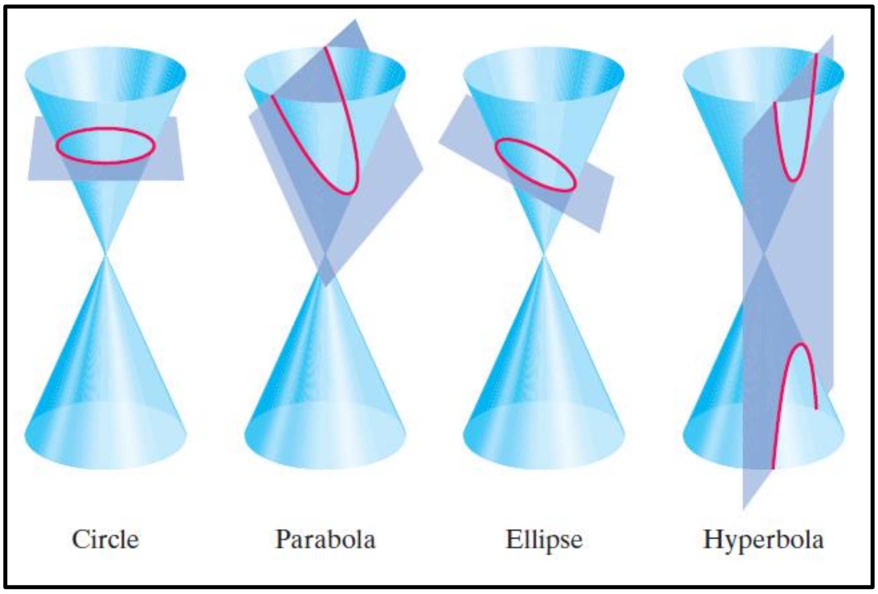
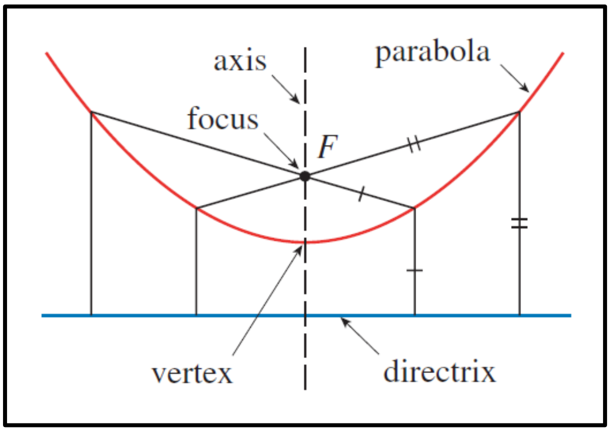
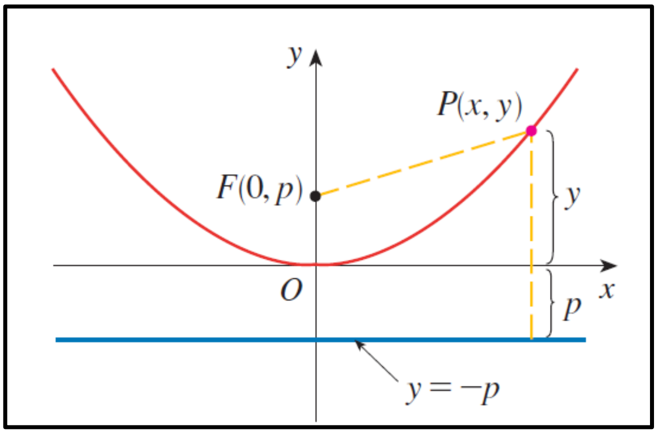
Conic sections are the curves obtained by intersecting a plane and a right circular cone. A plane perpendicular to the cone's axis cuts out a **circle**; a plane parallel to a side of the cone produces a **parabola**; a plane at an arbitrary angle to the axis of the cone forms an **ellipse**; and a plane parallel to the axis cuts out a **hyperbola**. If we extend the cone through its vertex and form a second cone, you find the second branch of the hyperbola. All these curves can be described as graphs of second-degree equations in two variables.



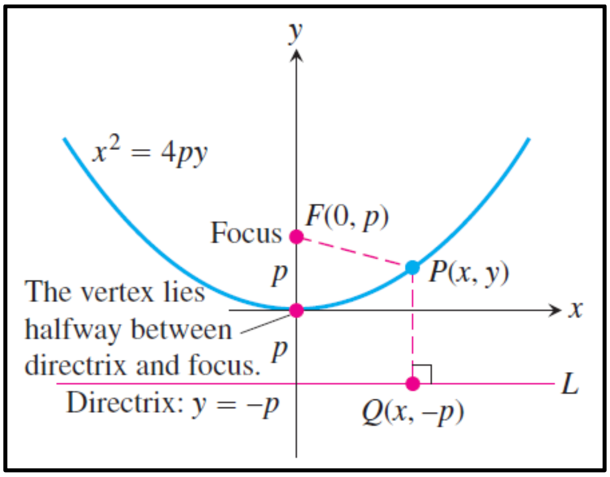
parabola

1. A **parabola** is defined as the set of all points in a plane that are equidistant from a line and a point not on the line. The line is called the **directrix** and the point is called the **focus** (plural, *foci*).





A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point F(0, p) on the positive y-axis and that the directrix is the line (y = - p)



In the notation of the figure, a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$ From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

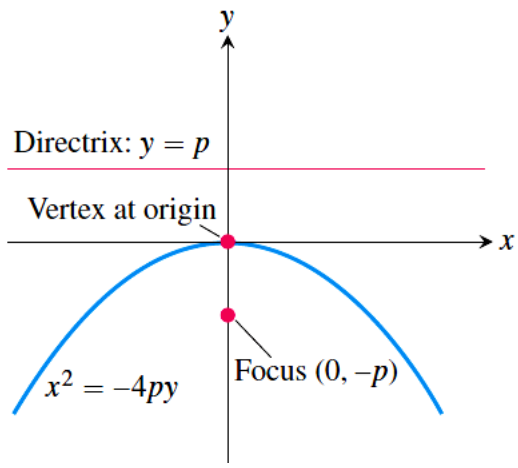
When we equate these expressions, square, and simplify, we get

$$\boxed{y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py}$$ **standard form** (1)

These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the **axis** of the parabola (short for "axis of symmetry"). The point where a parabola crosses its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin. The positive number p is the parabola's **focal length.**

If the parabola opens downward, with its focus at (0, - p) and its directrix the line y = p. Then Eq.(1) become

$$\boxed{y = -\frac{x^2}{4p} \quad \text{and} \quad x^2 = -4py}$$



The parabola $x^2 = -4py$, $p > 0$.

We can obtain similar equations for parabolas opening to the right or to the left

Standard-form equations for parabolas with vertices at the origin (p > 0)

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0, p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0, -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left

a) The parabola $y^2 = 4px$ (b) The parabola $y^2 = -4px$.

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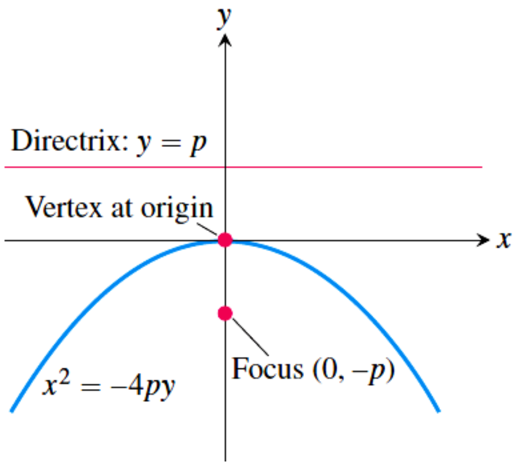

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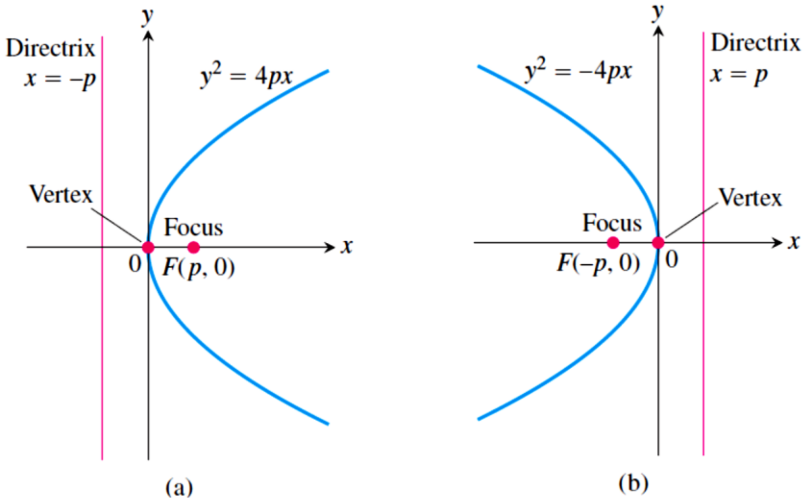
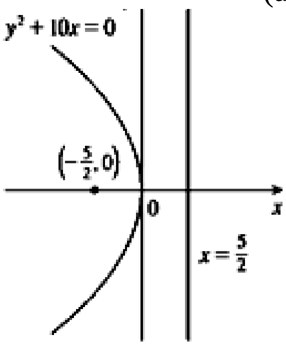


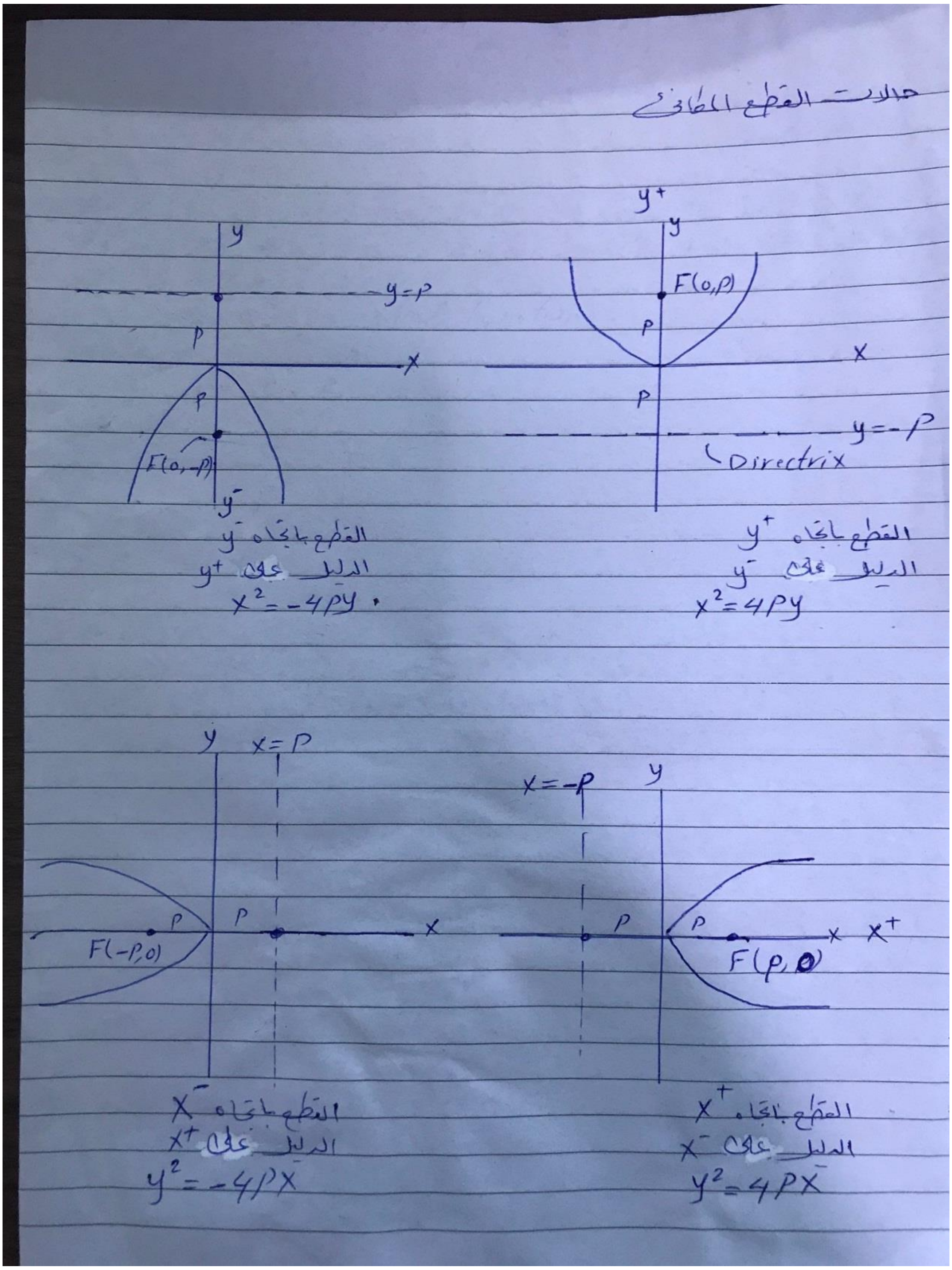
(a) The parabola $y^2 = 4px$ (b) The parabola $y^2 = -4px$.



which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the x-axis, which is the axis of the parabola.

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

SOLUTION If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = \left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5.



1.1 Writing Quadratic Equations in the Form y = a(x - h)^2 + k

The standard form of a quadratic function is $\boldsymbol{y = ax^2 + bx + c}$ In this section, we will write the quadratic equation in the form $\boldsymbol{y = a(x - h)^2 + k}$ where a, h, and k are real numbers. To do that, we will need to complete the square

➔ To write a quadratic equation in the form $\boldsymbol{y = a(x - h)^2 + k}$

1) Isolate the x-terms to one side of the equation.
2) Factor out the leading coefficient.
3) Add the value needed to complete the square to both sides of the equation.
4) Rewrite the trinomial as a binomial squared.
5) Solve the equation for y.

SUMMARY OF THE EFFECTS OF THE REAL NUMBERS a, h, AND k OF A QUADRATIC EQUATION ON A VERTICAL PARABOLA

The real numbers a, h, and k of a quadratic equation in the form $y = a(x - h)^2 + k$ affect the graph of the equation.

If $a > 0$, then the graph is concave upward (opens upward).
If $a < 0$, then the graph is concave downward (opens downward).
If $|a| > 1$, then the graph is narrower than it would be if $a = 1$.
If $|a| < 1$, then the graph is wider than it would be if $a = 1$.

The vertex of the graph is (h, k).
The axis of symmetry is the line graphed by $x = h$.

EXAMPLE 1 Write the quadratic equations in the form $y = a(x - h)^2 + k$. Identify a, h, and k.

a. $y = x^2 - 4x + 7$ **b.** $y = -2x^2 - 16x - 35$

Solution

a.

$y = x^2 - 4x + 7$

$y - 7 = x^2 - 4x$ Isolate the x-terms.

$y - 7 + 4 = x^2 - 4x + 4$ Add $(\frac{-4}{2})^2 = 4$ to both sides.

$y - 3 = (x - 2)^2$ Write the trinomial as a binomial squared.

$y = (x - 2)^2 + 3$ Solve for y.

In the equation $y = (x - 2)^2 + 3$, $a = 1$, $h = 2$, and $k = 3$.

b.

$y = -2x^2 - 16x - 35$

$y + 35 = -2x^2 - 16x$ Isolate the x-terms.

$y + 35 = -2(x^2 + 8x)$ Factor out the leading coefficient, -2.

$y + 35 + [-2(16)] = -2(x^2 + 8x + 16)$ Add $-2(\frac{8}{2})^2$, or $-2(16)$, to both sides.

$y + 3 = -2(x + 4)^2$ Write the trinomial as a binomial squared.

$y = -2(x + 4)^2 - 3$ Solve for y.

In the equation $y = -2(x + 4)^2 - 3$, or $y = -2[x - (-4)]^2 + (-3)$, $a = -2$, $h = -4$, and $k = -3$.

EXAMPLE 2 Determine the vertex and axis of symmetry for the graph of each equation. Describe the graph, but do not draw it.

a. $y = 2(x - 4)^2 - 3$ **b.** $y = -x^2 - 4x - 8$

Solution

a. $y = 2(x - 4)^2 - 3$

or $y = 2(x - 4)^2 + (-3)$ Write the equation in $y = a(x - h)^2 + k$ form.

We see that $a = 2$, $h = 4$, and $k = -3$.

Since $a = 2$ and $2 > 0$, the graph is concave upward.

Since $|2| = 2 > 1$, then the graph is narrower than it would be if $a = 1$.

The vertex is (h, k), or $(4, -3)$.

The axis of symmetry is the graph of $x = 4$.

b. $y = -x^2 - 4x - 8$

First, write the equation in the form $y = a(x - h)^2 + k$.

$y + 8 = -x^2 - 4x$ Isolate the x-terms.

$y + 8 = -1(x^2 + 4x)$ Factor out the leading coefficient.

$y + 8 + [-1(4)] = -1(x^2 + 4x + 4)$ Add $-1(\frac{4}{2})^2$, or $-1(4)$, to both sides.

$y + 4 = -1(x + 2)^2$ Write the trinomial as a binomial squared.

$y = -1(x + 2)^2 - 4$ Solve for y.

or $y = -1[x - (-2)]^2 + (-4)$

Therefore, $a = -1$, $h = -2$, and $k = -4$.

Since $a = -1$ and $-1 < 0$, the graph is concave downward.

Since $|-1| = 1$, the graph is the same width as it would be if $a = 1$.

The vertex is (h, k), or $(-2, -4)$.

The axis of symmetry is the graph of $x = -2$.

Hyperbola

A hyperbola is the set of all Points in a plane such that the absolute value of the difference of the distance between two fixed points stays Constant.

Each fixed Point is called a focus. and the Point midway between the foci is called the centre. The Line Containing the Foci is the transverse axis. The graph is made up of two Parts Called branches. Each branch intersects the transverse axis at a point Called the vertex. , figure 1, figure 2

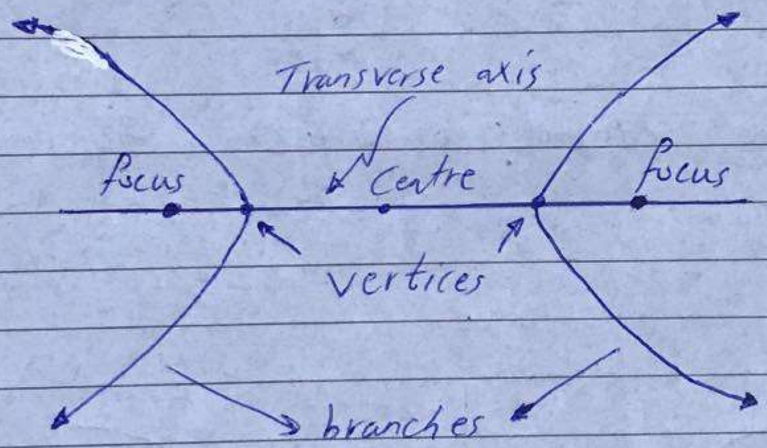


Fig 1

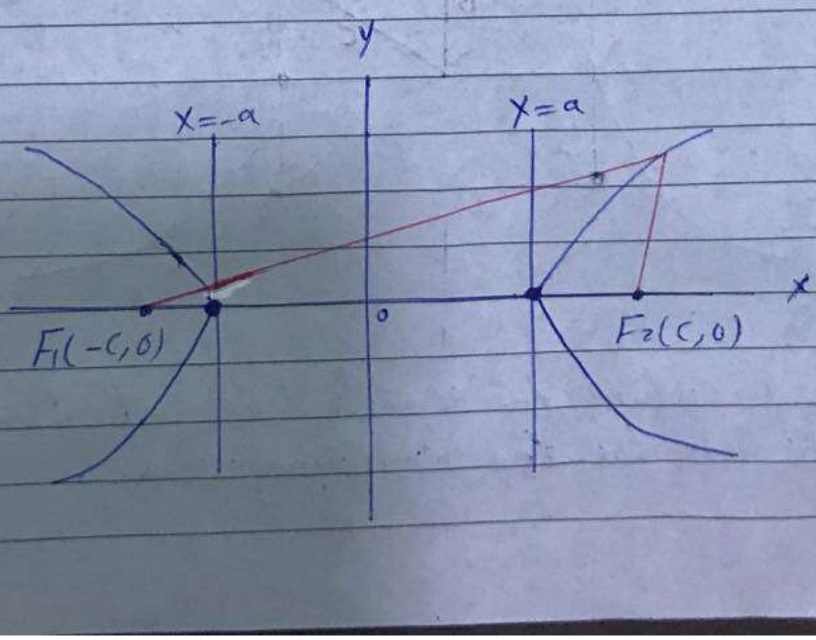


Fig 2

It is traditional in the study of hyperbolas to denote the distance between the vertices by 2a, the distance between the focily 2C, Figure 3, and to define the quantity b as $b=\sqrt{c^2-a^2}$
This relationship, which can also be expressed as

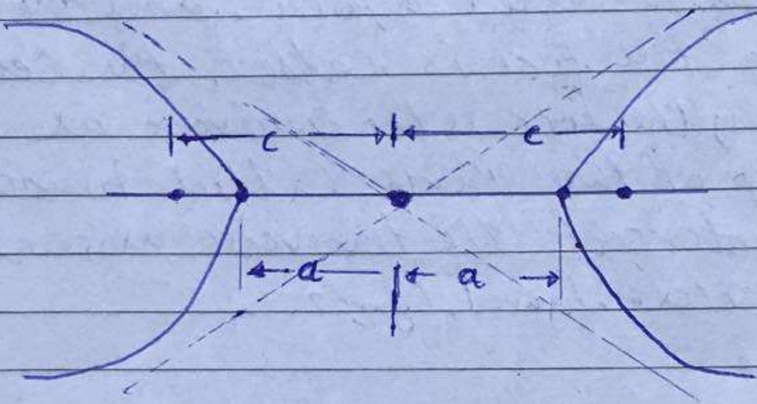


Fig 3

$C=\sqrt{a^2+b^2}$ is pictured geometrically in figure 4

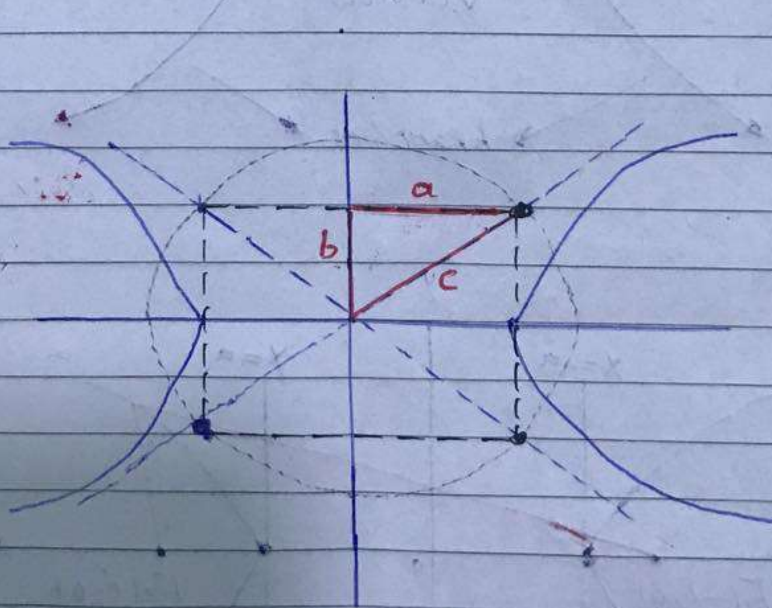
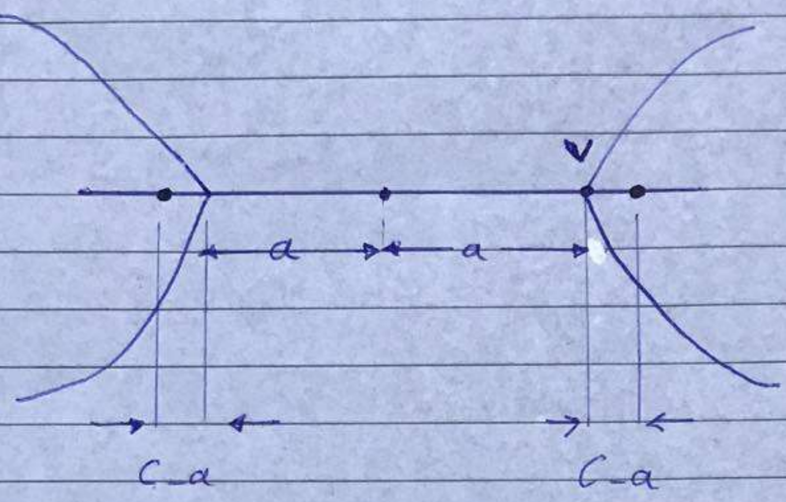


Fig 4

The number a is called the semifocal axis of the hyperbola and the number b the the Semiconjugate axis.

If V is one vertex of hyperbola, then as illustrated in figure 5

Fig 5



The distance from V to the farther focus minus the distance from V to the closer focus is

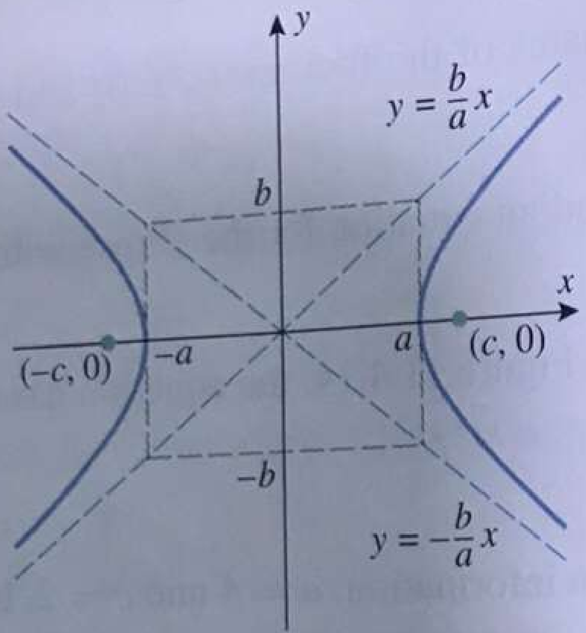
$$[(C-a)+2a]-(C-a)=2a$$

Thus, for all points on a hyperbola, the distance to the farther focus minus the distance to the closer focus is 2a

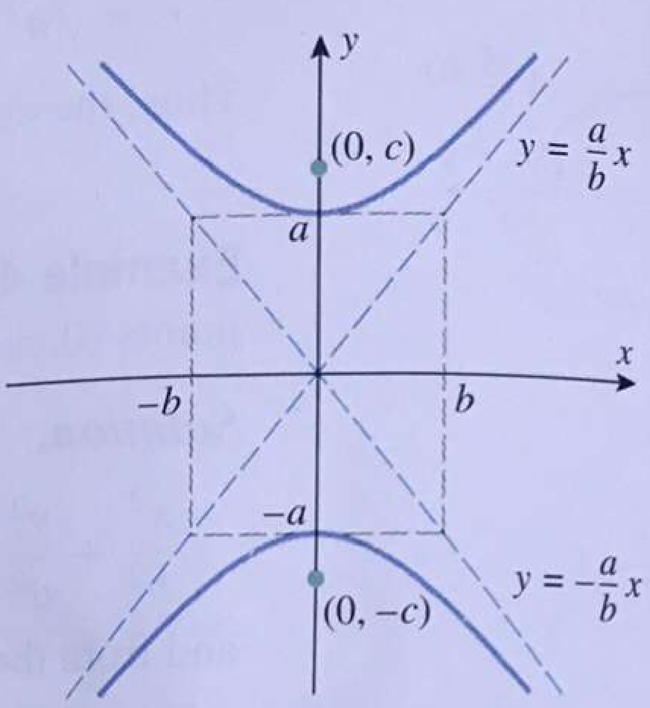
The equation of a hyperbola is simplest if the center of the hyperbola is at the origin and the foci are on the x-axis or y-axis.

The two possible such orientations are shown in figure 6. These are called the standard positions of a hyperbola, and the resulting equations are called the standard equations of a hyperbola.

HYPERBOLAS IN STANDARD POSITION



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Standard form equations for hyperbolas centered at the origin

Foci on the x-axis

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Centre to focus distance

$$c=\sqrt{a^2+b^2}$$

Foci $(\mp c, 0)$

Vertices $(\mp a, 0)$

Asymtotes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

or $y=\pm\frac{b}{a}x$

Foci on the y axis

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Centre to focus distance

$$c=\sqrt{a^2+b^2}$$

Foci $(0, \mp c)$

Vertices $(0, \mp a)$

Asymtotes

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=0$$

or $y=\pm\frac{a}{b}x$

Asymtotes

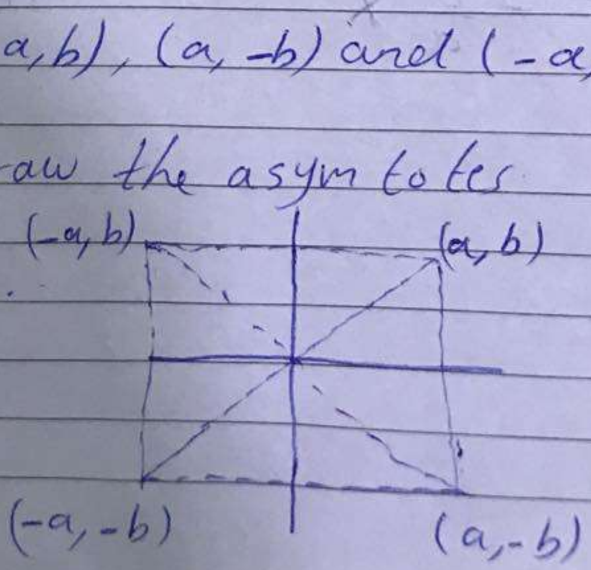
To graph a hyperbolas asymptotes, we need to draw a central rectangular with corners at (a,b), $(-a,b)$ $(a,-b)$ and $(-a,-b)$. The asymptotes are extended diagonals of this rectangle. Note the center is the intersection of the diagonals.

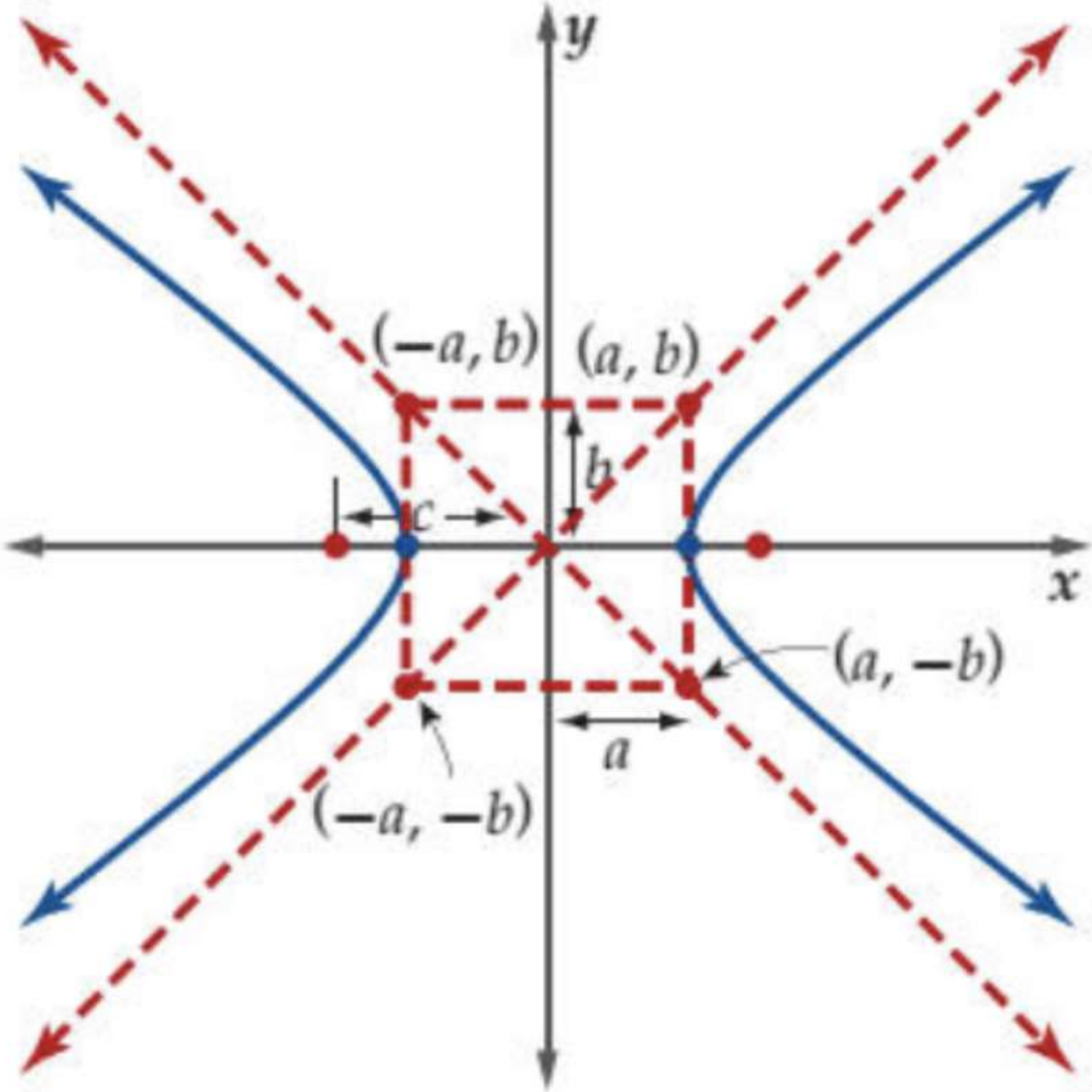
The equation of the asymptotes are

$$y = \pm \frac{b}{a} x$$

To graph a hyperbola with the centre at the origin:-

1. Find a, b, and c
2. With a dashed line, draw a central rectangle with corners of (a,b), $(-a,b)$, $(a,-b)$ and $(-a,-b)$
3. With a dashed line, draw the asymtotes
4. sketch the hyperbola.





y
x
(−a, b)
(a, b)
b
c
(a, −b)
a
(−a, −b)

Ex. 1 sketch the graphs of the equations:-

a. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$a = 5$, $b = 3$

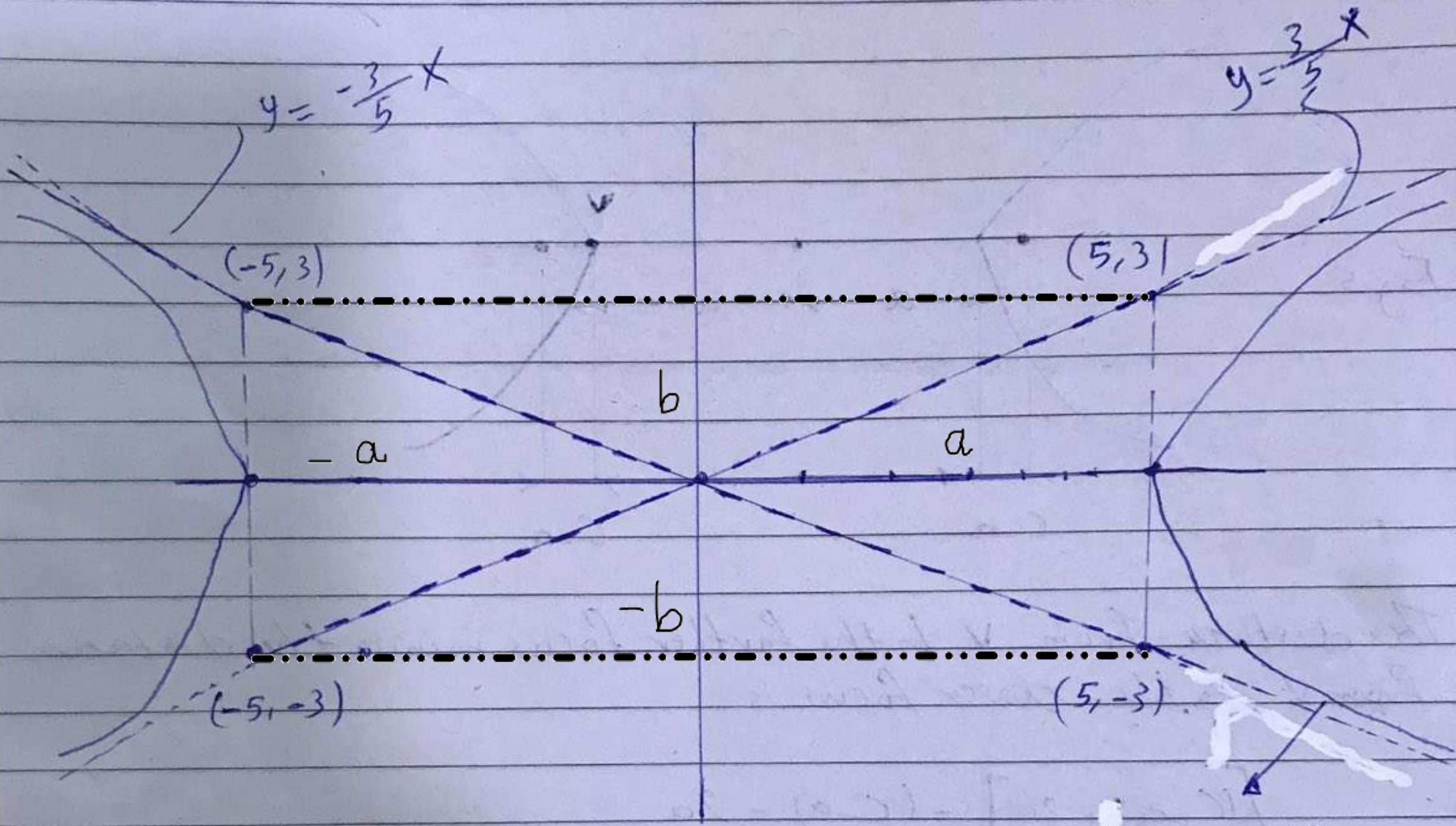
$c = \sqrt{a^2 + b^2}$ $\quad c = \sqrt{25 + 9} = \sqrt{34}$

Find (a, b) $(-a, b)$ $(a, -b)$ & $(-a, -b)$
$(5, 3)$ $(-5, 3)$ $(5, -3)$ & $(-5, -3)$
to draw the rectangle

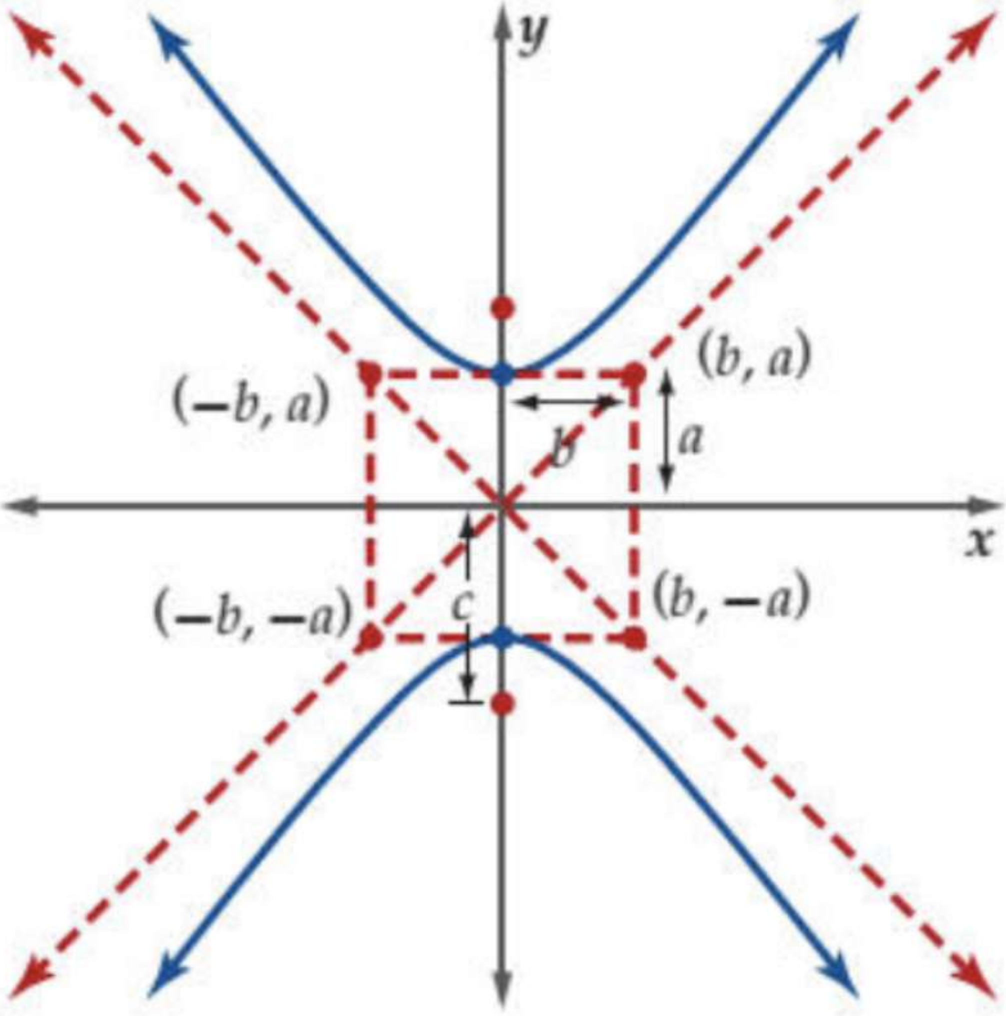
Draw the asymtotes

sketch the hyperbola

5



$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$



y
x
(b, a)
(−b, a)
a
b
c
(−b, −a)
(b, −a)

7

sketch the graph of the equation:

$-49x^2 + 36y^2 = 1764$

$\frac{-x^2}{36} + \frac{y^2}{49} = 1$

$\frac{y^2}{49} - \frac{x^2}{36} = 1.0$

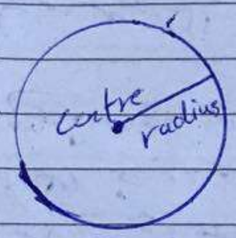
① Find a, b & c

$a = 7, \quad b = 6 \qquad c = \sqrt{a^2+b^2} = \sqrt{49+36} = 9.2$

Circle

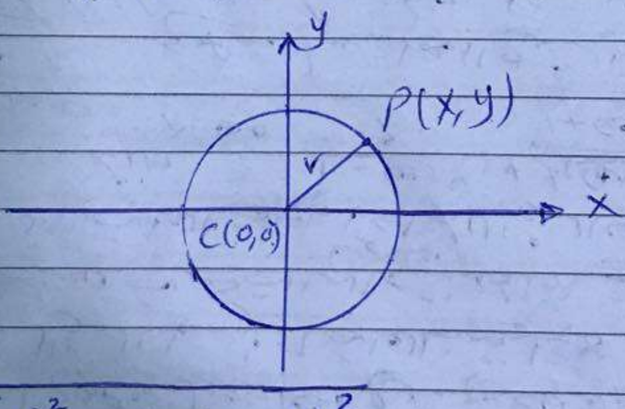
A circle is the set of Points in a plane that are equidistant from a given Point, called the centre

The radius (plural, radii) of the circle is the distance between each of its Points and the centre.



Graphing circles with the Centre at the origin

The equation of the distance can be used to determine the equation of the circle



$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$r = \sqrt{(x - 0)^2 + (y - 0)^2}$$

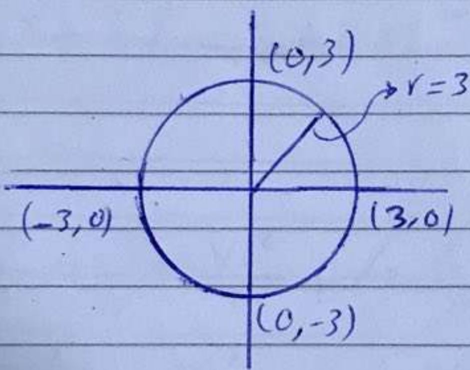
$$r^2 = x^2 + y^2$$

To graph a circle with the centre at the origin

- The origin of the circle is $(0;0)$
- Find the two x-intercepts $(r,0)$ and $(-r,0)$
- Find the two y-intercepts $(0,r)$ and $(0,-r)$
- sketch the circle

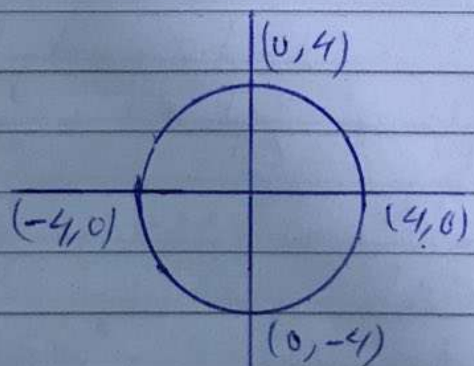
Ex 1

$x^2+y^2=9$



$x^2+y^2=9$

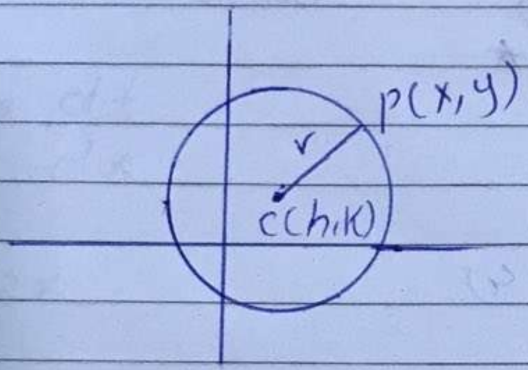
(b) $x^2+y^2=16$



$x^2+y^2=16$

Graphing circles with the Centre not at the origin

The distance formula can be use to determine the equation of the circle



$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

centre (0,0)

$$\boxed{r^2 = (x-h)^2 + (y-k)^2}$$

$$r^2 = x^2 + y^2$$

To graph a circle with its Centre not at the origin

Find h and K

Find $(h+r, K)$, $(h-r, K)$, $(h, K+r)$ & $(h, K-r)$

Draw the graph

④

Ex sketch the graph of the below equations

ⓐ $(x-3)^2 + (y-1)^2 = 4$

$(x-3)^2 + (y-1)^2 = 4$ The equation is similar to the formula

$(x-h)^2 + (y-k)^2 = r^2$

$r^2 = 4$, $r = 2$

Find h and k $\rightarrow h = 3$, & $k = 1$

find $(h+r, k)$ $(h-r, k)$ $(h, k+r)$ & $(h, k-r)$

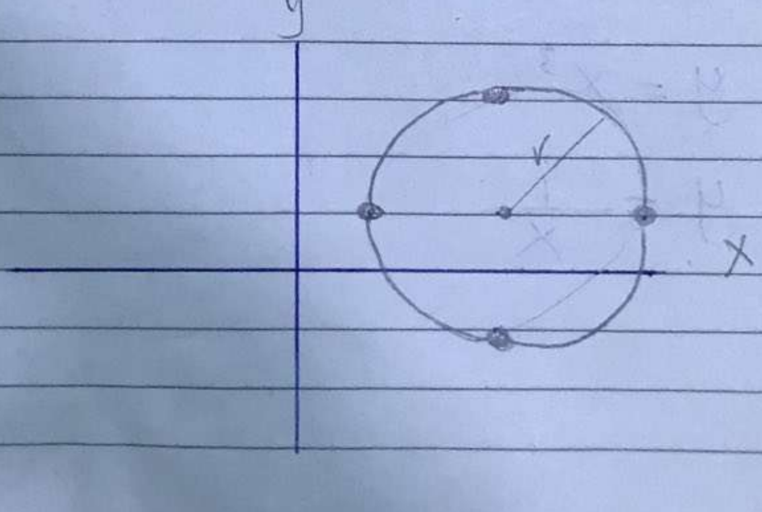
$(h+r, k) \rightarrow (5, 1)$

$(h-r, k) \rightarrow (1, 1)$

$(h, k+r) \rightarrow (3, 3)$

$(h, k-r) \rightarrow (3, -1)$

Draw the graph



If we have the Centre and (r), the equation can be found

Ex Write an equation of each circle if

(a) Centre at the origin and radius 7

Centre at the origin, then the formula is

$$x^2+y^2=r^2$$

$$x^2+y^2=7^2$$

$$x^2+y^2=49$$

(b) Centre at $(-1,3)$ & radius 4

The formula is

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-1))^2+(y-3)^2=4^2$$

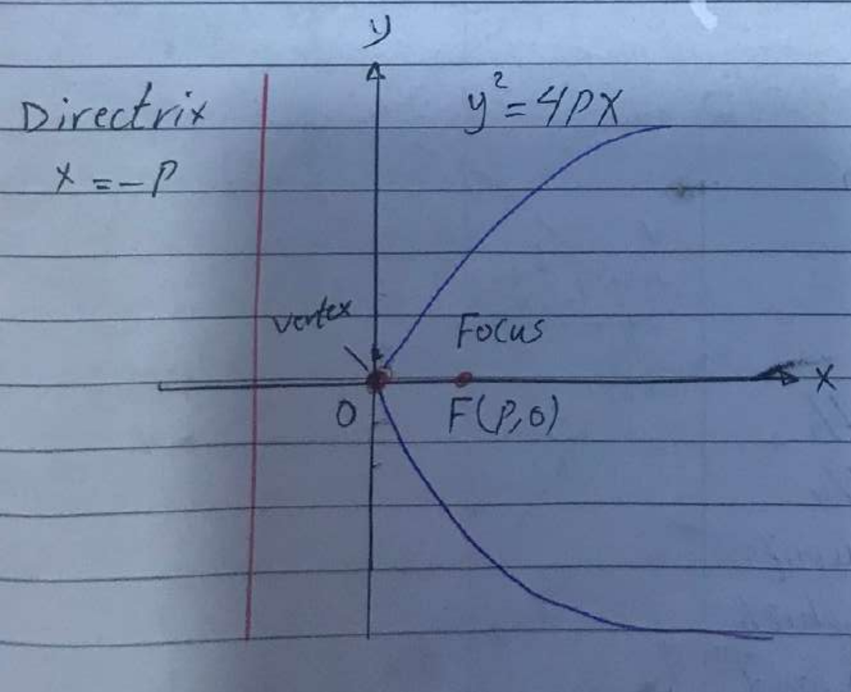
$$(x+1)^2+(y-3)^2=16$$

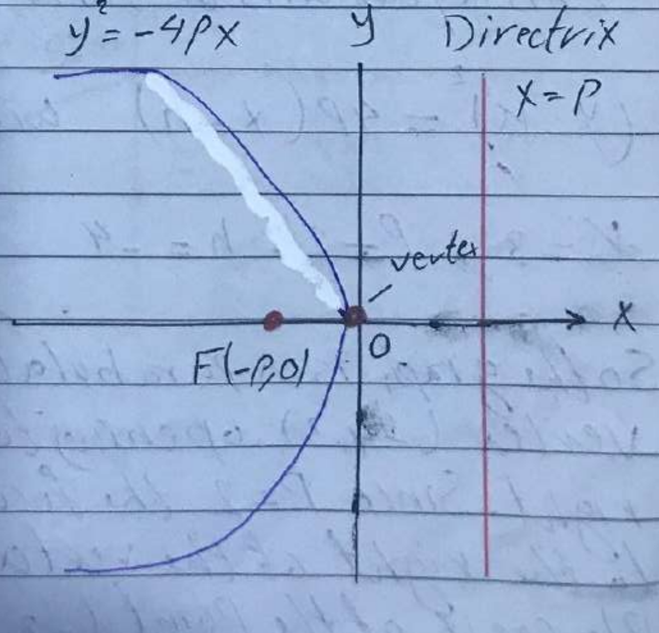
Translated Conics

Equations of conics that are translated from their standard positions can be obtained by replacing x by $x-h$ and y by $y-K$ in their standard equations. For a parabola this translates the vertex from the origin to the point (h,K), and for ellipses and hyperbolas this translates the centre from the origin to the point (h,K).

Parabolas with vertex (h,K) and axis parallel to x-axis :-

$(y-K)^2 = 4p(x-h)$	opens right
$(y-K)^2 = -4p(x-h)$	opens left





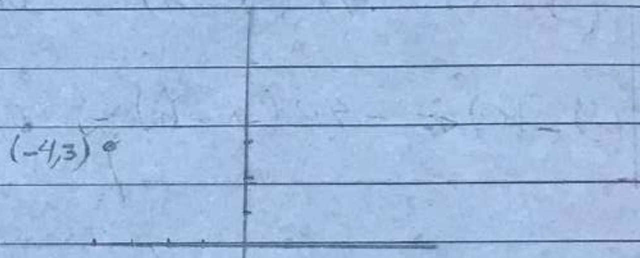
Ex Describe the graph of the equation

$$y^2 - 8x - 6y - 23 = 0$$

Solution:- The equation involves quadratic terms in y but none in x, so we first take all of the y-terms to one side:-

$$y^2 - 6y = 8x + 23$$

Next, we complete the square on the y-terms by adding 9 to both sides:-



$$y^2 - 6y + 9 = 8x + 23 + 9$$

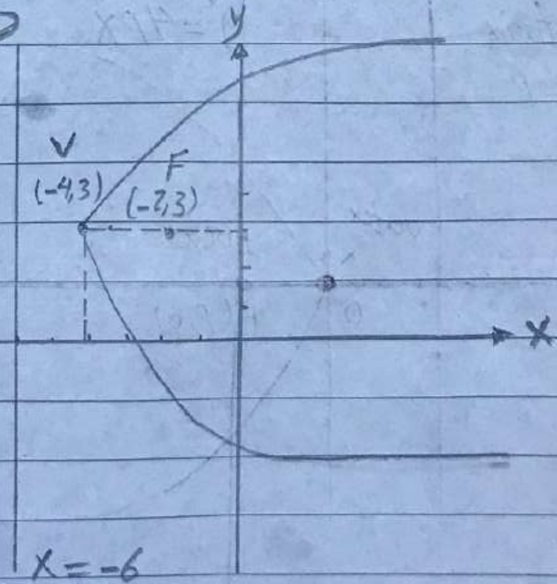
$$(y-3)^2 = 8x + 32$$

$$(y-3)^2 = 8(x+4)$$

This equation is of form

$$(y-k)^2 = 4P(x-h) \quad \text{with}$$

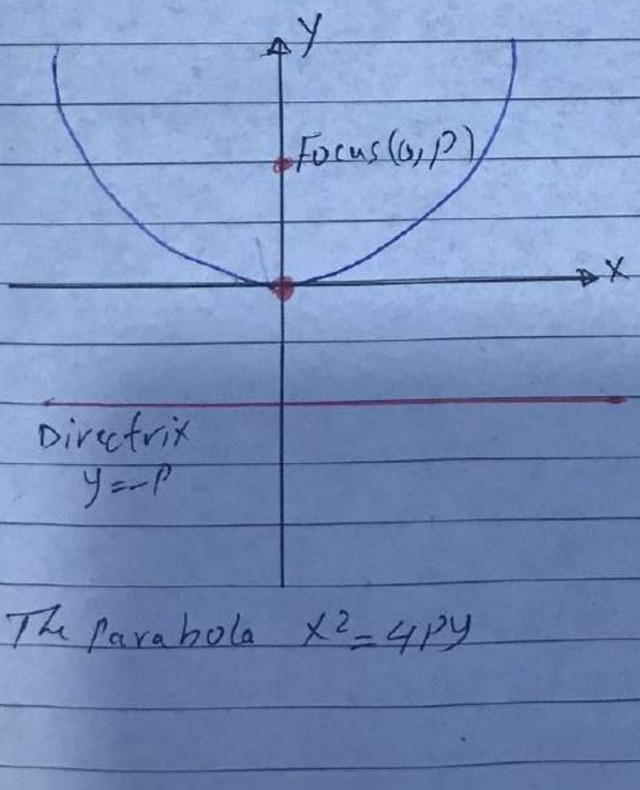
$$k = 3 \,,\; P = 2 \,,\; h = -4$$



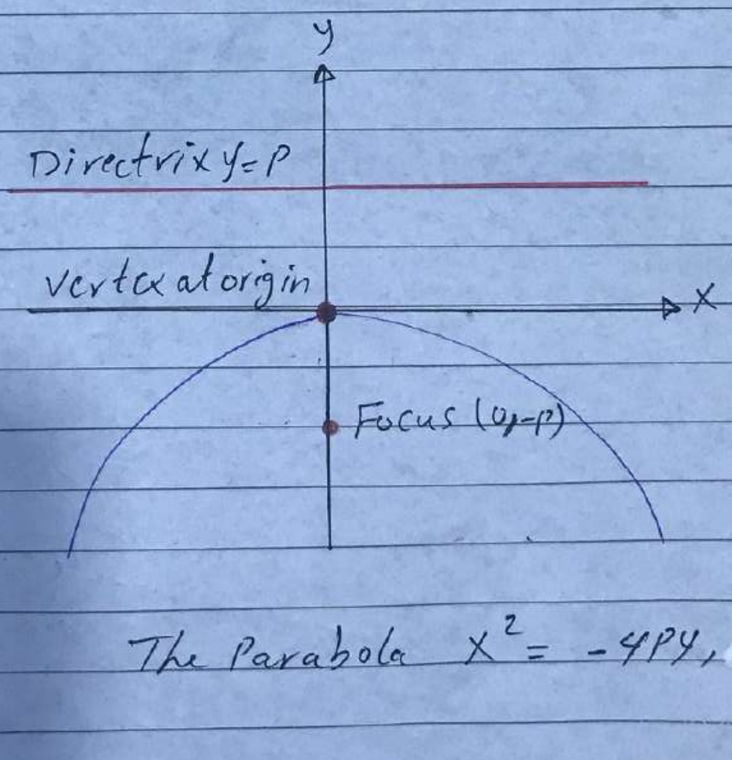
So the graph is a parabola with vertex $(-4,3)$, opening to the right. Since $P=2$, the focus units to the right of the vertex, which places it at the Point $(-2,3)$, and the directrix is 2 unit to the left of the vertex which means that its equation is $x = -6$

Parabolas with vertex (h, k) and axis parallel to y-axis

$$\boxed{\begin{array}{ll} (x-h)^2 = 4p(y-k) & \text{opens up} \\ (x-h)^2 = -4p(y-k) & \text{opens down} \end{array}}$$



The parabola $x^2 = 4py$



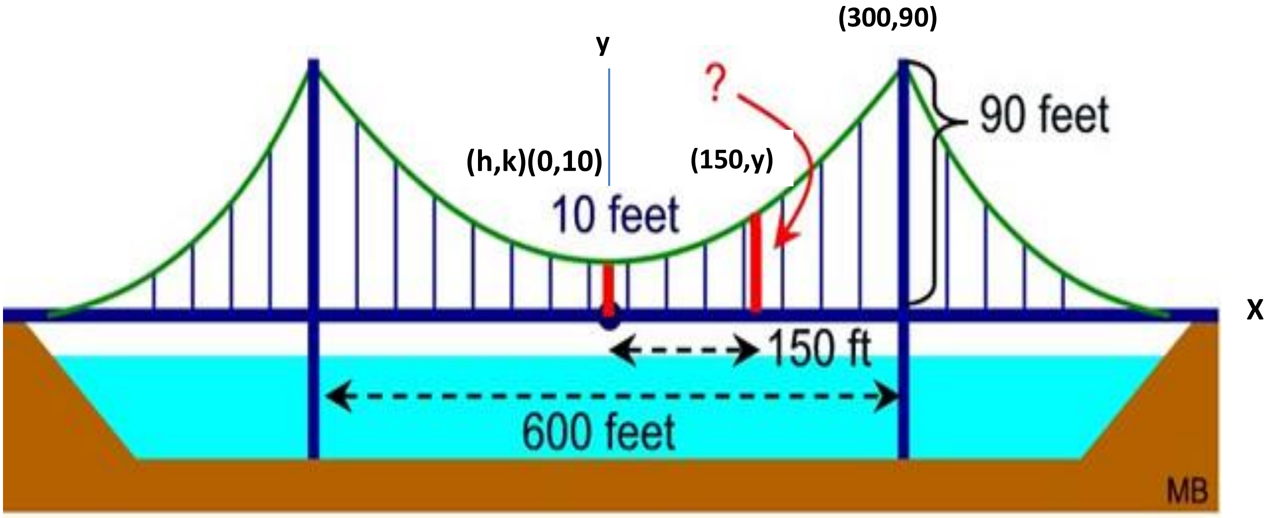
The parabola $x^2 = -4py$,

Application of Parabola in real life

An arch bridge: a parabola represents the profile of the supporting structure of an arch bridge. This concrete bridge transfers its weight horizontally into abutments

A suspension bridge: a parabola represents the profile of the cable of a suspended-deck suspension bridge. The curve of the cable created by the chains follows the curve of a parabola.

Example: The cables of a suspended-deck suspension bridge are in the shape of a parabola. The pillars supporting the cable are 600 feet apart and rise 90 feet above the road. The lowest height of the cable, which is 10 feet above the road, is reached halfway between the pillars. What is the height of the cable from the road at a point 150 feet (horizontally) from the center of the bridge?



ANSWER: Let the *x*-axis be the road and the *y*-axis be the location halfway between the pillars. The vertex of the parabola will be (0,10). The point on the pillar (300,90) lies on the parabola. We need an equation for the parabola to find (150,*y*) on the parabola.

Coordinate of vertex(h,k) (0,10)-Coordinate of pillar (x,y)(300,90)

$(x-h)^2 = 4p(y-k)$

$(x-0)^2=4p(y-10)$

$(300-0)^2=4p(90-10)$

$(300)^2=4p(90-10)$

$p=281.25$

Coordinate of vertex(h,k) (0,10)

Needed coordinate (x,y) (150,y)

$(x-h)^2 = 4p(y-k)$

$(x-0)^2=4p(y-10)$

$(x-0)^2=4(281.25)(y-10) \longrightarrow (150-0)^2=4(281.25)(y-10) \longrightarrow (150)^2=1125(y-10)$

$22500=1125y-11250$

Y=30 feet

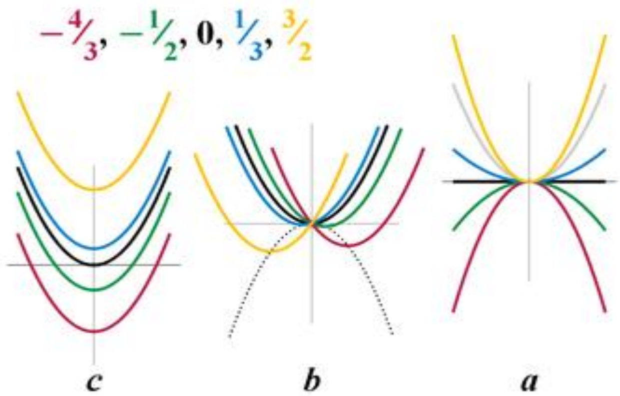
المعادلة التربيعية

المعادلة التربيعية (Quadratic equation) هي معادلة جبرية احادية المتغير من الدرجة الثانية تكتب وفق الصيغة العامة:

$$ax^2+bx+c=0$$

حيث يمثل x المجهول او المتغير اما ال a,b,c فيطلق عليها الثوابت او المعاملات

يطلق على a العامل الرئيسي وعلى c الحد الثابت ويشترط ان يكون $a \neq 0$ اما اذا كان $a=0$ عندها تصبح المعادلة معادلة خطية



رسم تخطيطي للدالة التربيعية $ax^2 + bx + c$. في كل مرة نقوم بتغيير قيمة أحد معاملات الدالة (بينما يكون المعاملان الآخران ثابتين) نلاحظ تغير المنحنى البياني فيها

تم إيجاد حلول (أو جذور) المعادلة التربيعية باستعمال عدة طرق: باستعمال الصيغة التربيعية أو طريقة إكمال المربع أو طريقة حساب المميز أو طريقة الرسم البياني

طريقة الرسم البياني

الدوال على الشكل $ax^2 + bx + c$ تسمى دوال تربيعية. جميع الدوال التربيعية لها شكل عام متشابه يسمى القطع المكافئ، موقع وحجم المقطع يرتبط بالقيم a,b,c

Writing Quadratic Equations in the Form y = a(x - h)^2 + k

The standard form of a quadratic function is $\boldsymbol{y = ax^2 + bx + c}$ In this section, we will write the quadratic equation in the form ***y = a(x - h)^2 + k*** where a, h, and k are real numbers. To do that, we will need to complete the square

➔ To write a quadratic equation in the form ***y = a(x - h)^2 + k***

1) Isolate the *x*-terms to one side of the equation.
2) Factor out the leading coefficient.
3) Add the value needed to complete the square to both sides of the equation.
4) Rewrite the trinomial as a binomial squared.
5) Solve the equation for *y*.

SUMMARY OF THE EFFECTS OF THE REAL NUMBERS *a*, *h*, AND *k* OF A QUADRATIC EQUATION ON A VERTICAL PARABOLA

The real numbers $a, h,$ and k of a quadratic equation in the form $y = a(x - h)^2 + k$ affect the graph of the equation.

If $a > 0$, then the graph is concave upward (opens upward).
If $a < 0$, then the graph is concave downward (opens downward).
If $|a| > 1$, then the graph is narrower than it would be if $a = 1$.
If $|a| < 1$, then the graph is wider than it would be if $a = 1$.

The vertex of the graph is (h, k).
The axis of symmetry is the line graphed by $x = h$.

EXAMPLE 1 Write the quadratic equations in the form $y = a(x - h)^2 + k$. Identify a, h, and k.

a. $y = x^2 - 4x + 7$ **b.** $y = -2x^2 - 16x - 35$

Solution

a.

$$y = x^2 - 4x + 7$$

$$y - 7 = x^2 - 4x \quad \text{Isolate the x-terms.}$$

$$y - 7 + 4 = x^2 - 4x + 4 \quad \text{Add } \left(\tfrac{-4}{2}\right)^2 = 4 \text{ to both sides.}$$

$$y - 3 = (x - 2)^2 \quad \text{Write the trinomial as a binomial squared.}$$

$$y = (x - 2)^2 + 3 \quad \text{Solve for } y.$$

In the equation $y = (x - 2)^2 + 3$, $a = 1$, $h = 2$, and $k = 3$.

b.

$$y = -2x^2 - 16x - 35$$

$$y + 35 = -2x^2 - 16x \quad \text{Isolate the x-terms.}$$

$$y + 35 = -2(x^2 + 8x) \quad \text{Factor out the leading coefficient, } -2.$$

$$y + 35 + [-2(16)] = -2(x^2 + 8x + 16) \quad \text{Add } -2\left(\tfrac{8}{2}\right)^2\text{, or } -2(16)\text{, to both sides.}$$

$$y + 3 = -2(x + 4)^2 \quad \text{Write the trinomial as a binomial squared.}$$

$$y = -2(x + 4)^2 - 3 \quad \text{Solve for } y.$$

In the equation $y = -2(x + 4)^2 - 3$, or $y = -2[x - (-4)]^2 + (-3)$, $a = -2$, $h = -4$, and $k = -3$.

EXAMPLE 2 Determine the vertex and axis of symmetry for the graph of each equation. Describe the graph, but do not draw it.

a. $y = 2(x - 4)^2 - 3$ **b.** $y = -x^2 - 4x - 8$

Solution

a. $y = 2(x - 4)^2 - 3$

or $y = 2(x - 4)^2 + (-3)$ Write the equation in $y = a(x - h)^2 + k$ form.

We see that $a = 2$, $h = 4$, and $k = -3$.

Since $a = 2$ and $2 > 0$, the graph is concave upward.

Since $|2| = 2 > 1$, then the graph is narrower than it would be if $a = 1$.

The vertex is (h, k), or $(4, -3)$.

The axis of symmetry is the graph of $x = 4$.

➔ Graphing a Vertical Parabola

To graph a parabola, we use the information that we can determine from its equation and add points to establish a pattern for the curve.

✓ To graph a vertical parabola,

- Locate and label the vertex, (h, k)
- Graph the axis of symmetry x = h, with a dashed line.
- Graph enough points to see a pattern. The *x*- and *y*-intercepts are important points to determine. Connect the points with a smooth curve.

EXAMPLE 3 Graph the vertical parabola for $y = 2(x - 4)^2 - 3$.

Solution

$$y = 2(x - 4)^2 - 3$$

$$y = 2(x - 4)^2 + (-3)$$ Write the equation in the form $y = a(x - h)^2 + k$.

Therefore, $a = 2$, $h = 4$, and $k = -3$.

The vertex is (h, k), or $(4, -3)$.

The axis of symmetry is the line $x = 4$.

The graph opens upward, because $a = 2 > 0$.

The graph is narrower than it would be if $a = 1$, because $|a| = |2| > 1$.

The y-intercept is the point on the graph where $x = 0$. Substitute 0 for x and solve for y.

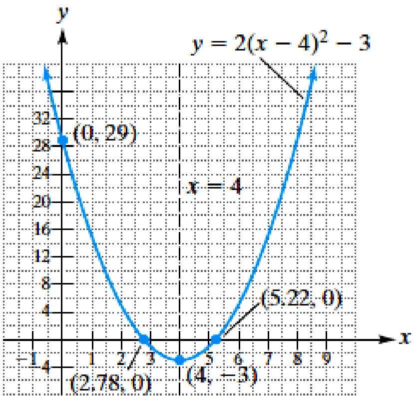
$$\begin{aligned} y &= 2(x-4)^2 - 3 \\ y &= 2(0-4)^2 - 3 \\ y &= 2(16) - 3 \\ y &= 32 - 3 \\ y &= 29 \end{aligned}$$

The y-intercept is $(0, 29)$.

The x-intercept is the point on the graph where $y = 0$. Therefore, substitute 0 for y and solve for x.

$$\begin{aligned} y &= 2(x-4)^2 - 3 \\ 0 &= 2(x-4)^2 - 3 \\ 2(x-4)^2 &= 3 \\ (x-4)^2 &= \frac{3}{2} \\ x - 4 &= \pm\sqrt{\frac{3}{2}} \\ x &= 4 \pm \sqrt{\frac{3}{2}} \\ x &= 4 \pm \sqrt{\frac{3}{2}\cdot\frac{2}{2}} \\ x &= 4 \pm \frac{\sqrt{6}}{2} \\ x &= \frac{8 \pm \sqrt{6}}{2} \end{aligned}$$

The x-intercepts are about $(5.22, 0)$ and $(2.78, 0)$.



➔ Graphing a Horizontal Parabola

Some parabolas open left or right. In such a case, the parabola has a horizontal axis of symmetry and is called a **horizontal parabola**.

The form of a quadratic equation that will graph a horizontal parabola is $\boldsymbol{x = a(y-k)^2 + h}$ the real numbers a, h, and k affect a vertical parabola, they also affect a horizontal parabola.

SUMMARY OF THE EFFECTS OF THE REAL NUMBERS a, h, AND k OF A QUADRATIC EQUATION ON A HORIZONTAL PARABOLA

The real numbers a, h, and k of a quadratic equation in the form $x = a(y - k)^2 + h$ affect its graph.

If $a > 0$, then the graph opens to the right.
If $a < 0$, then the graph opens to the left.
If $|a| > 1$, then the graph is narrower than it would be if $a = 1$.
If $|a| < 1$, then the graph is wider than it would be if $a = 1$.
The vertex of the graph is (h, k).
The axis of symmetry is the line graphed by $y = k$.

EXAMPLE 4 Graph the horizontal parabolas.

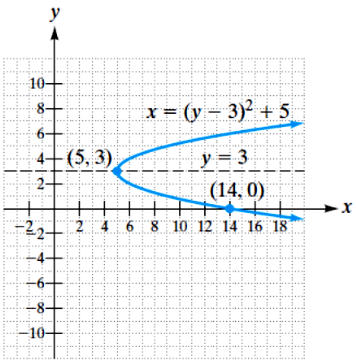
a. $x = (y - 3)^2 + 5$ **b.** $x = -2y^2 - 4y - 5$

Solution

a. $x = (y - 3)^2 + 5$

The equation is written in the form $x = a(y - k)^2 + h$. Therefore, $a = 1$, $h = 5$, and $k = 3$.

The vertex is (h, k), or $(5, 3)$.
The axis of symmetry is the graph of $y = k$, or $y = 3$.
The graph opens to the right, because $a = 1 > 0$.



The y-intercept is the point on the graph where $x = 0$. Substitute 0 for x and solve for y.

$$x = (y - 3)^2 + 5$$
$$0 = (y - 3)^2 + 5$$
$$(y - 3)^2 = -5$$
$$y - 3 = \pm\sqrt{-5}$$
$$y = 3 \pm i\sqrt{5}$$

y is an imaginary number. Therefore, the graph has no y-intercept. In fact, we know this is so because the vertex is located at (~~3, 5~~ 5,3) and opens to the right.

The x-intercept is the point on the graph where $y = 0$.

$$x = (y - 3)^2 + 5$$
$$x = (0 - 3)^2 + 5$$
$$x = 14$$

The x-intercept is (14, 0).

➔ Writing Quadratic Equations, Given the Vertex and a Point on the Graph

Earlier, we learned that although two points determine a straight line, you need three points to determine a curve. Thus, an infinite number of parabolas can be drawn through any two given points. However, if we know that one of these points is the vertex and if we know that the parabola is vertical or horizontal, then we can write an equation for the specific parabola that passes through these points.

EXAMPLE 5 **a.** Write an equation of a vertical parabola with a vertex of (2, 6) and passing through the point (−1, 4).

b. Write an equation of a horizontal parabola with a vertex of (−1, 1) and a y-intercept of (0, 2).

Solution

a. Since the vertex is (2, 6), it follows that $h = 2$ and $k = 6$. Also, we know that the point (−1, 4) is a solution of the equation. We will substitute −1 for x and 4 for y, as well as 2 for h and 6 for k, in the equation $y = a(x - h)^2 + k$ and solve for a.

$$y = a(x - h)^2 + k$$
$$4 = a[(-1) - 2]^2 + 6 \quad \text{Substitute.}$$
$$4 = a(-3)^2 + 6$$
$$4 = 9a + 6$$
$$a = -\frac{2}{9}$$

Now, we write an equation using the known values for a, h, and k.

$$y = a(x - h)^2 + k$$
$$y = -\frac{2}{9}(x - 2)^2 + 6$$

The graph of the equation $y = -\frac{2}{9}(x - 2)^2 + 6$ is a vertical parabola with a vertex of (2, 6) and passing through the point (−1, 4).

b. First, we substitute values for h, k, x, and y. Given the vertex (−1, 1), we know that $h = -1$ and $k = 1$.

We use the coordinates of the y-intercept for x and y, $x = 0$ and $y = 2$. Then we solve for a.

$$x = a(y - k)^2 + h$$
$$0 = a(2 - 1)^2 + (-1) \quad \text{Substitute.}$$
$$0 = a(1)^2 - 1$$
$$0 = a - 1$$
$$a = 1$$

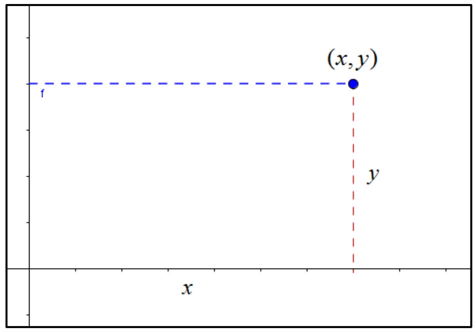
Write an equation using $a = 1$, $h = -1$, and $k = 1$.

$$x = a(y - k)^2 + h$$
$$x = 1(y - 1)^2 + (-1)$$
$$x = (y - 1)^2 - 1$$

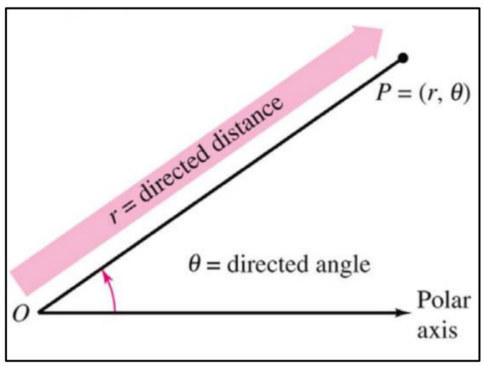
The graph of the equation $x = (y - 1)^2 - 1$ is a horizontal parabola with a vertex of (−1, 1) and a y-intercept of (0, 2).

Polar Coordinates

Consider the rectangular coordinate system.



We want to find another way to get to the point **(x, y)**. One way to do this is to use an angle **θ** and a distance **r**. It will look like this

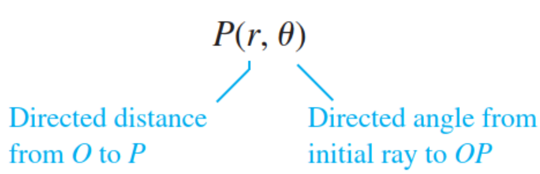


- To form the **polar coordinate system** in the plane, fix a point *O*, called the **pole** (or **origin**), and construct from *O* an initial ray called the **polar axis**, as shown in the above figure. Then, each point (*P)* in the plane can be assigned **polar coordinates** (*r*, θ).

Where

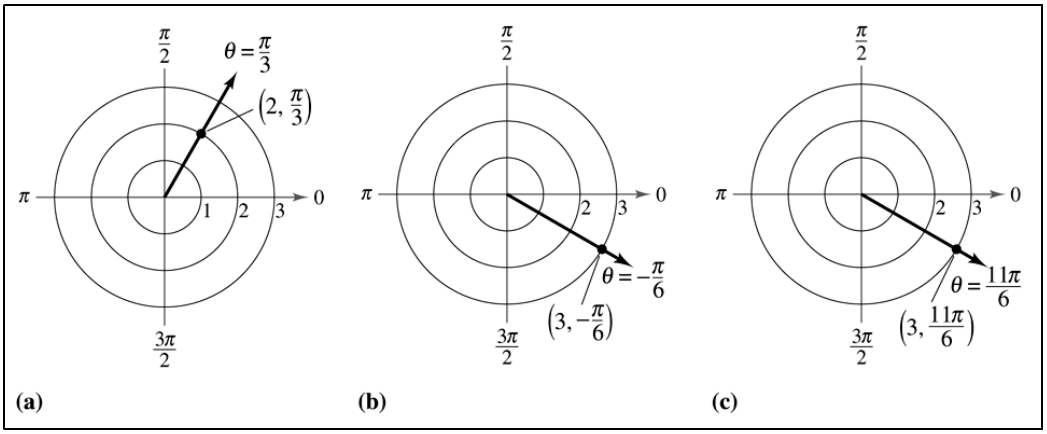
r = directed distance from O to P

θ = directed angle, counter clockwise from polar axis to $\overline{OP}$.



- The θ coordinate in (r, θ) is this angle, in degree or radian measure. **<u>The angle θ is positive if the rotation is counterclockwise and negative if the rotation is clockwise</u>**.

- The r coordinate in (r, θ) is the directed distance from the pole to the point P. **It is positive if measured from the pole along the terminal side of θ and negative if measured along the terminal side extended through the pole**.



With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates. For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point. Also because r is a directed distance, the coordinates (r, θ) and $(-r, \pi + \theta)$ represent the same point.

Sign conversion

θ*:*+ve when measured counter clockwise

θ*:*-ve when measured clockwise

r:+ in the direction of θ

r:- in the opposite direction of θ

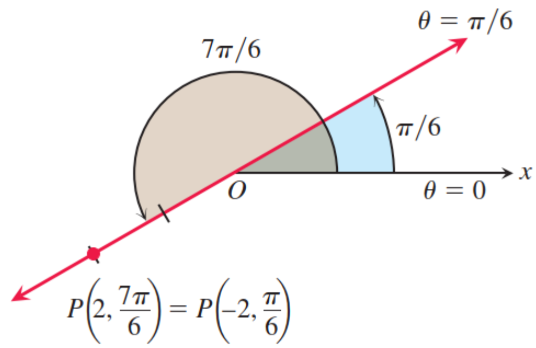
- Three are infinite pairs of polar coordinates of each point

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.22). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$



For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

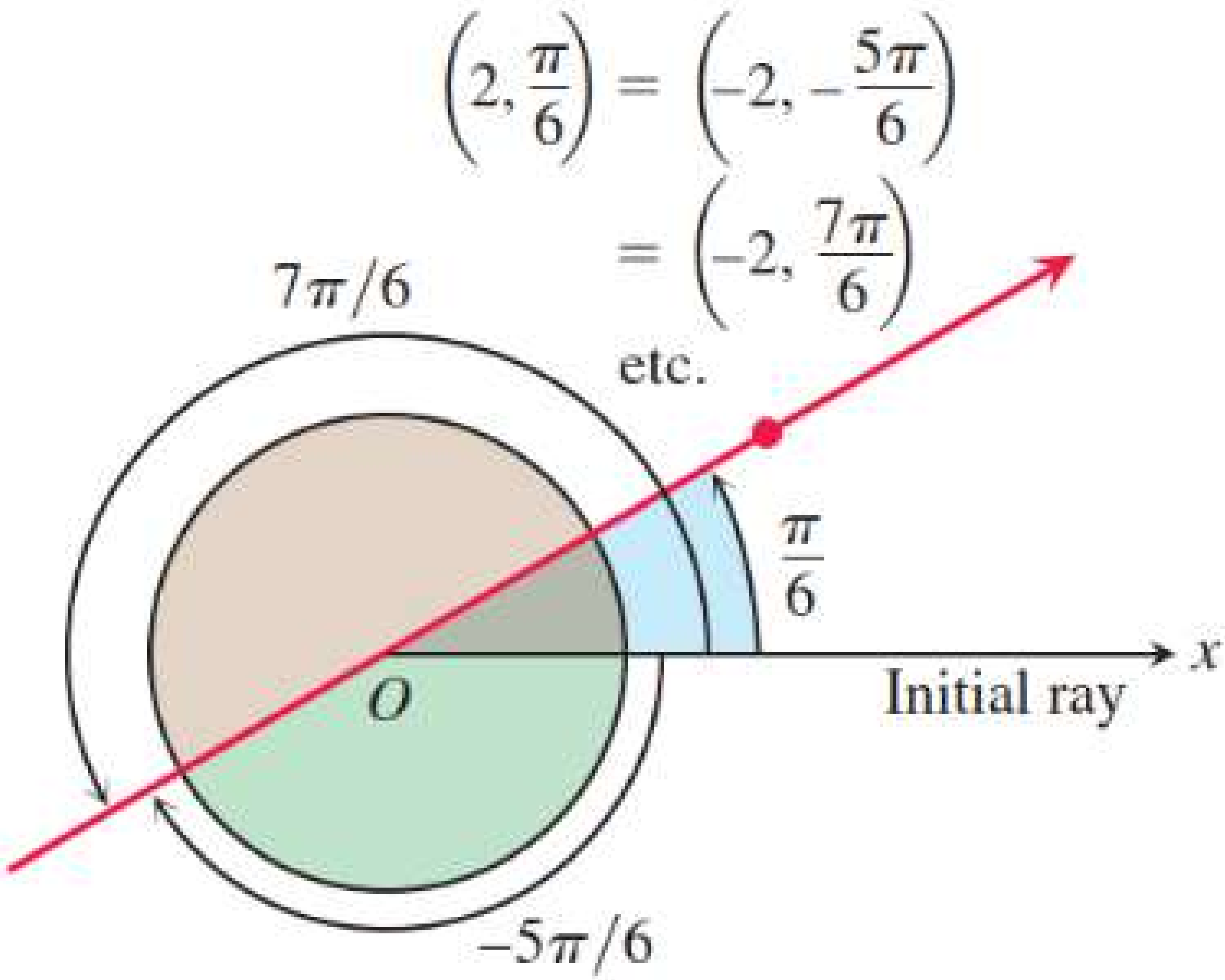
The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

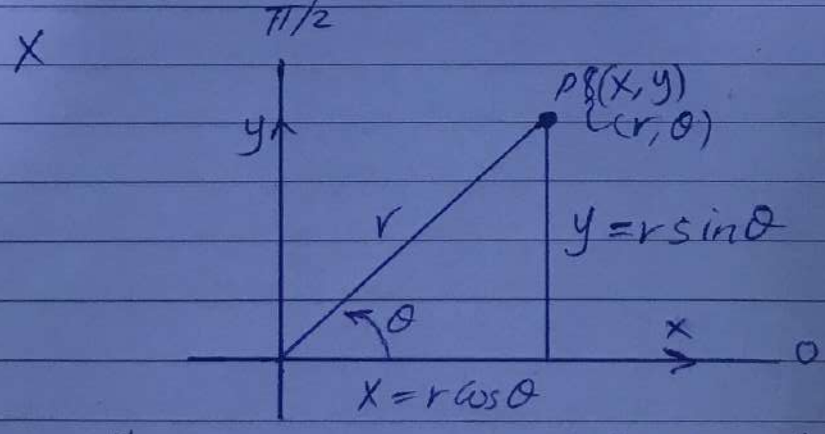
When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■



$\left(2, \frac{\pi}{6}\right) = \left(-2, -\frac{5\pi}{6}\right)$
$= \left(-2, \frac{7\pi}{6}\right)$
etc.
$7\pi/6$
$\frac{\pi}{6}$
x
O
Initial ray
$-5\pi/6$

for many Purposes it does not matter whether Polar angles are measured in degrees or radians. However, in problems that involve derivatives or integrals they must be measured in radians, since the derivatives of the trigonometric functions were derived under this assumption. Henceforth, we will use radian measure for Polar angles, except in certain application where it is not required and degree measure is more convenient.

Frequently, it will be useful to superimpose a rectangular xy coordinate system on top of a Polar coordinate system making the positive x-axis coincide with the polar axis. If this is done, then every point P will have both rectangular coordinates (x, y) and polar coordinates (r, θ). As suggested by figure 1, these coordinates are related by the equations



$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta, \quad \cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\tan\theta = \frac{y}{x}$$

These equations are well suited for finding x and y when r and θ are known. However to find r and θ when x and y are known, it is preferable to use $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta / \cos\theta$

$$r^2 = x^2 + y^2$$

Find the rectangular Coordinates of the Point P whose Polar Coordinates are $(6, 2\pi/3)$

Sol:

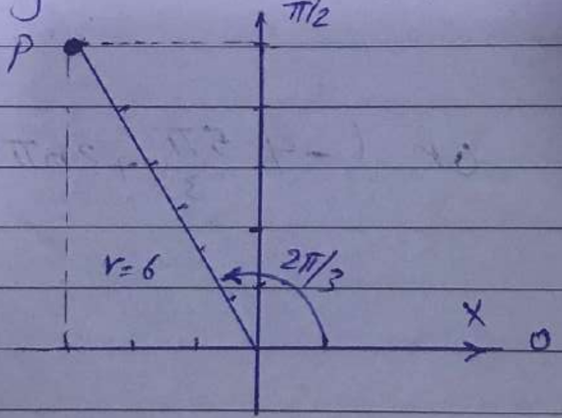
Substituting the polar Coordinates $r = 6$, and $\theta = \frac{2\pi}{3}$

$$x = r\cos\theta = 6\cos\frac{2\pi}{3} = 6\left(-\frac{1}{2}\right) = -3$$

$$y = r\sin\theta = 6\sin\frac{2\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 6 * 0.866 \simeq 5.2$$

$$3\sqrt{3}$$

Thus, the rectangular Coordinates of P are $(-3, 3\sqrt{3})$, Figure:



Find Polar Coordinates of the Point P whose rectangular Coordinates are $(-2, 2\sqrt{3})$

$$r^2 = x^2 + y^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16$$

So $r = 4$

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

from this and the fact that $(-2, 2\sqrt{3})$ lies in the second quadrant the angle $\theta = 2\pi/3$. Thus $(4, 2\pi/3)$ are Polar Coordinates of P. All other Polar Coordinates of P are expressible in the form

$$\left(4, \frac{2\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{5\pi}{3} + 2n\pi\right)$$