in practice, situations arise where multiple objectives are involved.
2. In an LP model, the objective function and constraints must be linear but these conditions are not satisfied in many practical problems.

3. Parameters such as costs, availabilities, and technological coefficients are assumed to be constant in an LP model, whereas they are neither constant nor deterministic in many practical problems.
4. An LP model does not always provide an integer solution. Therefore, when an integer solution is required, some other model should be used.

2.6 APPLICATIONS OF LINEAR PROGRAMMING PROBLEM IN BUSINESS AND INDUSTRIES

The following are some important applications of linear programming techniques or problems:

Assignment Linear programming technique is used to assign various jobs to available manpower to achieve maximum efficiency.

Transportation Transportation problems arise in businesses and industries where goods are required to be transported from the place of manufacturing or availability to other places of requirement. These problems require good application of the linear programming technique.

Product mix An LPP is used to determine the product mix for the production of a final product.

Diet An LPP is used in finding the best combination of daily requirement of nutrients that involve minimum cost.

Trim loss There are many situations where a commodity has to be prepared by cutting pieces from a metal sheet or a cloth bundle in some standard size. Linear programming technique provides the solution to problems where pieces have to be cut with minimum trim loss.

Blending By the application of the LP model, we can determine the most economical blend for a new product subject to the available ingredients.

Example 2.1 A pharmaceutical company produces three types of medicines M_1, M_2, and M_3. For manufacturing these medicines, three types of ingredients I_1, I_2, and I_3 are required. One M_1 requires 3 units of I_1 and 2 units of I_2; one M_2 requires 2 units of I_1, 3 units of I_2, and 4 units of I_3; and one M_3 requires 4 units of I_1 and 3 units of I_3. The company has a stock of 40 units of I_1, 30 units of I_2, and 45 units of I_3. The profit on selling one M_1 is ₹8, one M_2 is ₹13, and one M_3 is ₹10. If all the medicines produced are sold, formulate the problem as an LPP to maximize the profits.

Solution Tabulating the given data, we get

Ingredients ↓	Medicines			Total units available
	M_1	M_2	M_3	
I_1	3	2	4	40
I_2	2	3	0	30
I_3	0	4	3	45
Profit in ₹	8	13	10	

Let the company produce x_1, x_2, and x_3 number of medicines M_1, M_2, and M_3 respectively. The company would like to produce x_1, x_2, and x_3 such that the profit obtained as $(8x_1+13x_2+10x_3)$ is maximum. In producing x_1 units of M_1, x_2 units of M_2, and x_3 units of M_3, the required quantity of ingredient I_1 will be $(3x_1+2x_2+4x_3)$ units. Since the company has a stock of 40 units of ingredient I_1, we have $3x_1+2x_2+4x_3 \le 40$. Similarly, for ingredient I_2, we have $2x_1+3x_2 \le 30$, and for ingredient I_3, we have $4x_2+3x_3 \le 45$. Since the units of the manufactured product cannot be negative, we have $x_1, x_2, x_3 \ge 0$.

Hence, the LPP is formulated as follows:

$$\begin{aligned}
&\text{Maximize } Z = 8x_1+13x_2+10x_3\\
&\text{Subject to} \quad 3x_1+2x_2+4x_3 \le 40\\
&\qquad\qquad\quad\;\; 2x_1+3x_2 \le 30\\
&\qquad\qquad\quad\;\; 4x_2+3x_3 \le 45\\
&\text{and} \qquad\qquad x_1, x_2, x_3 \ge 0
\end{aligned}$$

Example 2.2 A steel company manufactures three products P_1, P_2, and P_3. Each product has to pass through two machines M_1 and M_2. Each unit of P_1 requires 3 hours (h) of M_1 and 2 h of M_2; each unit of P_2 requires 2 h of M_1 and 5 h of M_2; and each unit of P_3 requires 2 h of M_1 and 3 h of M_2. The machines M_1 and M_2 are available for 30 h and 40 h respectively. The profit on each unit of products P_1, P_2, and P_3 is ₹4, ₹2, and ₹3 respectively. If all the manufactured products are sold, formulate the problem as an LPP to maximize the profit.

Solution Tabulating the given data, we get

Machines ↓	Products			Total time available (in h)
	P_1	P_2	P_3	
M_1	3	2	2	30
M_2	2	5	3	40
Profit in ₹	4	2	3	

Let the company produce x_1, x_2, and x_3 units of products P_1, P_2, and P_3 respectively. The total profit Z of the steel company is $4x_1+2x_2+3x_3$. The total time required on machine M_1 is $(3x_1+2x_2+2x_3)$ h. Since the machine M_1 is available for a maximum of 30 h, we have $3x_1+2x_2+2x_3 \le 30$. Similarly, for the time restrictions on machine M_2, we have $2x_1+5x_2+3x_3 \le 40$. Since the units of the manufactured product cannot be negative, we have $x_1, x_2, x_3 \ge 0$.

Hence, the LPP is formulated as follows:

$$\begin{aligned}
&\text{Maximize } Z = 4x_1+2x_2+3x_3\\
&\text{Subject to} \quad 3x_1+2x_2+2x_3 \le 30\\
&\qquad\qquad\quad\;\; 2x_1+5x_2+3x_3 \le 40\\
&\text{and} \qquad\qquad x_1, x_2, x_3 \ge 0
\end{aligned}$$

Example 2.3 A furniture company manufactures tables and chairs. Each table and chair is processed on two machines M_1 and M_2. One table requires 2 minutes (min) on M_1 and 4 min on M_2, and one chair requires 1 min on M_1 and 3 min on M_2. The machines M_1 and M_2 are available for 6 h and 4 h 10 min respectively, during any working day. The profit on each table and chair is ₹5 and ₹3 respectively. How many tables and chairs are manufactured to achieve maximum profit? Formulate the problem as an LPP, assuming that all the tables and chairs manufactured are sold.

Solution Tabulating the given data, we get

Machine	Time of processing		Time available (min)
	Table	Chair	
M_1	2	1	360
M_2	4	3	250
Profit in ₹	5	3	

Let the company manufacture x_1 number of tables and x_2 number of chairs. The total profit Z is $5x_1+3x_2$. The total time required on machine M_1 is $2x_1+x_2$ min. Since the machine M_1 is available for not more than 6 h, that is, 360 min, we have $2x_1+x_2 \leq 360$. Similarly, for the total time required on machine M_2 and availability of machine M_2, we have $4x_1+3x_2 \leq 250$. Since the units of the manufactured product cannot be negative, we have $x_1, x_2 \geq 0$.

Hence, the LPP is formulated as follows:

$$\begin{aligned}
&\text{Maximize } Z = 5x_1 + 3x_2 \\
&\text{Subject to} \quad 2x_1 + x_2 \leq 360 \\
&\qquad\qquad\quad\; 4x_1 + 3x_2 \leq 250 \\
&\text{and} \qquad\qquad x_1, x_2 \geq 0
\end{aligned}$$

Example 2.4 A publisher deals in two items, namely books and dictionaries. The publisher has ₹70,000 to invest and space for not more than 600 pieces (including both books and dictionaries). A book costs ₹500 and a dictionary, ₹700. The publisher can sell all the items that he buys, earning a profit of ₹50 for each book and ₹140 for each dictionary. Formulate the problem as an LPP to maximize the profit.

Solution Let the publisher buy x_1 books and x_2 dictionaries. As the publisher can store a maximum of 600 pieces of books and dictionaries, we have $x_1+x_2 \leq 600$. The total cost incurred in purchasing x_1 books and x_2 dictionaries is $500x_1+700x_2$. However, since the publisher can invest only ₹70,000, we have $500x_1+700x_2 \leq 70{,}000$ or $5x_1+7x_2 \leq 700$. The total profit Z is $50x_1+140x_2$.

Hence, the LPP is formulated as follows:

$$\begin{aligned}
&\text{Maximize } Z = 50x_1 + 140x_2 \\
&\text{Subject to} \quad x_1 + x_2 \leq 600 \\
&\qquad\qquad\quad\; 5x_1 + 7x_2 \leq 700 \\
&\text{and} \qquad\qquad x_1, x_2 \geq 0
\end{aligned}$$

Example 2.5 The supervisor of Reliance oil refinery takes a decision to combine two types of crudes, say C_1 and C_2, to produce fuel gas and petroleum using two processes—1 and 2. The inputs and outputs per production run are given in the following table:

Process	Input		Output	
	Crude C_1	Crude C_2	Petroleum	Fuel gas
1	5	3	5	7
2	4	5	3	4

The maximum units available for crudes C_1 and C_2 are 300 and 200 respectively. The market survey shows that at least 200 units of petroleum and 150 units of fuel gas must be produced. The profits per production run from processes 1 and 2 are ₹30 and ₹45 respectively. Formulate the problem as an LPP to maximize the profit, so that market demand is fulfilled.

Solution Let x_1 and x_2 be the number of production runs of processes 1 and 2 respectively. The amount of crudes C_1 and C_2 required are $5x_1+4x_2$ and $3x_1+5x_2$. As the maximum amounts of these crudes are 300 and 200 respectively, we have $5x_1+4x_2 \le 300$ and $3x_1+5x_2 \le 200$. The output of petroleum and fuel gas is $5x_1+3x_2$ and $7x_1+4x_2$, whereas market demands are at least 200 and 150 units respectively. Therefore, we have $5x_1+3x_2 \ge 200$ and $7x_1+4x_2 \ge 150$. The total profit is $30x_1+45x_2$.

Hence, the LPP is formulated as follows:

$$\text{Maximize } Z = 30x_1+45x_2$$

$$\begin{aligned} \text{Subject to} \quad & 5x_1+4x_2 \le 300 \\ & 3x_1+5x_2 \le 200 \\ & 5x_1+3x_2 \ge 200 \\ & 7x_1+4x_2 \ge 150 \\ \text{and} \quad & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Example 2.6 A hotel operating 24 h has the following minimum requirement of servers:

Period	Time	Minimum number of servers required
1	7 a.m.–11 a.m.	6
2	11 a.m.–3 p.m.	12
3	3 p.m.–7 p.m.	8
4	7 p.m.–11 p.m.	16
5	11 p.m.–3 a.m.	5
6	3 a.m.–7 a.m.	3

A server reports to the hotel manager at the beginning of the period and continues to work for 8 h. The hotel manager wants to find the minimum number of servers available for each period. Formulate this as an LPP to minimize the total number of servers required.

Solution Let $x_1, x_2, \ldots, x_6$ be the number of servers reporting for duty at 7 a.m., 11 a.m., 3 p.m., 7 p.m., 11 p.m., and 3 a.m. respectively (i.e., at the beginning of the six periods). x_1 servers reporting at 7 a.m. will work up to 3 p.m. and x_2 servers reporting at 11 a.m. will work up to 7 p.m. Thus, the number of servers available between 11 a.m. and 3 p.m. is x_1+x_2. Similarly, x_2+x_3, x_3+x_4, x_5+x_6, and x_6+x_1 number of servers remain available between 11 a.m. to 3 p.m., 3 p.m. to 7 p.m., 7 p.m. to 11 p.m., 11 p.m. to 3 a.m., and 3 a.m. to 7 a.m. respectively. Therefore, the total number of servers reporting for duty in 24 h is $x_1+x_2+x_3+x_4+x_5+x_6$.

Hence, the LPP is formulated as follows:

$$\text{Minimize } Z = x_1+x_2+x_3+x_4+x_5+x_6$$

$$\begin{aligned} \text{Subject to} \quad & x_1+x_2 \ge 12 \\ & x_2+x_3 \ge 8 \\ & x_3+x_4 \ge 16 \\ & x_4+x_5 \ge 5 \end{aligned}$$

$$x_5+x_6 \geq 3$$
$$x_6+x_1 \geq 6$$
and $\quad x_i \geq 0, i=1, 2, 3, \ldots, 6$

Example 2.7 A biscuit company can produce three products P_1, P_2, and P_3. Each of these products requires three different operations O_1, O_2, and O_3. One unit of P_1 requires 2 h of O_1, 2 h of O_2, and 3 h of O_3; one unit of P_2 requires 3 h of O_1, 1 h of O_2, and 2 h of O_3; and one unit of P_3 requires 4 h of O_1, 3 h of O_2, and 4 h of O_3. O_1, O_2, and O_3 can operate for a maximum of 40 h, 50 h, and 150 h respectively. The profit on each unit of P_1, P_2, and P_3 is ₹2, ₹4, and ₹3 respectively. If all the produce is sold, formulate the problem as an LPP to maximize the profit.

Solution Tabulating the given data, we get

Operations ↓	Product			Available operation time (in h)
	P_1	P_2	P_3	
O_1	2	3	4	40
O_2	2	1	3	50
O_3	3	2	4	150
Profit in ₹	2	4	3	

Let x_1, x_2, and x_3 be the units of products, P_1, P_2, and P_3 respectively, produced by the company. The total time required for operation O_1 to produce the products is $2x_1+3x_2+4x_3$. Since the capacity of operation O_1 is 40 h per unit, we have $2x_1+3x_2+4x_3 \leq 40$. Similarly, considering the times for O_2 and O_3, we have $2x_1+x_2+3x_3 \leq 50$ and $3x_1+2x_2+4x_3 \leq 150$. Further as the units produced cannot be negative, we have $x_1, x_2, x_3 \geq 0$. The profit (in ₹) is $2x_1+4x_2+3x_3$.

Hence the LPP is formulated as follows:

Maximize $Z=2x_1+4x_2+3x_3$

Subject to $\quad 2x_1+3x_2+4x_3 \leq 40$
$$2x_1+x_2+3x_3 \leq 50$$
$$3x_1+2x_2+4x_3 \leq 150$$
and $\quad x_1, x_2, x_3 \geq 0$

Example 2.8 A dietician advises a wrestler to consume at least 90 grams of proteins, 100 grams of multivitamin, 150 grams of carbohydrates, and 70 grams of fats daily. The following table gives the analysis of the food items readily available in the market with their respective costs:

Food type	Food value (in grams) per 100 grams				Minimum daily requirement (grams)
	F_1	F_2	F_3	F_4	
Proteins	20	10	8	12	90
Multivitamins	10	16	18	12	100
Carbohydrates	5	15	25	10	150
Fats	12	10	20	14	70
Cost in ₹ (per kg)	5	4	8	6	

Formulate the problem as an LPP for an optimum diet.

Solution Let x_1, x_2, x_3, and x_4 kg of food F_1, F_2, F_3, and F_4 respectively be included in the daily diet. The total cost per day is $5x_1+4x_2+8x_3+6x_4$. The total amount of proteins in the daily diet is $(200x_1+100x_2+80x_3+120x_4)$. As the minimum daily requirement of proteins in the diet is 90 grams, we have $200x_1+100x_2+80x_3+120x_4 \geq 90$. Similarly, for the total amounts of multivitamins, carbohydrates, and fats in the diet, we have $100x_1+160x_2+180x_3+120x_4 \geq 100$, $50x_1+150x_2+250x_3+100x_4 \geq 150$, and $120x_1+100x_2+200x_3+140x_4 \geq 70$. Since the diet cannot contain negative quantity of any food items, we have $x_1, x_2, x_3, x_4 \geq 0$.

Hence, the LPP is formulated as follows:

Minimize $Z=5x_1+4x_2+8x_3+6x_4$

Subject to
$$200x_1+100x_2+80x_3+120x_4 \geq 90$$
$$100x_1+160x_2+180x_3+120x_4 \geq 100$$
$$50x_1+150x_2+250x_3+100x_4 \geq 150$$
$$120x_1+100x_2+200x_3+140x_4 \geq 70$$
and
$$x_1, x_2, x_3, x_4 \geq 0$$

Example 2.9 A company manufactures two products P_1 and P_2. Each product is processed on two machines M_1 and M_2. While P_1 requires 2 min on M_1 and 4 min on M_2, P_2 requires 3 min on M_1 and 3 min on M_2. The machines M_1 and M_2 are available for 1000 min and 1500 min respectively. The profit on each product P_1 and P_2 is ₹2 and ₹4 respectively to work. The company must manufacture 80 P_1's and 100 P_2's, but not more than 120 P_1's. Formulate the problem as an LPP to maximize the profit.

Solution Tabulating the given data, we get

Machine	Product		Total available time (in min)
	P_1	P_2	
M_1	2	3	1000
M_2	4	3	1500
Profit in ₹	2	4	

Let x_1 and x_2 be the number of P_1's and P_2's respectively produced by the company. The total time required to produce them on machines M_1 and M_2 are $2x_1+3x_2$ and $4x_1+3x_2$ min respectively. Since the time available with the machines M_1 and M_2 are 1000 and 1500 min respectively, we have $2x_1+3x_2 \leq 1000$ and $4x_1+3x_2 \leq 1500$. Moreover, the firm must manufacture 80 P_1's and 100 P_2's, but not more than 120 P_1's. Therefore, we have $80 \leq x_1 \leq 120$ and $x_2 \geq 100$. The profit (in ₹) is $2x_1+4x_2$.

Hence, the LPP is formulated as follows:

Maximize $Z=2x_1+4x_2$

Subject to
$$2x_1+3x_2 \leq 1000$$
$$4x_1+3x_2 \leq 1500$$
$$80 \leq x_1 \leq 120$$
and
$$x_2 \geq 100$$

Example 2.10 A doctor recommends two foods F_1 and F_2 to a patient for his diet, which includes at least 1000 units of vitamin, 850 units of protein, and 700 units of fat. While each unit of food F_1 contains 15 units of vitamin, 18 units of protein, and 12 units of fat, each unit of food F_2 contains 22 units of vitamin, 15 units of protein, and 16 units of fat. The cost of F_1 and F_2 is ₹5 and ₹8 respectively. Formulate the problem as an LPP to obtain the minimized cost for a diet.

Solution Tabulating the given data, we get

Food	F_1	F_2	Requirement (in units)
Vitamin	15	22	1000
Protein	18	15	850
Fat	12	16	700
Cost in ₹	5	8	

Let us consider x_1 and x_2 units of foods F_1 and F_2 respectively. In a diet, vitamin, protein, and fat units are $15x_1+22x_2$, $18x_1+15x_2$, and $12x_1+16x_2$. As the diet must have at least 1000, 850, and 700 units of vitamins, proteins, and fats respectively, we have $15x_1+22x_2 \geq 1000$, $18x_1+15x_2 \geq 850$, and $12x_1+16x_2 \geq 700$. Since the daily diet cannot contain negative quantity of any food items, we have $x_1, x_2 \geq 0$. The cost (in ₹) is $5x_1+8x_2$.

Hence, the LPP is formulated as follows:

Minimize $Z=5x_1+8x_2$

$$\begin{aligned} \text{Subject to} \quad & 15x_1+22x_2 \geq 1000 \\ & 18x_1+15x_2 \geq 850 \\ & 12x_1+16x_2 \geq 700 \\ \text{and} \quad & x_1, x_2 \geq 0 \end{aligned}$$

RECAPITULATION

1. An optimization problem in which the objective function and constraints are linear and all decision variables are non-negative is called a linear programming problem (LPP).
2. No constraint should hold with strict inequalities such as '<' or '>' in any LPP.
3. In an LPP, there is only one objective function.
4. An LP model does not always provide an integer solution to the problem. To obtain an integer solution, we have to use other techniques like the integer programming technique.

EXERCISES

Multiple-choice Questions

2.1 Which of the following is required to be optimized in an LPP?
(a) Objective function
(b) Constraints
(c) Decision variables
(d) All of these

2.2 In an LPP, the restrictions under which the objective function is to be optimized are called
(a) constraints
(b) decision variables
(c) objective function
(d) none of these

2.3 Who invented the simplex method of linear programming?
(a) George B. Dantzig
(b) Cauchy
(c) Gauss
(d) Newton

2.4 A decision taken considering all the circumstances into account is called
(a) optimum (b) maximum (c) minimum (d) none of these

Review Questions

2.1 Define an LPP.
2.2 Write the basic assumptions needed in formulating a problem in LPP form.
2.3 What are the limitations of an LPP?
2.4 What are the applications of LPP in businesses and industries?

Numerical Problems

2.1 A newly established company has purchased sufficient amount of curtain cloth to furnish the staff offices. The curtain cloths are in pieces of length 20 feet each and can be cut in three sizes of length 3, 4, and 5 feet. The curtain requirement of the three sizes is given in the following table:

Curtain length (in feet)	Number required
3	1000
4	1500
5	3000

The company wishes to minimize the trim loss.
(a) Identify the patterns of cutting a piece of 20 feet length into requisite sizes.
(b) Identify the decision variables.
(c) Find the objective function.
(d) Write all the constraints.
(e) Formulate the problem as an LP model.

2.2 A cloth company manufactures T-shirts and pairs of jeans. Each cloth has to pass through three machines M_1, M_2, and M_3. One pair of jeans requires 3 h, 1 h, and 2 h on machines M_1, M_2, and M_3 respectively and one T-shirt requires 1 h each on machines M_1 and M_2 and 3 h on machine M_3. The profit on selling of one pair of jeans is ₹50, whereas for a T-shirt it is ₹55. The machines M_1, M_2, and M_3 are available for 60 h, 50 h, and 80 h per week respectively. How many pairs of jeans and T-shirts should be manufactured per week so as to maximize the profit? Formulate the problem as an LPP.

2.3 Asian Paints produces both exterior and interior paints from two raw materials P and Q as per the following table:

Raw material	Tonnes of raw material per tonne of		Maximum weekly availability (tonnes)
	Exterior paint	Interior paint	
P	7	5	35
Q	2	3	6
Profit per tonne (₹ in thousands)	6	4	

The market survey indicates that the maximum weekly demand of exterior paints is 3 tonnes and its weekly demand cannot exceed that of interior paints by more than 2 tonnes. The company wants to determine the best product mix of interior and exterior paints to maximize the total weekly profit. Answer the following with respect to the LP model of the aforementioned problem:
(a) What are the decision variables?

(b) The number of constraints are
(i) two (ii) three (iii) four (iv) five
(c) The objective function is ________.
(d) The formulation is ________.

2.4 A company manufactures two products A and B. Each product has to pass through two machines M_1 and M_2. The machines M_1 and M_2 are available for 100 h and 80 h per week respectively. To manufacture one unit of product A, 4 h of M_1 and 2 h of M_2 are required, whereas for one unit of product B, 3 h of M_1 and 1 h of M_2 are required. The profit on the sale of each unit of A and B is ₹15 and ₹10 respectively. If the manufacturer can sell all the items produced, how many of each should be produced to get maximum profit per week? Formulate the problem as an LPP.

2.5 A blacksmith who manufactures axes and knives takes 2 h to make an axe and 1 h to make a knife. It is assumed that he can work for a maximum of 15 h a day. The maximum number of axes and knives that he can handle per day is 20. While the profit on an axe is ₹15, the profit on a knife is ₹5. If all that is manufactured is sold, formulate the problem as an LPP to maximize the profit.

2.6 A pen manufacturing company manufactures two types of pens, ordinary and deluxe, say O and D. Each pen D takes twice as long to manufacture than one pen O. The company would have time to make a maximum of 500 pens per day if it produces only the ordinary version. The supply of plastic is sufficient to produce 400 pens per day (both O and D combined). The deluxe version requires fancy pens of which there are only 200 pieces per day available. The profit on the sale of each pen of O and D is ₹2 and ₹5 respectively. If all that is manufactured is sold, formulate the problem as an LPP to maximize the profit.

2.7 A company manufactures two types of cloths for curtains C_1 and C_2. Both curtains go through two machines M_1 and M_2. The details are given in the following table:

Types of machines	**Processing time per piece (in h)**		**Total time required (in h)**
	C_1	C_2	
M_1	3	1	100
M_2	1	4	80
Profit in ₹	5	8	

How many of each type of curtain should be manufactured per month to obtain the best returns? If all that is manufactured is sold, formulate the problem as an LPP.

2.8 A pharmaceutical company produces two medicines M_1 and M_2. The company has stocks of ingredients to make 30,000 bottles of M_1 and 40,000 bottles of M_2. However, there are only 50,000 bottles into which either of the medicines can be filled. Further, it takes 3 h and 1h to prepare enough material to fill 1000 bottles of medicine M_1 and M_2 respectively and 70 h are available for this operation. The profit on the sale of each bottle of M_1 and M_2 is ₹10 and ₹8 respectively. If all that is manufactured is sold, formulate the problem as an LPP to maximize the profit.

2.9 A leather factory manufactures two types of leather purses, L_1 (higher quality) and L_2 (lower quality). Each purse of type L_1 requires three times the time required for type L_2, and if all are of type L_2, the company could make 1200 purses per day. However, the supply of leather is sufficient for only 1000 purses per day. Purse L_1 requires a fancy buckle and only 600 buckles are available per day, whereas 900 buckles are available for purse L_2. The profit on each purse L_1 and L_2 is ₹5 and ₹2 respectively. What should the daily production of each type of purse be to get the maximum profit? Formulate the problem as an LPP.

2.10 A landlord owns 150 acres of land. He uses this land for growing grapes, coconuts, and apples. The price he can obtain is ₹1 per kg for grapes, ₹0.75 ahead for coconuts, and ₹2 per kg for apples. The average yield per acre is 2000 kg of grapes, 3000 kg of coconuts, and 1000 kg of apples. Fertilizer is available at ₹0.50 per kg and the amount required per acre is 100 kg each for grapes and coconuts and 50 kg for apples. Labour required for sowing, cultivating, and harvesting per acre is 5 man-days for grapes and apples, and 6 man-days for coconuts. A total of 500 man-days of labour are available at ₹20 per man-day. Formulate this problem as an LPP to maximize the profit.

2.11 A cloth manufacturing company produces three types of cloths C_1, C_2, and C_3, which require two types of wool, red and black. Assume one unit length of C_1 needs 1 yard of red wool and 2 yards of black wool, that of C_2 needs 3 yards of red wool and 4 yards of black wool, and that of C_3 needs 3 yards of red wool and 1 yard of black wool. The company has a stock of only 12 yards of red wool and 20 yards of black wool. The profit on the sale of each cloth C_1, C_2, and C_3 is ₹5, ₹10, and ₹15 respectively. If all that is manufactured is sold, formulate the problem as an LPP to maximize the profit.

2.12 A family doctor advises a minimum intake of 20, 25, and 15 units of vitamins V_1, V_2, and V_3 respectively. The doctor suggests two type of foods F_1 and F_2. F_1 contains 5, 4, and 3 units per gram and F_2 contains 3, 4, and 6 units per gram of vitamins V_1, V_2, and V_3 respectively. The costs of F_1 and F_2 are ₹4 and ₹5 per gram. How many grams of each food type should the family purchase every day to keep the food expense at a minimum? Formulate the problem as an LPP.

2.13 A computer company has two types of operators O_1 and O_2 to check the computer pieces produced by the company. It is required that at least 1000 pieces be inspected in an 8 h-day. Operator O_1 can check the pieces at the rate of 40 per h with an accuracy of 95% and operator O_2 can check pieces at the rate of 20 per h with an accuracy of 90%. The wage rate of operator O_1 is ₹5 per h and that of operator O_2 is ₹3 per h. Each time an error is made by an operator, the cost to company is ₹2. The company has 20 types of O_1 and 15 types of O_2 operators. Formulate the problem as an LPP to minimize the total cost of inspection.

2.14 A complete unit of a certain product consists of four and three units of components A and B respectively. A and B are manufactured from two different raw materials of which 100 and 200 units respectively are available with the company. Three departments are engaged in the production and they use different methods for manufacturing the component. The objective is to determine the number of production runs for each department that will maximize the total number of the final product. The following table gives the raw material requirement per production run and the resulting output of each component:

Department	Input per run (units)		Output per run (units)	
	Raw material		Component	
	R_1	R_2	A	B
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate the problem as an LPP.

2.15 A firm manufacturing two types of electrical items A and B can make a profit of ₹160 per unit of A and ₹240 per unit of B. Both A and B make use of two essential components: a motor and a transformer. While each unit of A requires 3 motors and 2 transformers, each unit of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 210 motors and 300 transformers. Further, type B is an export model requiring a voltage stabilizer, which has a supply restricted to 65 units per month. How many units each of A and B should the firm manufacture per month to maximize its profit? Formulate the problem as an LPP.

2.16 The data for two foods A and B is given in the following table:

Food value	**One unit of food**		**Requirement**
	Type A	**Type B**	
Calories	1000	2000	3000
Protein	25 grams	10 grams	100
Price (₹)	60	21	

Formulate the LPP for minimizing the expenditure on food satisfying the calorie and protein requirements.

CONCEPTUAL PROBLEM

A company wants to engage itself in the manufacturing of three kinds of paints. As a first step, the company has observed and analysed the working of other companies engaged in the manufacture and supply of these varieties of paints. The company has considered all the factors and wants to manufacture the best quality of these varieties at a competitive price. The company is in the process of developing a fortnight manufacturing programme to come out with three grades of paints P_1, P_2, and P_3. The plant works in three shifts and the following data is available:

Requirement	**Grade**			**Availability**
	P_1	P_2	P_3	
Special additive (per kilolitre)	0.25	0.2	0.70	700 tonnes
Packing (kilolitre per shift)	15	15	15	85 shifts

There are no restrictions on other resources. The particulars of the sale forecasts and the estimated profit are given in the following table:

Sales forecasts and estimated profit	**Grade**		
	P_1	P_2	P_3
Maximum possible sale per fortnight (kilolitre)	150	450	700
Profit (₹ per kilolitre)	4000	3600	2000

Due to prior commitments, a minimum of 200 kilolitres per fortnight of grade P_2 has to be supplied during the next year. The company plans to compare its products with other companies with respect to the quality and prices.

Formulate the LP model for developing the fortnight manufacturing programme to maximize the profit.

(Contd)

(*Contd*)

Solution Here the company is planning to produce three kinds of paints that can fetch different amounts of revenue for the company. Thus, the first task of the company is to find quantities of different grades of paints to be produced in such a way that the total revenue obtained is maximum. For this let x_1, x_2, and x_3 be the quantity in kilolitres of paint grades P_1, P_2, and P_3 respectively, which the company finally decides to manufacture fortnightly.

The total profit in producing these quantities of different grade paints is $4000x_1 + 3600x_2 + 2000x_3$.

The company would like to produce x_1, x_2, and x_3 such that the profit $4000x_1 + 3600x_2 + 2000x_3$ is maximum.

In producing x_1 units of P_1, x_2 units of P_2, and x_3 units of P_3, the required quantity of special additive will be $(0.25x_1 + 0.2x_2 + 0.7x_3)$ kilolitres.

Since the company has a stock of special additive of 700 tonnes, we have

$$0.25x_1 + 0.2x_2 + 0.7 \leq 700$$

Similarly, for packaging, we have $\frac{x_1}{15} + \frac{x_2}{15} + \frac{x_3}{15} \leq 85$

and $x_2 \geq 200$, $x_1 \leq 150$, $x_2 \leq 450$, $x_3 \leq 700$

Further, x_1, x_2, and x_3, being quantities in kilolitres of the paint, cannot be negative.

Thus, we have $x_1, x_2, x_3 \geq 0$.

Hence, the problem formulated is

$$\text{Maximize } Z = 4000x_1 + 3600x_2 + 2000x_3$$

$$\begin{aligned}
\text{Subject to} \quad & 5x_1 + 4x_2 + 14x_3 \leq 14000 \\
& \frac{x_1}{15} + \frac{x_2}{15} + \frac{x_3}{15} \leq 85 \\
& x_1 \leq 150 \\
& x_2 \leq 450 \\
& x_2 \geq 200 \\
& x_3 \leq 700 \\
\text{and} \quad & x_1, x_2, x_3 \geq 0
\end{aligned}$$

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Chapter Two

Linear Programming Problem II–Graphical Method

2.1 Introduction Graphical Method

The graphical technique provides a geometric picture of the solution to an LPP, and explains many other relevant aspects geometrically. It reveals that the maximization or minimization of an LPP occurs at the vertices of the set of feasible solutions (SF), provided SF is bounded. If the optimum occurs at more than one vertex of SF, then it also occurs at every point on the line segment joining these vertices. In the graphical method, we first identify the decision variables, then recognize the underlying objective of the problem, and ultimately obtain the given problem in the form of an LPP

2.2 Concept of Graphical technique and its Methods

This section discusses how a linear programming model (LP model) in two variables is solved graphically. This method cannot be used when more than two variables are involved. Although LP models in two variables rarely occur in practice (where a typical LP model may involve many variables), the ideas revealed by the graphical procedure establish the foundation of the general solution technique (known as the simplex method), and Any LP model includes three basic elements:

1. The non-negative decision variables whose values are to be determined

2. The objective function that we target to optimize

3. The restrictions (constraints) to be satisfied by the decision variables. The set of non-negative decision variables satisfying all the constraints is known as the set of feasible solutions. The feasible solution for which the value of the objective function is optimum (minimum in case of minimization problems and maximum in case of maximization problems) is called the optimum feasible solution of the LPP.

In the graphical method, we retain the inequality form of the constraints. An inequality constraint in two variables represents a half-plane, which is the set of points on one side of the line, including the points on the line. We sketch all these half planes created by all the constraints and take their intersection to get the set of feasible solutions (SF). The following two steps are used in the graphical technique for solving an LP model in two variables:

1. Determination of the set of feasible solutions (solution space) that satisfies all the restrictions placed on the decision variables involved in the problem (including non-negativity restrictions).

2. Determination of the optimal solution(s) among all the points in the set of feasible solutions. For obtaining the optimum solution in the graphical method, two methods are employed: (a) Corner point method and (b) Iso-profit method (Iso value line method). There are Graphical Methods to find the optimal solution.

2.2.1 Corner Point Method

We take the following steps to obtain the graphical solution to an LPP using the corner point method:

1. Sketch the inequalities (constraints).
2. Find the set of feasible solutions (SF), by taking the intersection of inequalities.
3. Obtain the corner points (vertices) of SF.
4. Select the optimum value of the objective function. This gives the value of the decision variables along with the optimum value of the objective function.

2.2.2 Iso-profit Method

In this method, the following steps are taken to obtain the optimal solution to an LPP:

1. Sketch all the constraints involved in the problem.
2. Identify the set of feasible solutions (SF).
3. Draw parallel lines with different values of the objective function, moving in the direction in which the objective function is tending to increase or decrease, according to whether the problem is of maximization or minimization type.
4. Continue to draw the parallel lines obtained by taking different values of the objective function in the desired direction, until one the following conditions holds good:

(a) The last line passes through a vertex of SF, where the optimum solution is obtained and any further movement of the parallel line pushes it out of SF.

(b) The parallel line surpasses the last vertex in the desired direction and still contains the points of SF with no bounds on the objective function value. Hence, the solution is unbounded.

It is preferable to use the corner point method to obtain the optimal solution when the set of feasible solutions (SF) is bounded.

2.3 Special Cases in Graphical Method

While solving an LPP using the graphical method, we come across some special situations where the problem has more than one optimal solution, no solution, or sometimes even an infinite number of solutions. In this section, we will discuss all such scenarios.

2.3.1 Alternate Optimal Solution

An LPP is said to have an alternate optimal solution, if it has an optimal solution at more than one vertex. In Fig. 2.1, if the optimal solution exists at two vertices, say P_1 and P_2, then it will exist at every point X of the line segment joining the two vertices P_1 and P_2. Thus, the problem will have an infinite number of solutions.

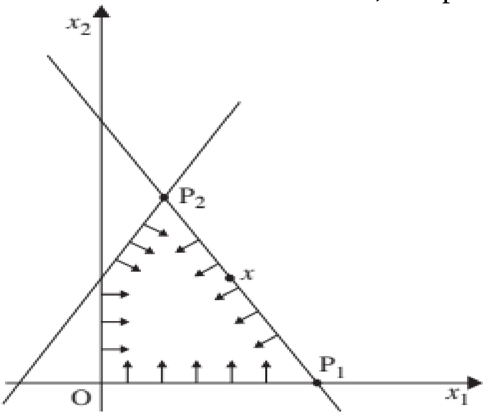


Fig. 2.1 Optimal solution at two vertices P_1 and P_2

2.3.2 No Feasible Solution

Consider the following LPP:

Max Z = 2x1 + 5x2

Subject to $\quad x_1 + x_2 \le 1 \; (2.1)$

$3x_1 + 2x_2 \ge 6 \; (2.2)$

And $x_1, x_2 \ge 0$

As shown in Fig. 2.2, S1 is the feasible solution space of Eq. (2.1) and S2 is the solution space of Eq. (2.1). Therefore, the solution space of the problem as a whole is SF = S1 ∩ S2 = φ. Thus, as the solution space is an empty set, the problem has no solution.

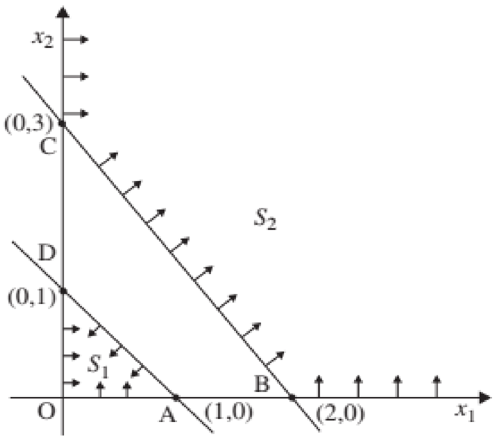


Fig. 2.2 Empty solution space

2.3.3 Unbounded Solution Space but Bounded Optimal Solution

Consider the following LPP:

Min $Z = X_1 + 2X_2$

Subject to $X_1 + X_2 \geq 3$ (2.3)

$X_1 - 3X_2 \leq 3$ (2.4)

And $X_1, X_2 \geq 0$

In Fig. 2.3, **(AB is the line $X_1 + X_2 = 3$)**, **(BC is the line $X_1 - 3X_2 = 3$)**, and SF is the set of feasible solutions. Now, the problem is to find $(X_1, X_2) \in SF$ that makes Z minimum. For this, consider the Iso-profit parallel lines corresponding to the different values of Z. From the figure, it is clear that as we move in the direction of arrows, the Z value decreases. At the vertex B (3, 0), Z = 3. If we try to decrease Z further by moving in the direction of the arrow, the Iso-profit line would not pass through SF (we should remain in SF) and hence, a further decrease in the Z value is not possible. Thus, Min Z = 3, at vertex B when $X_1 = 3$ and $X_2 = 0$.

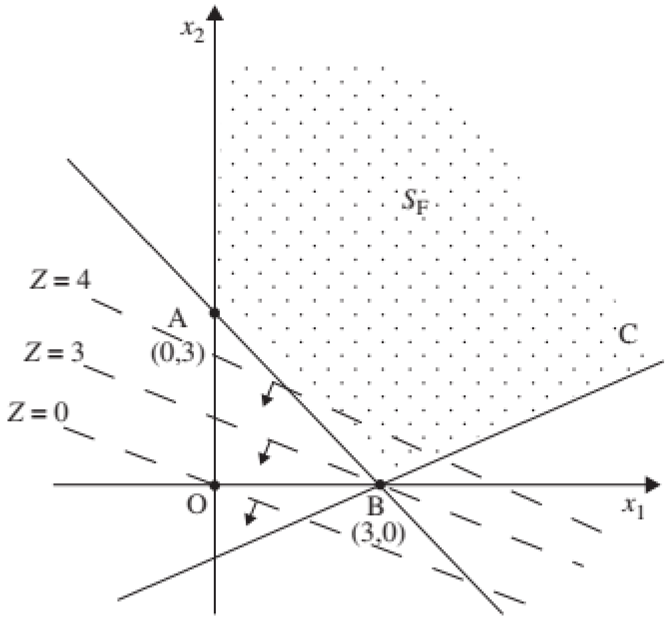


Fig. 2.3 Bounded solutions with unbounded solution space

2.3.4 Unbounded Solution Space and Unbounded Solution

Consider the LPP

Max $Z = 2X_1 + X_2$

Subject to $-3X_1 + X_2 \leq 3$ (2.5)

$X_1 + X_2 \geq 3$ (2.6)

$X_1 - 3x_2 \leq 3$ (2.7)

And $X_1, X_2 \geq 0$

In Fig. 2.4, **(line AB is $-3X_1 + X_2 = 3$)**, **(line BC is $X_1 + X_2 = 3$), (line CD is $X_1 - 3X_2 = 3$)**, and SF is the set of feasible solutions. Now the problem is to find point $(X_1, X_2) \in SF$, which makes the Z value maximum.

For this, consider the iso-profit parallel lines corresponding to the different values of Z. From the figure, it is clear that as we move in the direction of the arrows through the parallel iso-profit lines, the Z value is increasing. At the vertex B, Z = 3 and at C, Z = 6. If we move further in the direction of the arrows, the parallel iso-profit lines beyond vertex C would still pass across the set of feasible solutions (SF), making the Z values larger with no upper bound. Thus, the solution is unbounded with unbounded solution space (as is clear from Fig. 2.4).

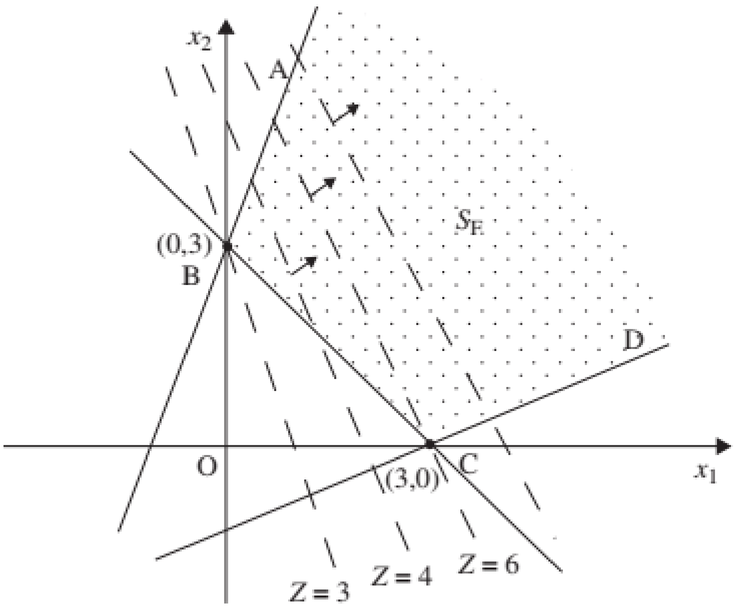


Fig. 2.4 Unbounded solution with unbounded solution space

Example 2.1

A leather company manufactures two types of belts, B1 and B2. Each of these sell at a profit of $ 2 per belt. Each belt is processed on two machines, M1 and M2. The B1 type belt requires one minute of processing time on machine M1 and two minutes on machine M2. The B2 type belt requires one minute of processing time on each of the machines M1 and M2. While machine M1 is available for a maximum of 6 h and 40 min, machine M2 is available for 10 h on all working days. Formulate the given problem as an LPP and solve it using the graphical method.

Solution Let x_1, x_2 be the number of belts of types B1 and B2 to be manufactured. The LPP formulation of the given problem is as follows:

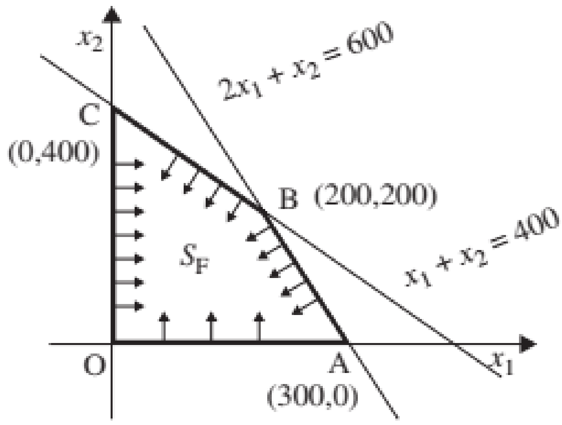
Max $Z = 2x_1 + 2x_2$

Subject to $X_1 + X_2 \leq 400$

$2X_1 + X_2 \leq 600$

And $X_1, X_2 \geq 0$

Points	Z
O(0, 0)	$Z = 2 \times 0 + 2 \times 0 = 0$
A(300, 0)	$Z = 2 \times 300 + 2 \times 0$ $= 600$
B(200, 200)	$Z = 2 \times 200 + 2 \times 200$ $= 800$ (Max)
C(0, 400)	$Z = 2 \times 0 + 2 \times 400$ $= 800$ (Max)



Alternate optimal solution at B and C

Corner point method

Now the problem is to find a point X = (x1, x2)T for satisfying Eqs, where Z=2x1+ 2x2 has the maximum value. We plot the inequalities in Eqs to get the set of feasible solutions SF, as shown in Fig.. Next, we find the coordinates of points O, A, B, and C, which are vertices (corner points) of SF and calculate the Z values as given in the following table. Thus, Max Z = 800, when x1 = 200 and x2 = 200. Further, Max Z = 800 when x1 = 0 and x2 = 400, which is another optimal solution called the alternate optimal solution of the problem. In fact, the optimum solution will occur at every point on the line segment joining the points B and C.

Example 2

Solve the following LPP using the graphical method:

Maximize Z = −X1 + 2X2

Subject to −X1 + X2 ≤ 1

X1 + X2 ≤ 2

And X1, X2 ≥ 0

Solution the given LPP is

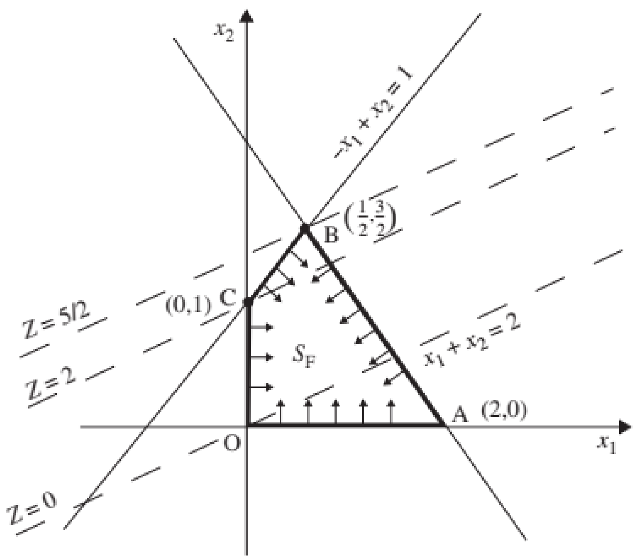
Maximize Z = −X1 + 2X2 (3.23)

Subject to −X1 + X2 ≤ 1 (3.24)

X1 + X2 ≤ 2 (3.25)

And X1, X2 ≥ 0 (3.26)

Points	Z
O(0, 0)	Z = 0 + 2.0 = 0
A(2, 0)	Z = −2 + 2.0 = −2
B(1/2, 3/2)	Z = −(1/2) + 2(3/2) = 5/2 (Max)
C(0, 1)	Z = 0 + 2.1 = 2



Bounded solution with bounded solution space

Thus optimum solution is Max Z = 5/2, when x1 = 1/2 and x2 = 3/2.

Now the problem is to find a point X = (X1, X2) T in the first quadrant, satisfying Eqs (3.24) and (3.25), where Z = −X1 + 2X2 has the maximum value. We sketch the inequalities in Eqs (3.24), (3.25), and (3.26) to get the region OABC (set of feasible solutions) as shown in Fig. 3.10. We shall determine the solution to the LPP using both the iso-profit and the corner point methods.

Iso-profit method Let us consider the parallel lines that correspond to different values of Z, say Z = 0, 2, 5/2. We see that Z increases as the movement of parallel lines is in the direction of the arrow, as shown in Fig. 3.10. The last such line passes through B(1/2, 3/2), with Z = −(1/2) + 2(3/2) = 5/2. Thus, the optimum solution is Max Z = 5/2, when X1 = ½ and X2 = 3/2.

Corner point method For this we find the coordinates of points O, A, B, and C, which are vertices (corner points) of SF and calculate the Z values.

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- **المقدمة**

توفر التقنية الرسومية صورة هندسية للحل إلى LPP ، وتشرح العديد من الجوانب الأخرى ذات الصلة هندسيًا. ويكشف أن تكبير أو تصغير LPP يحدث عند رؤوس مجموعة الحلول الممكنة (SF) ، بشرط أن يكون SF محددًا. إذا حدث الحد الأقصى عند أكثر من قمة واحدة لـ SF ، فإنه يحدث أيضًا في كل نقطة على قطعة الخط التي تنضم إلى هذه القمم. في الطريقة الرسومية ، نحدد أولاً متغيرات القرار ، ثم نتعرف على الهدف الأساسي للمشكلة ، ونحصل في النهاية على المشكلة المحددة في شكل LPP.

- **مفهوم الطريقة البيانية**

ناقش هذا القسم كيفية حل نموذج البرمجة الخطية (نموذج LP) في متغيرين بيانياً. لا يمكن استخدام هذه الطريقة عندما يتعلق الأمر بأكثر من متغيرين. على الرغم من أن نماذج LP في متغيرين نادرًا ما تحدث في الممارسة (حيث قد يتضمن نموذج LP النموذجي العديد من المتغيرات) ، فإن أي نموذج LP يتضمن ثلاثة عناصر أساسية:

١. متغيرات القرار غير السلبية التي سيتم تحديد قيمها

٢. الوظيفة الموضوعية التي نستهدف تحسينها

٣. القيود (القيود) الواجب توافرها في متغيرات القرار.

تُعرف مجموعة متغيرات القرار غير السلبية التي تلبي جميع القيود بمجموعة الحلول الممكنة. الحل المجدي الذي تكون قيمة الوظيفة الهدف له هي الأمثل (الحد الأدنى في حالة مشاكل التقليل والحد الأقصى في حالة مشكلات التعظيم) يسمى الحل المجدي الأمثل لـ LPP.

في الطريقة الرسومية ، نحتفظ بشكل عدم المساواة من القيود. يمثل قيد عدم المساواة في متغيرين نصف مستوي ، وهو مجموعة من النقاط على جانب واحد من الخط ، بما في ذلك النقاط على الخط. نحن نرسم كل هذه الأنصاف التي تم إنشاؤها بواسطة جميع القيود ونأخذ تقاطعها للحصول على مجموعة من الحلول الممكنة (SF). يتم استخدام الخطوتين التاليتين في التقنية الرسومية لحل نموذج LP في متغيرين:

١. تحديد مجموعة الحلول الممكنة (مساحة الحل) التي تفي بجميع القيود المفروضة على متغيرات القرار التي تنطوي عليها المشكلة (بما في ذلك القيود غير السلبية).

٢. تحديد الحل (الحلول) الأمثل بين جميع النقاط في مجموعة الحلول الممكنة. للحصول على الحل الأمثل في الطريقة الرسومية ، يتم استخدام طريقتين: (أ) طريقة نقطة الزاوية و (ب) طريقة الربح من Iso (طريقة خط قيمة Iso).

- **طريقة الحل الاولى – نقطة الزاوية**

نتخذ الخطوات التالية للحصول على الحل الرسومي لـ LPP باستخدام طريقة نقطة الزاوية:

١. رسم عدم المساواة (القيود).

٢. أوجد مجموعة الحلول الممكنة (SF) ، عن طريق أخذ تقاطع التفاوتات.

٣. الحصول على نقاط الزاوية (قمم) SF.

٤. حدد القيمة المثلى لوظيفة الهدف. وهذا يعطي قيمة متغيرات القرار مع القيمة المثلى لوظيفة الهدف.

- **طريقة الحل الثانية – طريقة Iso-الربحية أو طريقة خط Isovalue**

في هذه الطريقة ، يتم اتخاذ الخطوات التالية للحصول على الحل الأمثل لـ LPP:

١. رسم جميع المعوقات التي تنطوي عليها المشكلة.

٢. تحديد مجموعة الحلول الممكنة (SF).

٣. ارسم خطوطًا متوازية بقيم مختلفة للدالة الهدف ، تتحرك في الاتجاه الذي تميل فيه الوظيفة الهدف إلى الزيادة أو النقصان ، وفقًا لما إذا كانت المشكلة من نوع التكبير أو الحد الأدنى.

٤. استمر في رسم الخطوط المتوازية التي تم الحصول عليها بأخذ قيم مختلفة لدالة الهدف في الاتجاه المطلوب ، حتى تصبح الشروط التالية جيدة:

(أ) يمر الخط الأخير من خلال قمة SF ، حيث يتم الحصول على الحل الأمثل وكل حركة أخرى للخط الموازي تدفعه إلى خارج SF.

(ب) يتجاوز الخط المتوازي القمة الأخيرة في الاتجاه المطلوب ولا يزال يحتوي على نقاط SF بدون حدود على قيمة الوظيفة الهدف. وبالتالي ، فإن الحل لا حدود له.

يفضل استخدام طريقة نقطة الزاوية للحصول على الحل الأمثل عندما تكون مجموعة الحلول الممكنة (SF) محدودة.

- **الطريقة الثالثة -حالات خاصة في الطريقة الرسومية**

أثناء حل LPP باستخدام الطريقة الرسومية ، صادفنا بعض المواقف الخاصة حيث تحتوي المشكلة على أكثر من حل أمثل ، أو لا يوجد حل ، أو حتى في بعض الأحيان عدد لا حصر له من الحلول. في هذا القسم ، سنناقش كل هذه السيناريوهات.

١. الحل الأمثل البديل

يقال أن LPP لديها حل مثالي بديل ، إذا كان لديه الحل الأمثل في أكثر من قمة واحدة. في الشكل ٢,١ ، إذا كان الحل الأمثل موجودًا في ذروتين ، على سبيل المثال P1 و P2 ، فسيكون موجودًا في كل نقطة X من جزء الخط الذي ينضم إلى الذروتين P1 و P2. وبالتالي ، سيكون للمشكلة عدد لا نهائي من الحلول.

٢. لا يوجد حل عملي

خذ بعين الاعتبار LPP التالية:

كما هو موضح في الشكل ٢,٢ ، S1 هي مساحة الحل الممكنة للمعادلة. (٢,١) و S2 هي مساحة حل المعادلة. (٢,١). لذلك ، فإن مساحة حل المشكلة ككل هي $SF = S1 \cap S2 = \varphi$. وبالتالي ، بما أن مساحة الحل هي مجموعة فارغة ، فليس للمشكلة حل.

٣. مساحة حل غير محدودة ولكن الحل الأمثل محدود

في الشكل ٢,٣ ، AB هو الخط $x1 + x2 = 3$، BC هو الخط $x1 - 3x2 = 3$ ، و SF هي مجموعة الحلول الممكنة. الآن ، تكمن المشكلة في إيجاد $(X1، X2) \in SF$ الذي يجعل Z كحد أدنى. لهذا ، ضع في اعتبارك الخطوط المتوازية iso-profit المقابلة للقيم المختلفة لـ Z. من الشكل، من الواضح أنه عندما نتحرك في اتجاه الأسهم ، تنخفض قيمة Z. في القمة B (3 ،0)، $Z = 3$. إذا حاولنا تقليل Z بشكل أكبر من خلال التحرك في اتجاه السهم ، فلن يمر خط iso-الربح عبر SF (يجب أن نبقى في SF) وبالتالي ، مزيد من الانخفاض في قيمة Z غير ممكن. وبالتالي ، $Min\ Z = 3$ عند الرأس B عندما $x1 = 3$ و $x2 = 0$.

٤. مساحة حل غير محدودة وحل غير محدود

في الشكل ٢,٤ ، الخط AB هو $x1 + x2 = 3$ ، الخط BC هو $x1 + x2 = 3$ ، الخط CD هو $x1 - 3x2 = 3$ ، و SF هي مجموعة الحلول الممكنة. الآن تكمن المشكلة في إيجاد النقطة $(x1، x2) \in SF$ ، مما يجعل قيمة Z كحد أقصى. لهذا ، ضع في اعتبارك الخطوط المتوازية الاحترافية المقابلة للقيم المختلفة لـ Z. من الشكل ، من الواضح أنه بينما نتحرك في اتجاه الأسهم عبر خطوط الربح المتساوي الأيزو ، فإن قيمة Z تزداد. في القمة B، $Z = 3$ ، وفي C، $Z = 6$. إذا تحركنا أكثر في اتجاه الأسهم ، فإن الخطوط المتوازية المتوازنة وراء القمة C ستمر عبر مجموعة من الحلول الممكنة (SF) ، جعل قيم Z أكبر بدون حد أعلى. وبالتالي ، فإن المحلول غير محدود مع مساحة حل غير محدودة (كما هو واضح في الشكل ٢,٤).

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Chapter Four

Linear Programming Problem IIII—Assignment Problems

4.1 Introduction Assignment Problems

The assignment of jobs (activities) to different people (facilities) can be done if there exists a one-to-one correspondence between them and the efficiency of every person on the job is known. Our objective here is to create an assignment that maximizes the efficiency or minimizes the inefficiency. To accomplish this objective, the Hungarian algorithm for assignment is used.

4.2 Procedure of Method

Example: A company have three works can be completed each of all in three machines as shown in table. The first work has done with three machine by costs 23, 13, 19 thousands IQ, the second work has done with three machines by costs 5, 10, 7 thousand IQ and the third machines by 12, 3, 8 thousand IQ.

Machines / Product	1	2	3
A	23	13	19
B	5	10	7
C	12	3	8

First step for solution

نختار اقل رقم في كل عمود ونطرح منه باقي عناصر العمود نفسه .... نعمل هذا مع كل الاعمدة ..

Machines / Product	1	2	3
A	23-5	13-3	19-7
B	5-5	10-3	7-7
C	12-5	3-3	8-7

Machines / Product	1	2	3
A	18	10	12
B	0	7	0
C	7	0	1

Second step for solution

نختار اقل رقم في كل صف ونطرح منه باقي عناصر الصف نفسه .... نعمل هذا مع كل الصفوف ..

Machines / Product	1	2	3
A	18-10	10-10	12-10
B	0-0	7-0	0-0
C	7-0	0-0	1-0

Product \ Machines	1	2	3
A	8	0	2
B	0	7	0
C	7	0	1

Third step for solution

نرسم خطوط مستقيمة لكي تمر بالاصفار..

Product \ Machines	1	2	3
A	8	0	2
B	0	7	0
C	7	0	1

Fourth step for solution

يتم اختيار اصغررقم(بحيث لا يمر فيه اي مستقيم من المستقيمات المتقاطعة) في المصفوفة بعد التقاطع الخطوط <<< الرقم هو 1 و يطرح الرقم (1) من عناصر المصفوفة التي لا يمر بها خط ويجمع الرقم (1) مع النقطة التي تتقاطع الخطوط بها لتصبح ..

Product \ Machines	1	2	3
A	7	0	1
B	0	8	0
C	6	0	0

Fifth step for solution

بعد ذلك ولاختيار الحل الامثل في طريقة التخصيص نضع هذه المصوفة التي حصلنا عليها بالمقارنة مع المصفوفة الرئيسية قبل اي شي ونقاطع اي العمود الاول قيمة الصفر فيه تقابل بالمصفوفة الرئيسية 5 والصفر الاول في العمود الثاني يقابل 13 في المصفوفة الرئيسية

Machines / Product	1	2	3
A	7	0	1
B	0	8	0
C	6	0	0

لذلك فأن التخصيص الامثل للمعدات هي كما يلي

A	13	ويكون افضل عمل على الماكنة الثانية
B	5	ويكون افضل عمل على الماكنة الاولى
C	8	ويكون افضل عمل على الماكنة الثالثة

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Chapter Three- 2

Linear Programming Problem III–Simplex Method

3.1 Introduction Simplex Method

In Chapter 2, we discussed the graphical method for solving an LPP, which was limited to two-variable problems only. We observed that the optimum value of the objective function of an LPP occurs at the corner points of the feasible region (the region where the constraints and non-negativity restrictions hold well), provided the feasible region is bounded. In the simplex method, we start with a vertex (corner point of the feasible region) and move to another vertex in a prescribed manner, so that the value of the objective function improves. In this way, we reach the vertex that gives the optimum value in a few steps. This method, developed by Danzig, is such that we omit many vertices of SF while going from one vertex to the other and then terminates in a few steps.

3.2 Standard form of linear Programming Problem

An LPP is said to be in standard form, if the following four conditions are satisfied:

1. The objective function must be either maximization or minimization.
2. All the constraints should hold with an equality sign.
3. The right-hand side (RHS) of all constraints is non-negative.
4. All the variables involved in the problem are non-negative.

3.3 Procedure of Method

Solve the following LPPs using Simplex M.

$$\text{Max } Z = 5x_1 + 7x_2$$

Subjected to

$$x_1 + x_2 \leq 4$$
$$3x_1 - 8x_2 \leq 24$$
$$10x_1 + 7x_2 \leq 35$$
$$x_1, x_2 \geq 0$$

Solution.

Find x_1, x_2, x_3, x_4, x_5

تم فرض ثلاث متغيرات
لوجود ثلاث معادلات .

$$Z = 5x_1 + 7x_2 + 0(x_3 + x_4 + x_5)$$

a_{ij}

$$x_1 + x_2 + x_3 + 0(x_4 + x_5) = 4$$
$$3x_1 - 8x_2 + 0x_3 + x_4 + 0x_5 = 24$$
$$10x_1 + 7x_2 + x_5 + 0x_3 + 0x_4 = 35$$

b_{ij}

P-1

x_1	x_2 (a_{ij})	x_3	x_4	x_5	b_{ij}	b_{ij}/a_{ij}
1	1	1	0	0	4	$4/1 = 4$
3	-8	0	1	0	24	$24/-8 = -3$
10	7	0	0	1	35	$35/7 = 5$
5	7	0	0	0		

الصف الأمثل

العمود الأمثل

∵ المسألة ايجاد الربح (Max) ⟸ ① اختيار العمود الأمثل بالاعتماد على أعلى رقم في معادلة الربح

② ثم اختيار الصف الأمثل بالاعتماد على أقل قيمة بالصف الأمثل .

الخطوة التالية

نطرح الصفوف من الصف الأمثل .

$1-3$	$1+8$	$1-0$	$0-1$	$0-0$	$4-24$
-2	9	1	-1	0	-20

الصف الأول الجديد

$10-3$	$7+8$	$0-0$	$0-1$	$1-0$	$35-24$
7	15	0	-1	1	11

الصف الثالث الجديد

P-2

5+3	7+8	0-0	0-1	0-0
2	15	0	-1	0

صف معادلة الهدف

x_1	x_2	x_3	x_4	x_5	المصفوفة الجديدة b_{ij}
-2	9	1	-1	0	-20
3	-8	0	1	0	24
7	15	0	-1	1	11
2	15	0	-1	0	

وصلنا إلى الخطوة الأخيرة للحل

وهي كيف يتم الحصول على قيم x_2 و x_1 من خلال

أن تكون قيمها بالمصفوفة مساوية إلى 1

وتأكد الحل؟ اختبار رياضي بسيط

$Z = 13$

P-3

Solve Using Simplex M.

مسألة تكاليف

Min $Z = x_1 + x_2 + 3x_3$

Subj to $3x_1 + 2x_2 + x_3 \leq 3$

$2x_1 + x_2 + 2x_3 \leq 2$

Sol. after let. x_4, x_5

$Z = x_1 + x_2 + 3x_3 + 0(x_4 + x_5)$

$3x_1 + 2x_2 + x_3 + x_4 + 0x_5 = 3$

$2x_1 + x_2 + 2x_3 + 0x_4 + x_5 = 2$

* عمود الامثل في هذا الحل

العمود الامثل

or

تكاليف

x_1	x_2	x_3	x_4	x_5	b_{ij}	b_{ij}/a_{ij}
3	2	1	1	0	3	1
2	1	2	0	1	2	1
	العمود الامثل				الصف الامثل	
1	1	3	0	0		

* اختيار العمود الامثل يعتمد على اقل قيمة لكون المسألة تكاليف

* [illegible] تم اختيار x العمود الامثل

P-4

بعد أكمال المصفوفة الجديدة لتصبح

العمود الأمثل أو اما

x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
1	1	-1	1	-1	3	3
2	1	2	0	1	2	1
-1	1	3	0	0		

بعدها يكون العمود الأمثل هو أقل قيمة بالنسبة للتكاليف

$$x_1 = x_4 = 1$$
$$x_2 = x_5 = 1$$

$$Z = 1 + 1 + 3(-1) = 2 - 3 = -1$$

Cost

P-5.