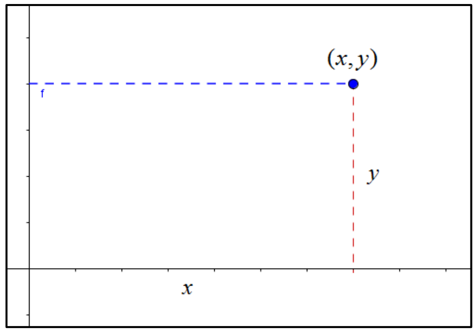
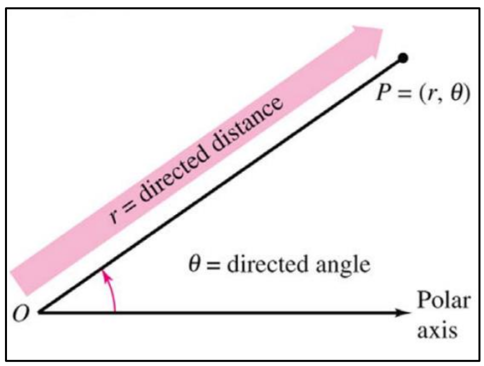
Polar Coordinates

Consider the rectangular coordinate system.



We want to find another way to get to the point **(x, y)**. One way to do this is to use an angle **θ** and a distance **r**. It will look like this

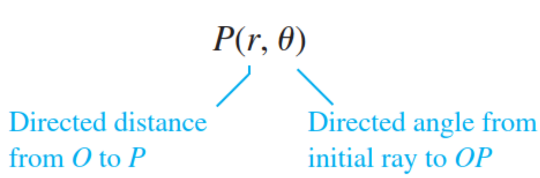


- To form the **polar coordinate system** in the plane, fix a point *O*, called the **pole** (or **origin**), and construct from *O* an initial ray called the **polar axis**, as shown in the above figure. Then, each point (*P)* in the plane can be assigned **polar coordinates** (*r*, θ).

Where

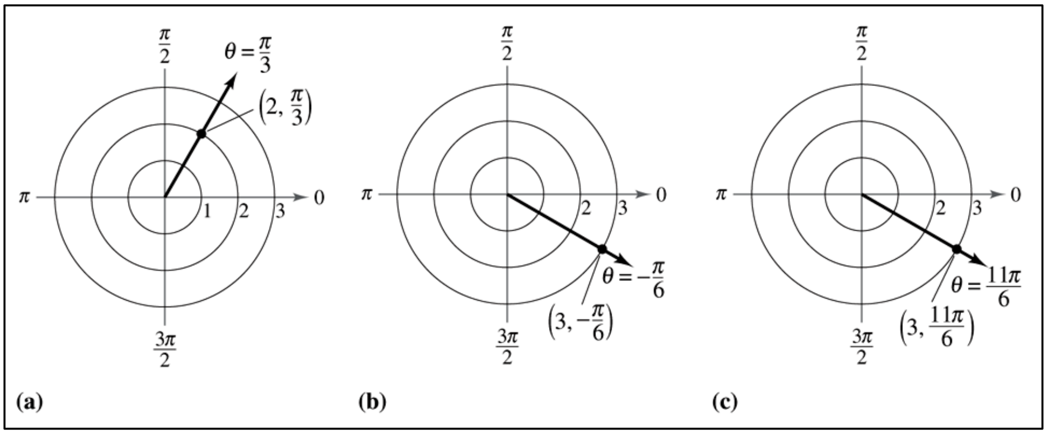
r = directed distance from O to P

θ = directed angle, counter clockwise from polar axis to $\overline{OP}$.



- The θ coordinate in (r, θ) is this angle, in degree or radian measure. **<u>The angle θ is positive if the rotation is counterclockwise and negative if the rotation is clockwise</u>**.

- The r coordinate in (r, θ) is the directed distance from the pole to the point P. **It is positive if measured from the pole along the terminal side of θ and negative if measured along the terminal side extended through the pole**.



With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates. For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point. Also because r is a directed distance, the coordinates (r, θ) and $(-r, \pi + \theta)$ represent the same point.

Sign conversion

θ*:*+ve when measured counter clockwise

θ*:*-ve when measured clockwise

r:+ in the direction of θ

r:- in the opposite direction of θ

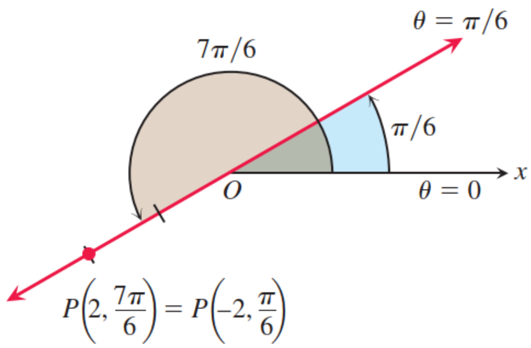
- Three are infinite pairs of polar coordinates of each point

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.22). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$



For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

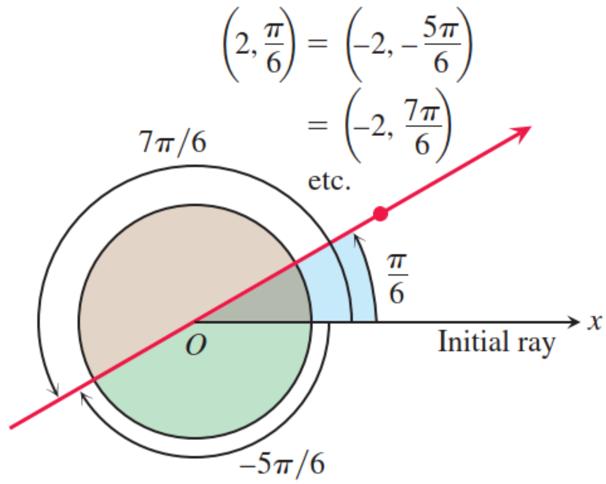
The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

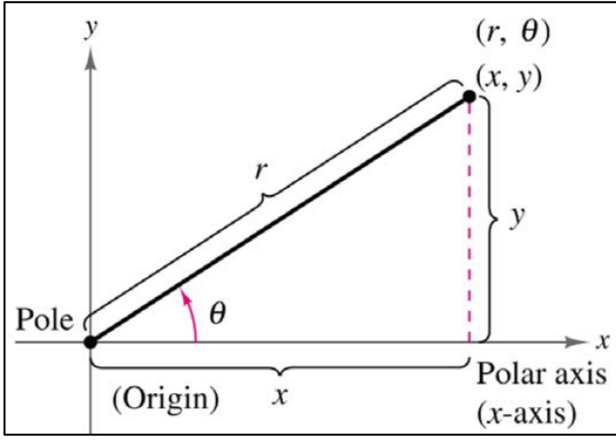
$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■



Coordinates conversion

To establish the relationship between polar and rectangular (Cartesian) coordinates, let the polar axis to coincide with the positive *x-axis* and the pole with the origin.



$$\tan(\theta) = \frac{y}{x}$$

$$\cos(\theta) = \frac{x}{r}, \quad \text{and} \quad \sin(\theta) = \frac{y}{r}$$

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

1. $x = r\cos\theta$ **2.** $\tan\theta = \frac{y}{x}$

$y = r\sin\theta$ $r^2 = x^2 + y^2$

Example 2: Convert each of the following points into the given coordinate system.

a) $\left(-4, \dfrac{2\pi}{3}\right)$ into Cartesian coordinates

b) $(-1, -1)$ into Polar coordinates

Solution

a) r = -4, θ = 2π/3

x =-4 cos (2π/3) = 4 cos (π/3) = 2

y = =-4 sin (2π/3) = =-4 sin (π/3) = -2$\sqrt{3}$

The point is (2, -2$\sqrt{3}$)

b) (-1, -1) is on the Ⅲ quadrant, and tan(θ) =1, $\Rightarrow$ θ =5π/4

$r^2 = 2 \Rightarrow r = \sqrt{2}$

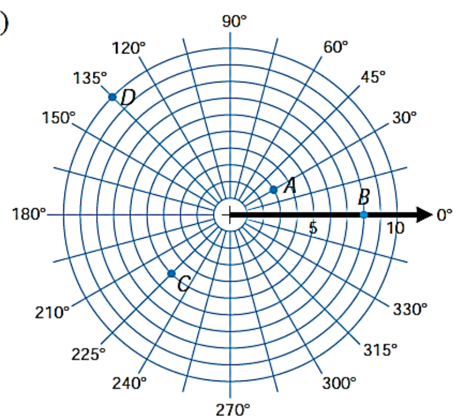
The point is ($\sqrt{2}$, 5π/4).

EXAMPLE 3 Plotting Points in a Polar Coordinate System

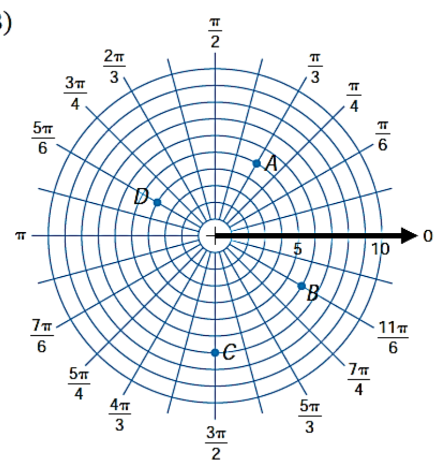
Plot the following points in a polar coordinate system:

(A) $A(3, 30°)$, $B(-8, 180°)$, $C(5, -135°)$, $D(-10, -45°)$
(B) $A(5, \pi/3)$, $B(-6, 5\pi/6)$, $C(7, -\pi/2)$, $D(-4, -\pi/6)$

Solutions (A)



(B)



Note: **From the signs of x and y the quadrant for θ can determined.**

EXAMPLE 4 Converting from Polar to Rectangular Form, and Vice Versa

(A) Convert the polar coordinates $(-4, 1.077)$ to rectangular coordinates to three decimal places.

(B) Convert the rectangular coordinates $(-3.207, -5.719)$ to polar coordinates with θ in degree measure, $-180° < \theta \le 180°$, and $r \ge 0$.

Solution (A) Use a calculator set in radian mode.

$$(r, \theta) = (-4, 1.077)$$

$$x = r\cos\theta = (-4)\cos 1.077 = -1.896$$

$$y = r\sin\theta = (-4)\sin 1.077 = -3.522$$

Rectangular coordinates are $(-1.\ 896, -3.522)$

(B) Use a calculator set in degree mode.

$$(x, y) = (-3.207, -5.719)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.207)^2 + (-5.719)^2} = 6.557$$

$$\tan\theta = \frac{y}{x} = \frac{-5.719}{-3.207}$$

θ is a third-quadrant angle and is to be chosen so that

$$\theta = -180° + \tan^{-1}\frac{-5.719}{-3.207} = -119.28°$$

Polar coordinates are $(6.557, -119.28°)$.

EXAMPLE 5 Converting an Equation from Rectangular Form to Polar Form

Change $x^2 + y^2 - 4y = 0$ to polar form.

Solution Use $r^2 = x^2 + y^2$ and $y = r \sin \theta$.

$$x^2 + y^2 - 4y = 0$$

$$r^2 - 4r \sin \theta = 0$$

$$r(r - 4 \sin \theta) = 0$$

$$r = 0 \qquad \text{or} \qquad r - 4 \sin \theta = 0$$

The graph of $r = 0$ is the pole. Because the pole is included in the graph of $r - 4 \sin \theta = 0$ (let $\theta = 0$), we can discard $r = 0$ and keep only

$$r - 4 \sin \theta = 0$$

or

$$r = 4 \sin \theta \qquad \text{The polar form of } x^2 + y^2 - 4y = 0$$

EXAMPLE 6 Converting an Equation from Polar Form to Rectangular Form

Change $r = -3 \cos \theta$ to rectangular form.

Solution The transformation of this equation as it stands into rectangular form is fairly difficult. With a little trick, however, it becomes easy. We multiply both sides by r, which simply adds the pole to the graph. But the pole is already part of the graph of $r = -3 \cos \theta$ (let $\theta = \pi/2$), so we haven't actually changed anything.

$$r = -3 \cos \theta$$

$$r^2 = -3r \cos \theta \qquad \text{Multiply both sides by } r.$$

$$x^2 + y^2 = -3x \qquad r^2 = x^2 + y^2 \text{ and } r \cos \theta = x$$

$$x^2 + y^2 + 3x = 0$$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a+bu}$ OR ITS RECIPROCAL

102. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

103. $\displaystyle\int u^2\sqrt{a+bu}\,du = \frac{2}{105b^3}(15b^2u^2-12abu+8a^2)(a+bu)^{3/2}+C$

104. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2u^n(a+bu)^{3/2}}{b(2n+3)} - \frac{2an}{b(2n+3)}\int u^{n-1}\sqrt{a+bu}\,du$

105. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

106. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2-4abu+8a^2)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C & (a>0) \\[2ex] \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}} + C & (a<0) \end{cases}$

109. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bu}\,du}{u^{n-1}}$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{2au-u^2}$ OR ITS RECIPROCAL

112. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

113. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

114. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2} + a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

115. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u} - \sin^{-1}\left(\frac{u-a}{a}\right)+C$

116. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\left(\frac{u-a}{a}\right)+C$

117. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au}+C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

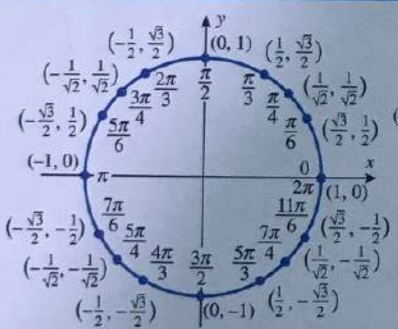
INTEGRALS CONTAINING $(2au-u^2)^{3/2}$

120. $\displaystyle\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}+C$

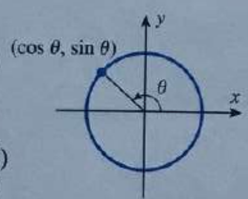
121. $\displaystyle\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}+C$

THE WALLIS FORMULA

122. $\displaystyle\int_0^{\pi/2}\sin^n u\,du = \int_0^{\pi/2}\cos^n u\,du = \frac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot6\cdot\cdots\cdot n}\cdot\frac{\pi}{2}\ \left(\begin{array}{l}n \text{ an even}\\ \text{integer and}\\ n\ge 2\end{array}\right)$ or $\displaystyle\frac{2\cdot4\cdot6\cdot\cdots\cdot(n-1)}{3\cdot5\cdot7\cdot\cdots\cdot n}\ \left(\begin{array}{l}n \text{ an odd}\\ \text{integer and}\\ n\ge 3\end{array}\right)$



TRIGONOMETRY REVIEW



PYTHAGOREAN IDENTITIES

$\sin^2\theta+\cos^2\theta=1 \qquad \tan^2\theta+1=\sec^2\theta \qquad 1+\cot^2\theta=\csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta)=-\sin\theta \qquad \cos(-\theta)=\cos\theta \qquad \tan(-\theta)=-\tan\theta$

$\csc(-\theta)=-\csc\theta \qquad \sec(-\theta)=\sec\theta \qquad \cot(-\theta)=-\cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi-\theta)=\sin\theta \qquad \cos(\pi-\theta)=-\cos\theta \qquad \tan(\pi-\theta)=-\tan\theta$

$\csc(\pi-\theta)=\csc\theta \qquad \sec(\pi-\theta)=-\sec\theta \qquad \cot(\pi-\theta)=-\cot\theta$

$\sin(\pi+\theta)=-\sin\theta \qquad \cos(\pi+\theta)=-\cos\theta \qquad \tan(\pi+\theta)=\tan\theta$

$\csc(\pi+\theta)=-\csc\theta \qquad \sec(\pi+\theta)=-\sec\theta \qquad \cot(\pi+\theta)=\cot\theta$

ADDITION FORMULAS

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha=2\sin\alpha\cos\alpha \qquad \cos 2\alpha=2\cos^2\alpha-1$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha \qquad \cos 2\alpha=1-2\sin^2\alpha$

HALF-ANGLE FORMULAS

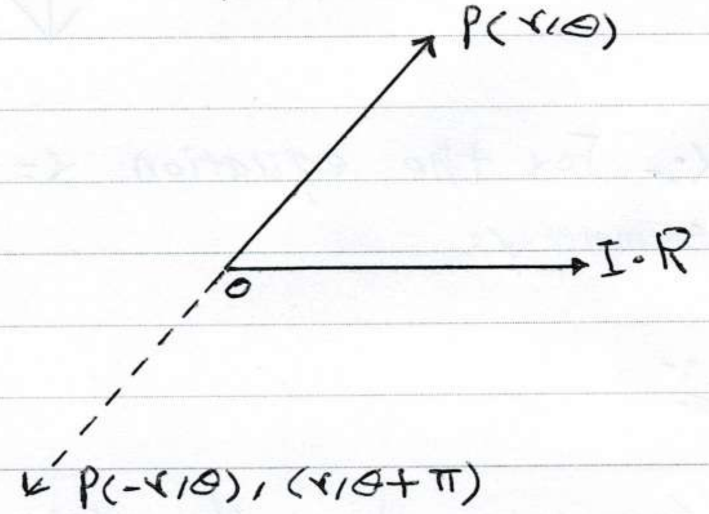
$\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2} \qquad \cos^2\dfrac{\alpha}{2}=\dfrac{1+\cos\alpha}{2}$

Symmetry of Polar Coordinate

There are three types of symmetry in Polar coordinate system:-

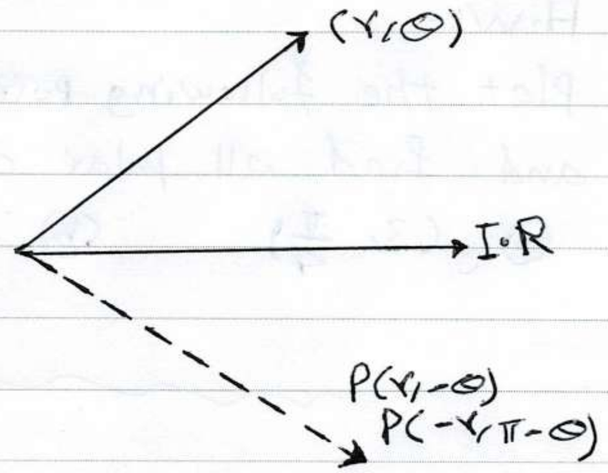
1) Symmetry about the origin

If the equation is unchanged when "r" is replaced by "$-r$" or "θ" is replaced by "$\pi+\theta$".

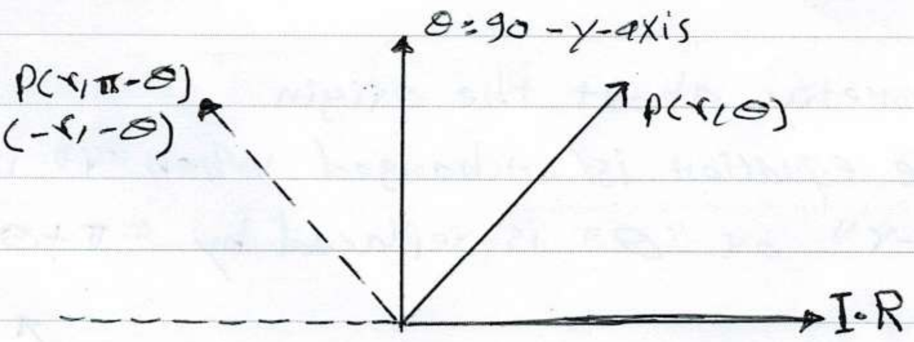


2) Symmetry about the x-axis:-

If the equation is unchanged when (θ) is replaced by $(-\theta)$ or the pair (r,θ) by $(-r, \pi-\theta)$.



3) Symmetry about y-axis
If the equation is unchanged when (θ) is replaced by $(\pi-\theta)$ or the pair (r,θ) by $(-r,-\theta)$



EX:- For the equation $r = a\sin\theta$, check the symmetry.

Sol:-

① Symmetry about the origin
replace (r) by $(-r)$

$\therefore -r = a\sin\theta \Rightarrow r = -a\cdot\sin\theta$ $\therefore$ changed
$\therefore$ Not Symmetry

2) Symmetry about x - axis

replace (θ) by $(-\theta)$

$r = a\sin(-\theta) = -a\cdot\sin\theta$ ∴ changed

∴ No symmetry about x-axis

3) Symmetry about the y-axis

replace (θ) by $(\pi-\theta)$

or replace (r,θ) by $(-r,-\theta)$

∴ $r = a\sin\theta$

$r = a\sin(\pi-\theta)$

$r = a[\sin\pi\cos\theta - \cos\pi\sin\theta]$

$= a\sin\theta$ ∴ Unchanged

∴ Symmetry about y-axis

or

$r = a\sin\theta \Rightarrow -r = a\sin(-\theta)$

$r = a\sin\theta$ Unchanged

∴ Symmetry about y-axis

GRAPHS IN POLAR COORDINATES

We will now consider the problem of graphing equations of the form $r = f(\theta)$ in polar coordinates, where θ is assumed to be measured in radians. Some examples of such equations are

$$r = 2\cos\theta, \quad r = \frac{4}{1 - 3\sin\theta}, \quad r = \theta$$

In a rectangular coordinate system the graph of an equation $y = f(x)$ consists of all points whose coordinates (x, y) satisfy the equation. However, in a polar coordinate system, points have infinitely many different pairs of polar coordinates, so that a given point may have some polar coordinates that satisfy the equation $r = f(\theta)$ and others that do not. Taking this into account, we define the ***graph of*** $r = f(\theta)$ ***in polar coordinates*** to consist of all points with *at least one* pair of coordinates (r, θ) that satisfy the equation.

The most elementary way to graph an equation $r = f(\theta)$ in polar coordinates is to plot points. The idea is to choose some typical values of θ, calculate the corresponding values of r, and then plot the resulting pairs (r, θ) in a polar coordinate system. Here are some examples.

Example 3 Sketch the graph of the equation $r = \sin\theta$ in polar coordinates by plotting points.

Solution. Table 11.1.1 shows the coordinates of points on the graph at increments of $\pi/6$ $(= 30°)$.

Table 11.1.1

θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
(r, θ)	$(0, 0)$	$\left(\frac{1}{2}, \frac{\pi}{6}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$	$\left(\frac{1}{2}, \frac{5\pi}{6}\right)$	$(0, \pi)$	$\left(-\frac{1}{2}, \frac{7\pi}{6}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{4\pi}{3}\right)$	$\left(-1, \frac{3\pi}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{3}\right)$	$\left(-\frac{1}{2}, \frac{11\pi}{6}\right)$	$(0, 2\pi)$

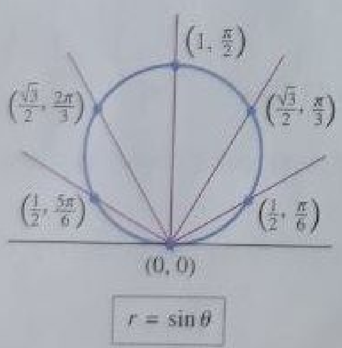


Figure 11.1.7

These points are plotted in Figure 11.1.7. Note, however, that there are 13 points listed in the table but only 6 distinct plotted points. This is because the pairs from $\theta = \pi$ on yield duplicates of the preceding points. For example, $(-1/2, 7\pi/6)$ and $(1/2, \pi/6)$ represent the same point. ◀

Observe that the points in Figure 11.1.7 appear to lie on a circle. We can confirm that this is so by expressing the polar equation $r = \sin\theta$ in terms of x and y. To do this, we multiply the equation through by r to obtain

$$r^2 = r\sin\theta$$

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

$$x^2 + y^2 = y$$

Rewriting this equation as $x^2 + y^2 - y = 0$ and then completing the square yields

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

which is a circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$ in the xy-plane.

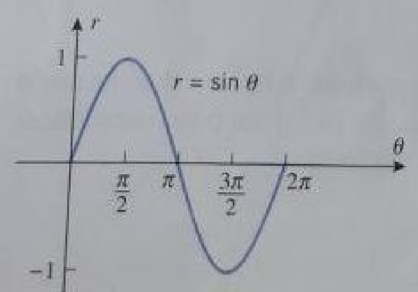


Figure 11.1.8

Just because an equation $r = f(\theta)$ involves the variables r and θ does not mean that it has to be graphed in a polar coordinate system. When useful, this equation can also be graphed in a rectangular coordinate system. For example, Figure 11.1.8 shows the graph of $r = \sin\theta$ in a rectangular θr-coordinate system. This graph can actually help to visualize how the polar graph in Figure 11.1.7 is generated:

- At $\theta = 0$ we have $r = 0$, which corresponds to the pole $(0, 0)$ on the polar graph.
- As θ varies from 0 to $\pi/2$, the value of r increases from 0 to 1, so the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$.
- As θ varies from $\pi/2$ to π, the value of r decreases from 1 back to 0, so the point (r, θ) moves along the circle from the high point back to the pole.
- As θ varies from π to $3\pi/2$, the values of r are negative, varying from 0 to -1. Thus, the point (r, θ) moves along the circle from the pole to the high point at $(1, \pi/2)$, which is the same as the point $(-1, 3\pi/2)$. This duplicates the motion that occurred for $0 \le \theta \le \pi/2$.
- As θ varies from $3\pi/2$ to 2π, the value of r varies from -1 to 0. Thus, the point (r, θ) moves along the circle from the high point back to the pole, duplicating the motion that occurred for $\pi/2 \le \theta \le \pi$.

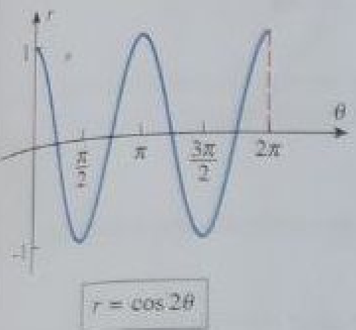
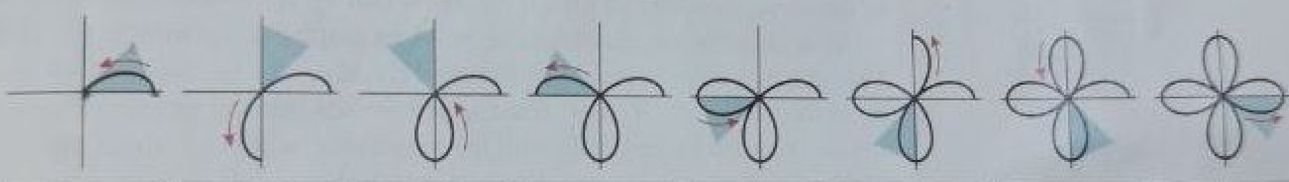


Figure 11.1.9

Example 4 Sketch the graph of $r = \cos 2\theta$ in polar coordinates.

Solution. Instead of plotting points, we will use the graph of $r = \cos 2\theta$ in rectangular coordinates (Figure 11.1.9) to visualize how the polar graph of this equation is generated. The analysis and the resulting polar graph are shown in Figure 11.1.10. This curve is called a *four-petal rose.* ◀



r varies from 1 to 0 as θ varies from 0 to $\pi/4$.	r varies from 0 to -1 as θ varies from $\pi/4$ to $\pi/2$.	r varies from -1 to 0 as θ varies from $\pi/2$ to $3\pi/4$.	r varies from 0 to 1 as θ varies from $3\pi/4$ to π.	r varies from 1 to 0 as θ varies from π to $5\pi/4$.	r varies from 0 to -1 as θ varies from $5\pi/4$ to $3\pi/2$.	r varies from -1 to 0 as θ varies from $3\pi/2$ to $7\pi/4$.	r varies from 0 to 1 as θ varies from $7\pi/4$ to 2π.

Figure 11.1.10

SYMMETRY TESTS

Observe that the polar graph of $r = \cos 2\theta$ in Figure 11.1.10 is symmetric about the x-axis and the y-axis. This symmetry could have been predicted from the following theorem, which is suggested by Figure 11.1.11 (we omit the proof).

> **11.1.1 THEOREM (*Symmetry Tests*).**
>
> (*a*) *A curve in polar coordinates is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation* (Figure 11.1.11*a*).
>
> (*b*) *A curve in polar coordinates is symmetric about the y-axis if replacing θ by $\pi - \theta$ in its equation produces an equivalent equation* (Figure 11.1.11*b*).
>
> (*c*) *A curve in polar coordinates is symmetric about the origin if replacing θ by $\theta + \pi$, or by replacing r by $-r$ in its equation produces an equivalent equation* (Figure 11.1.11*c*).

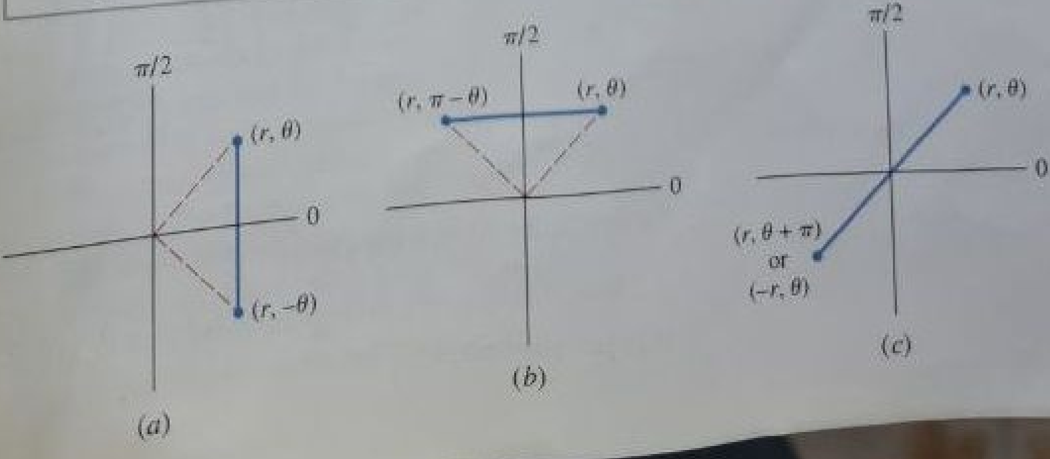


Figure 11.1.11

Example 5 Use Theorem 11.1.1 to confirm that the graph of $r = \cos 2\theta$ in Figure 11.1.10 is symmetric about the x-axis and y-axis.

Solution. To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields

$$r = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $-\theta$ does not alter the equation.

To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta$$

Thus, replacing θ by $\pi - \theta$ does not alter the equation. ◀

FOR THE READER. A graph that is symmetric both about the x-axis and about the y-axis is also symmetric about the origin. Use Theorem 11.1.1(c) to verify that the curve of Example 5 is also symmetric about the origin.

Example 6 Sketch the graph of $r = a(1 - \cos\theta)$ in polar coordinates, assuming a to be a positive constant.

Solution. Observe first that replacing θ by $-\theta$ does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis.

As in our previous examples, we will first graph the equation in rectangular coordinates. This graph, which is shown in Figure 11.1.12a, can be obtained by rewriting the given equation as $r = a - a\cos\theta$, from which we see that the graph in rectangular coordinates can be obtained by first reflecting the graph of $r = a\cos\theta$ about the x-axis to obtain the graph of $r = -a\cos\theta$, and then translating that graph up a units to obtain the graph of $r = a - a\cos\theta$. Now we can see that:

- As θ varies from 0 to $\pi/3$, r increases from 0 to $a/2$.
- As θ varies from $\pi/3$ to $\pi/2$, r increases from $a/2$ to a.
- As θ varies from $\pi/2$ to $2\pi/3$, r increases from a to $3a/2$.
- As θ varies from $2\pi/3$ to π, r increases from $3a/2$ to $2a$.

This produces the polar curve shown in Figure 11.1.12b. The rest of the curve can be obtained by continuing the preceding analysis from π to 2π or, as noted above, by reflecting the portion already graphed about the x-axis (Figure 11.1.12c). This heart-shaped curve is called a ***cardioid*** (from the Greek word "kardia" for heart). ◀

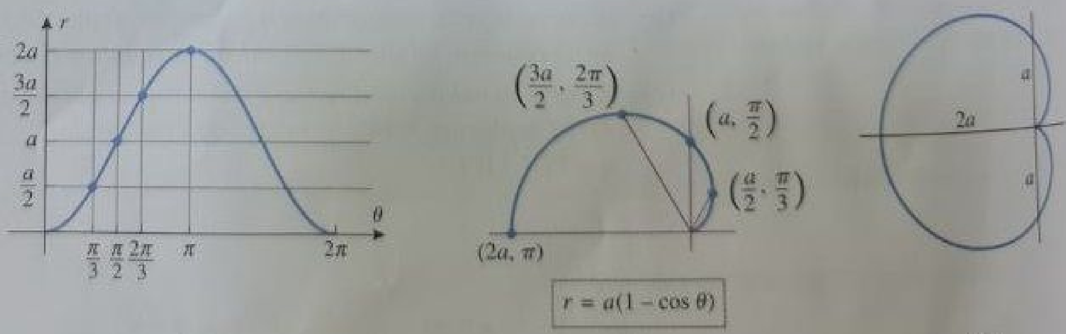


Figure 11.1.12

Example 7 Sketch the curves

(a) $r = 1$ (b) $\theta = \dfrac{\pi}{4}$ (c) $r = \theta \quad (\theta \geq 0)$

in polar coordinates.

Graphing Polar Equations

Ex:- Graph the curve $r = a(1-\cos\theta)$

Sol

Check the symmetric

1. About the origin

$-r = a(1-\cos\theta) \Rightarrow$ changed $\therefore$ unsymmetric about the origin

2. About the x-axis

$r = a(1-\cos -\theta) \Rightarrow r = a(1-\cos\theta)$

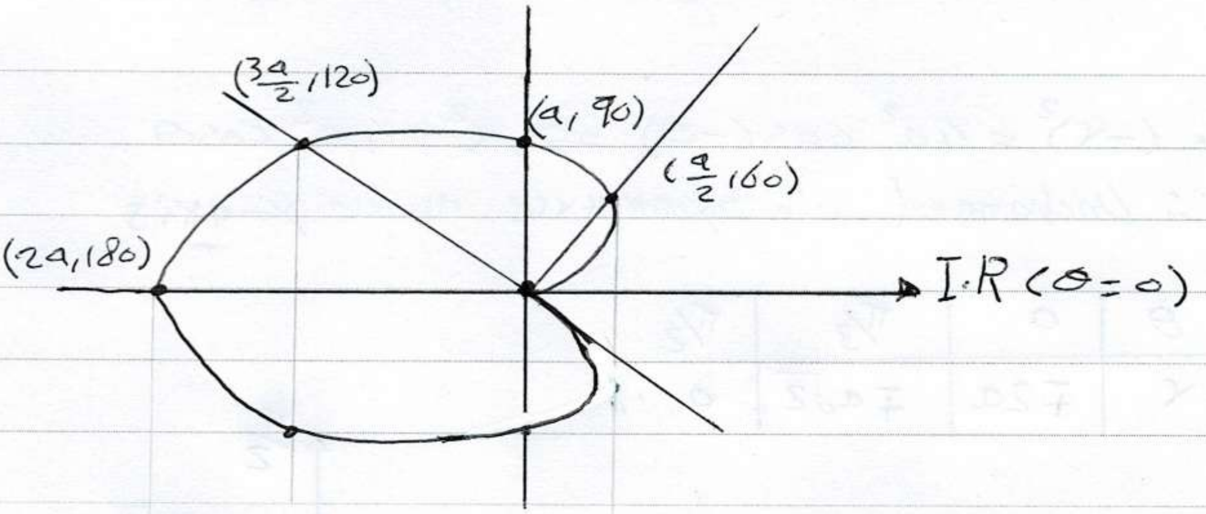
Unchanched $\therefore$ symmetric

3. About the y-axis

$r = a(1-\cos\theta) \Rightarrow -r = a(1-\cos -\theta)$

$\therefore$ Changed $\Rightarrow$ Unsymmetric

θ	0	60	90	120	180
r	0	$a/2$	a	$\frac{3a}{2}$	$2a$



$(\frac{3a}{2}, 120)$
$(a, 90)$
$(\frac{a}{2}, 60)$
$(2a, 180)$
I.R ($\theta = 0$)

Solution (*a*). For all values of θ, the point $(1, \theta)$ is 1 unit away from the pole. Thus, the graph is the circle of radius 1 centered at the pole (Figure 11.1.13*a*).

Solution (*b*). For all values of r, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 11.1.13*b*). Positive values of r correspond to points on the line in the first quadrant and negative values of r to points on the line in the third quadrant. Thus, in absence of any restriction on r, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant.

Solution (*c*). Observe that as θ increases, so does r; thus, the graph is a curve that spirals out from the pole as θ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the intersections with the x- and y-axes for values of θ that are multiples of $\pi/2$, keeping in mind that the value of r is always equal to the value of θ (Figure 11.1.13*c*). ◀

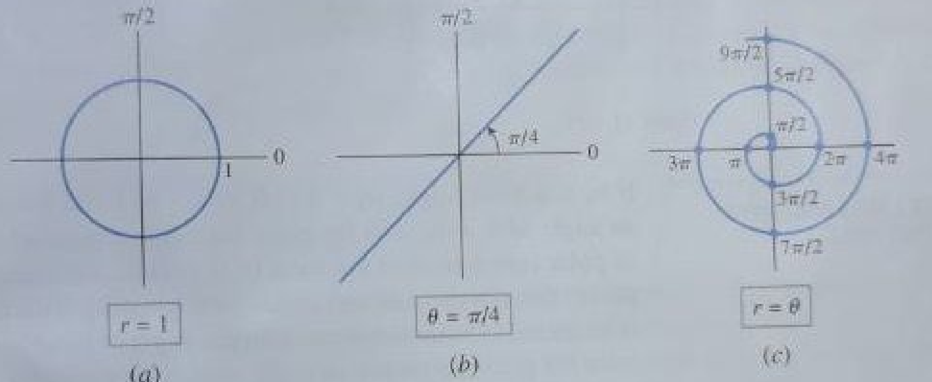


Figure 11.1.13

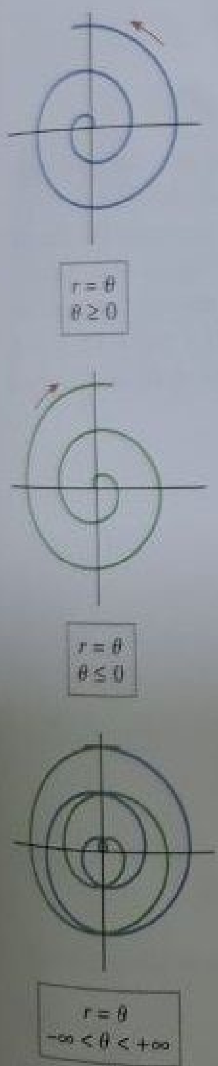


Figure 11.1.14

REMARK. The spiral in Figure 11.1.13*c*, which belongs to the family of ***Archimedean spirals*** $r = a\theta$, coils counterclockwise around the pole because of the restriction $\theta \geq 0$. Had we made the restriction $\theta \leq 0$, the spiral would have coiled clockwise, and had we allowed both positive and negative values of θ, the clockwise and counterclockwise spirals would have been superimposed to form a double Archimedean spiral (Figure 11.1.14).

Example 8 Sketch the graph of $r^2 = 4\cos 2\theta$ in polar coordinates.

Solution. This equation does not express r as a function of θ, since solving for r in terms of θ yields two functions:

$$r = 2\sqrt{\cos 2\theta} \quad \text{and} \quad r = -2\sqrt{\cos 2\theta}$$

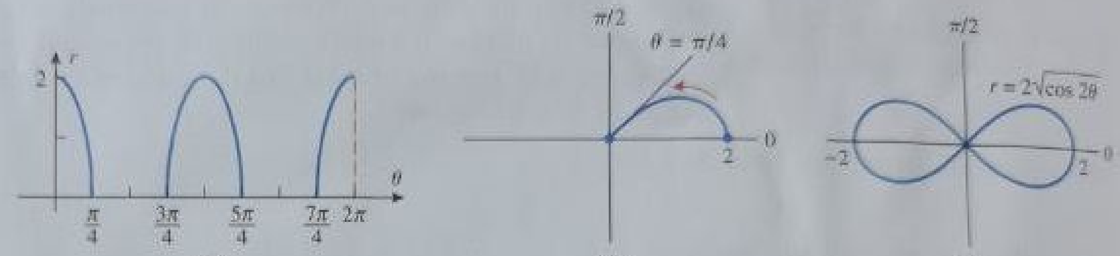
Thus, to graph the equation $r^2 = 4\cos 2\theta$ we will have to graph the two functions separately and then combine those graphs.

We will start with the graph of $r = 2\sqrt{\cos 2\theta}$. Observe first that this equation is not changed if we replace θ by $-\theta$ or if we replace θ by $\pi - \theta$. Thus, the graph is symmetric about the x-axis and the y-axis. This means that the entire graph can be obtained by graphing the portion in the first quadrant, reflecting that portion about the y-axis to obtain the portion in the second quadrant and then reflecting those two portions about the x-axis to obtain the portions in the third and fourth quadrants.

To begin the analysis, we will graph the equation $r = 2\sqrt{\cos 2\theta}$ in rectangular coordinates (see Figure 11.1.15*a*). Note that there are gaps in that graph over the intervals $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ because $\cos 2\theta$ is negative for those values of θ. From this graph we can see that:

- As θ varies from 0 to $\pi/4$, r decreases from 2 to 0.
- As θ varies from $\pi/4$ to $\pi/2$, no points are generated on the polar graph.

This produces the portion of the graph shown in Figure 11.1.15*b*. As noted above, we can complete the graph by a reflection about the y-axis followed by a reflection about the x-axis (11.1.15*c*). The resulting propeller-shaped graph is called a ***lemniscate*** (from the Greek word "lemniscos" for a looped ribbon resembling the number 8). We leave it for you to verify that the equation $r = 2\sqrt{\cos 2\theta}$ has the same graph as $r = -2\sqrt{\cos 2\theta}$, but traced in a diagonally opposite manner. Thus, the graph of the equation $r^2 = 4\cos 2\theta$ consists of two identical superimposed lemniscates. ◀

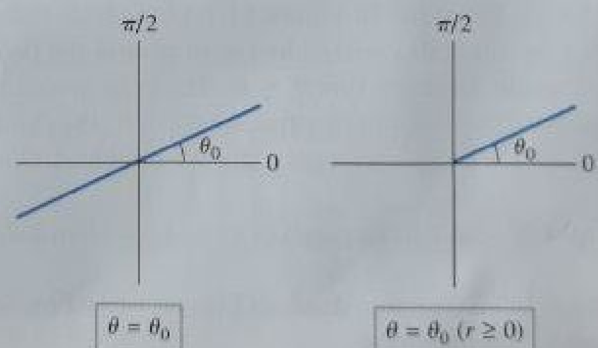


(*a*) (*b*) (*c*)

Figure 11.1.15

FAMILIES OF LINES AND RAYS THROUGH THE POLE

If θ_0 is a fixed angle, then for all values of r the point (r, θ_0) lies on the line that makes an angle of $\theta = \theta_0$ with the polar axis; and, conversely, every point on this line has a pair of polar coordinates of the form (r, θ_0). Thus, the equation $\theta = \theta_0$ represents the line that passes through the pole and makes an angle of θ_0 with the polar axis (Figure 11.1.16*a*). If r is restricted to be nonnegative, then the graph of the equation $\theta = \theta_0$ is the ray that emanates from the pole and makes an angle of θ_0 with the polar axis (Figure 11.1.16*b*). Thus, as θ_0 varies, the equation $\theta = \theta_0$ produces either a family of lines through the pole or a family of rays through the pole, depending on the restrictions on r.



(*a*) (*b*)

Figure 11.1.16

FAMILIES OF CIRCLES

We will consider three families of circles in which a is assumed to be a positive constant:

$$r = a \qquad r = 2a\cos\theta \qquad r = 2a\sin\theta \tag{3–5}$$

The equation $r = a$ represents a circle of radius a centered at the pole (Figure 11.1.17*a*). Thus, as a varies, this equation produces a family of circles centered at the pole. For families (4) and (5), recall from plane geometry that a triangle that is inscribed in a circle with a diameter of the circle for a side must be a right triangle. Thus, as indicated in Figures 11.1.17*b* and 11.1.17*c*, the equation $r = 2a\cos\theta$ represents a circle of radius a, centered on the x-axis and tangent to the y-axis at the origin; similarly, the equation $r = 2a\sin\theta$ represents a circle

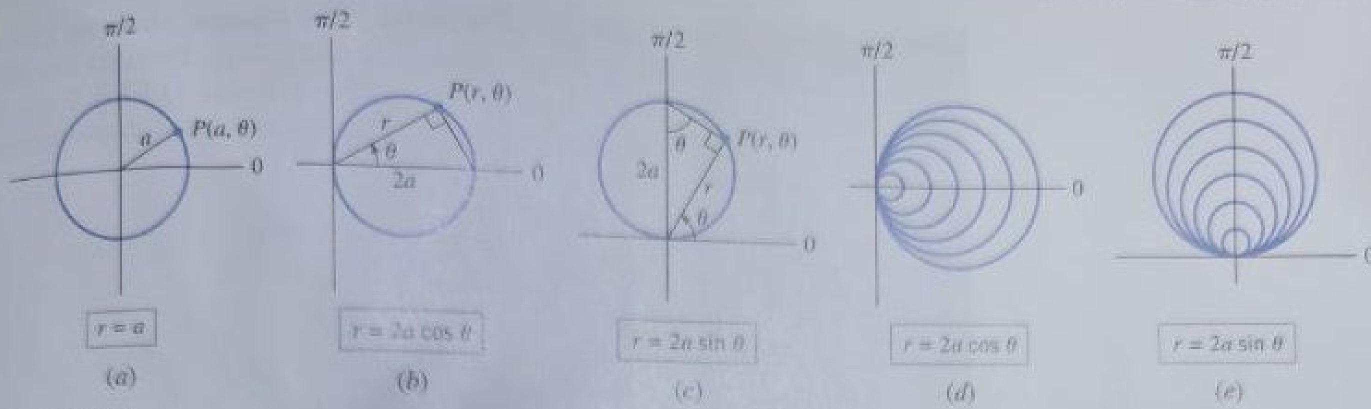


Figure 11.1.17

of radius a, centered on the y-axis and tangent to the x-axis at the origin. Thus, as a varies, Equations (4) and (5) produce the families illustrated in Figures 11.1.17*d* and 11.1.17*e*.

REMARK. Observe that replacing θ by $-\theta$ does not change the equation $r = 2a\cos\theta$, and replacing θ by $\pi - \theta$ does not change the equation $r = 2a\sin\theta$. This explains why the circles in Figure 11.1.17*d* are symmetric about the x-axis and those in Figure 11.1.17*e* are symmetric about the y-axis.

FAMILIES OF ROSE CURVES

In polar coordinates, equations of the form

$$r = a\sin n\theta \qquad r = a\cos n\theta \tag{6–7}$$

in which $a > 0$ and n is a positive integer represent families of flower-shaped curves called ***roses*** (Figure 11.1.18). The rose consists of n equally spaced petals of radius a if n is odd and $2n$ equally spaced petals of radius a if n is even. It can be shown that a rose with an even number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < 2\pi$ and a rose with an odd number of petals is traced out exactly once as θ varies over the interval $0 \le \theta < \pi$ (Exercise 73). A four-petal rose of radius 1 was graphed in Example 4.

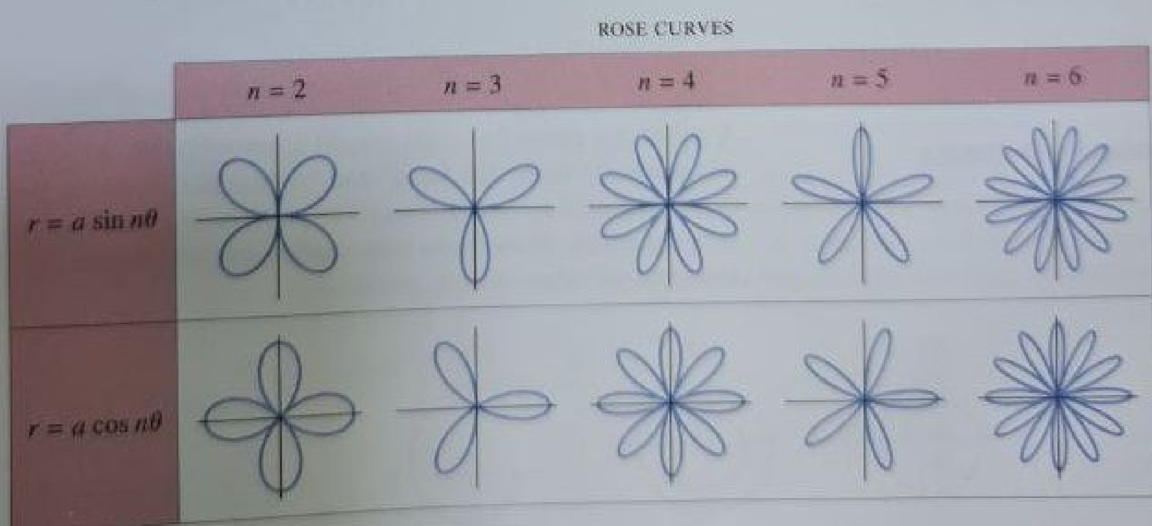


Figure 11.1.18

• FOR THE READER. What do the graphs of the one-petal roses look like?

FAMILIES OF CARDIOIDS AND LIMAÇONS

Equations with any of the four forms

$$r = a \pm b\sin\theta \qquad r = a \pm b\cos\theta \tag{8–9}$$

in which $a > 0$ and $b > 0$ represent polar curves called ***limaçons*** (from the Latin word "limax" for a snail-like creature that is commonly called a slug). There are four possible shapes for a limaçon that are determined by the ratio a/b (Figure 11.1.19). If $a = b$ (the case $a/b = 1$), then the limaçon is called a ***cardioid*** because of its heart-shaped appearance, as noted in Example 6.

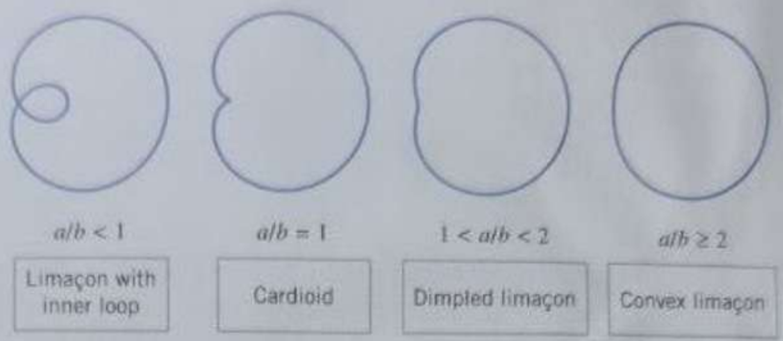


Figure 11.1.19

Example 9 Figure 11.1.20 shows the family of limaçons $r = a + \cos\theta$ with the constant a varying from 0.25 to 2.50 in steps of 0.25. In keeping with Figure 11.1.19, the limaçons evolve from the loop type to the convex type. As a increases from the starting value of 0.25, the loops get smaller and smaller until the cardioid is reached at $a = 1$. As a increases further, the limaçons evolve through the dimpled type into the convex type. ◀

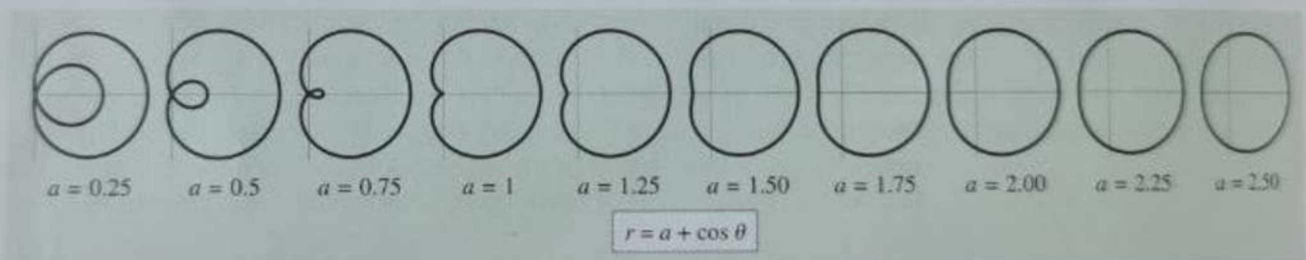


Figure 11.1.20

FAMILIES OF SPIRALS

A ***spiral*** is a curve that coils around a central point. As illustrated in Figure 11.1.14, spirals generally have "left-hand" and "right-hand" versions that coil in opposite directions, depending on the restrictions on the polar angle and the signs of constants that appear in their equations. Some of the more common types of spirals are shown in Figure 11.1.21 for nonnegative values of θ, a, and b.

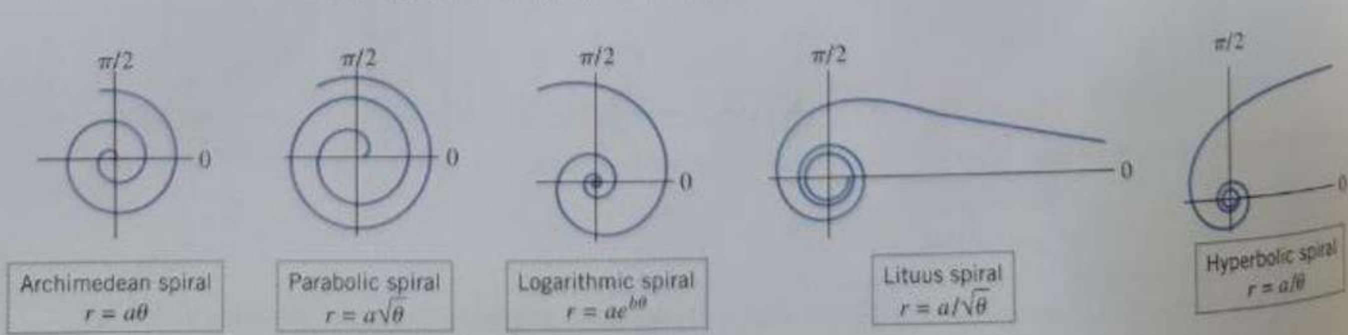


Figure 11.1.21

SPIRALS IN NATURE

Spirals of many kinds occur in nature. For example, the shell of the chambered nautilus (*below*) forms a logarithmic spiral, and a coiled sailor's rope forms an Archimedean spiral. Spirals also occur in flowers, the tusks of certain animals, and in the shapes of galaxies.

The shell of the chambered nautilus reveals a logarithmic spiral. The animal lives in the outermost chamber.

A sailor's coiled rope forms an Archimedean spiral.

Steps for finding points of intersection of two polar curves

To find the points of intersection of two polar curves,

solve both curves for r,

set the two curves equal to each other

solve for θ

Using these steps, we might get more intersection points than actually exist, or fewer intersection points than actually exist. To verify that we've found all of the intersection points, and only real intersection points, we graph our curves and visually confirm the intersection points.

We can also convert our polar equations to rectangular equations, solve for the points of intersection of the rectangular curves, and then convert the rectangular points of intersection back into polar coordinates. Even though it's extra work to convert everything from polar to rectangular, using this method guarantees that we'll find all of the points of intersection, and only the real points of intersection

How to find the points of intersection of polar curves

Two examples of finding intersection points of the polar curves

Example

Find the points of intersection of the polar curves.

1. $r=\sin\theta$
2. $r=1-\sin\theta$

To find the points of intersection of these polar curves, we'll set them equal to each other and solve for θ.

$$\text{Sin } \theta=1-\sin\theta$$

$2\sin\theta=1$

$\sin\theta=1/2$

$\theta=\pi/6,\ 5\pi/6$

To find the values of r that are associated with these values of θ, we'll plug the θ values back into either of the original polar curves; we'll choose r=Sinθ.

For $\theta=\pi/6$

$r=\sin\pi/6$

$r=1/2$

For $\theta=5\pi/6$

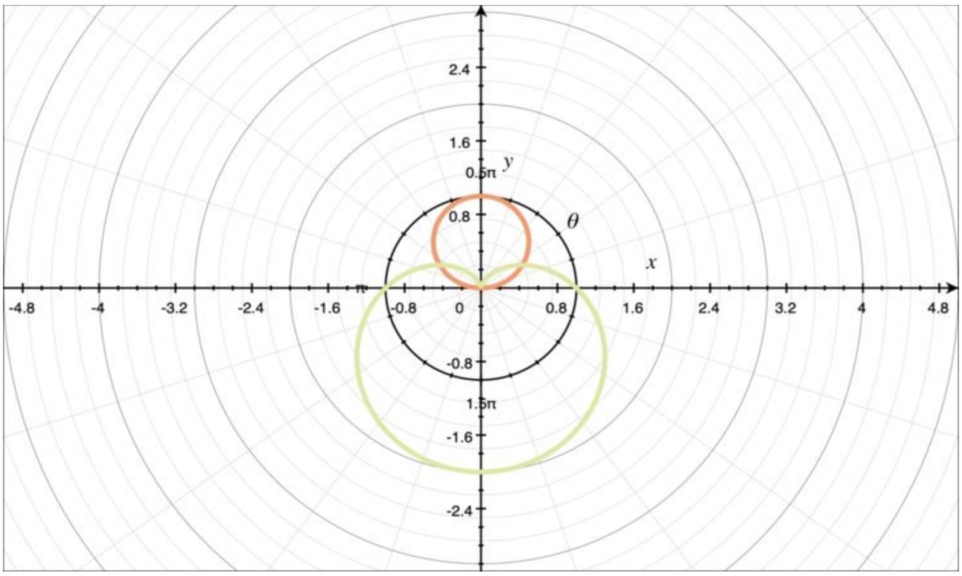
$r=\sin 5\pi/6$

$r=1/2$

Putting these values together, the points of intersection are

$(1/2,\pi/6)$ and $(1/2,5\pi/6)$

To confirm that these are the points of intersection, we can graph both curves.



Looking at the graph, we see that $(0,0)$ is also a point of intersection, so in total, the graphs intersect each other at

$$(1/2,\pi/6) \text{ and } (1/2,5\pi/6) \text{ and } (0,0)$$

In the previous example, we had to graph the polar curves in order to find all of the points of intersection. That's because we left everything in polar form.

Let's try another example where we convert our polar curves into rectangular coordinates.

Example

Find the points of intersection of the polar curves.

$$r=\cos\theta$$

$$r=2-\cos\theta$$

We'll convert the polar curves to rectangular coordinates using the conversion formula

$$x=r\cos\theta$$

$$\cos\theta=x/r$$

Plugging x/r into the given polar equations for $\cos\theta$, we get

$$r=\cos\theta$$

$$r=x/r$$

$$x=r^2 \text{ and } r=2-\cos\theta$$

$$r=2-x/r$$

$r^2 = 2r - x$

$x = 2r - r^2$

We've gotten rid of θ, but now we need to get rid of r, which we'll do using the conversion formula

$r^2 = x^2 + y^2$

$r = \sqrt{x2 + y2}$

Plugging x^2+y^2 into the given polar equations for r^2, and $\sqrt{x2 + y2}$ in for r, we get

$x = r^2$

$x = x2 + y2$

$x^2 + y^2 - x = 0$ ------------1

and

$x = 2r - r^2$

$x = 2\sqrt{x2 + y2} - (x^2 + y^2)$

$x = 2\sqrt{x2 + y2} - x^2 - y^2$

$x^2 + y^2 - 2\sqrt{x2 + y2} + x = 0$-----------2

Since both of our rectangular equations are equal to 0, we can set them equal to each other.

$x^2 + y^2 - x = x^2 + y^2 - 2\sqrt{x2 + y2 +} X$

$-x = -2\sqrt{x2 + y2} + x$

$-2\sqrt{x2 + y2} = -2x$

$\sqrt{x2 + y2} = X$

Since we found that $x^2+y^2=x$ when we were converting $r=\cos\theta$

to rectangular coordinates, we can say

$\sqrt{x}=x$

$x=x^2$

$x^2-x=0$

$x(x-1)=0$

$x(x-1)=0$

$x=0, 1$

To find the y values associated with these x-values, we'll plug them into $x^2+y^2-x=0$

For $x=0$

$(0)^2+y^2-(0)=0$

$y=0$

For $x=1$

$(1)^2+y^2-(1)=0$

$y=0$

Putting our values together, we know that the points of intersection are (0,0) and (1,0)

We need to convert these rectangular coordinate points back into polar coordinates, which we'll do using the conversion formulas

$r=\sqrt{x2+y2}$

$\theta=\tan-1(y/x)$

Plugging the rectangular coordinate points into these formulas, we get

For (0,0)

$r=\sqrt{(0)2 + (0)2}$ so r=0

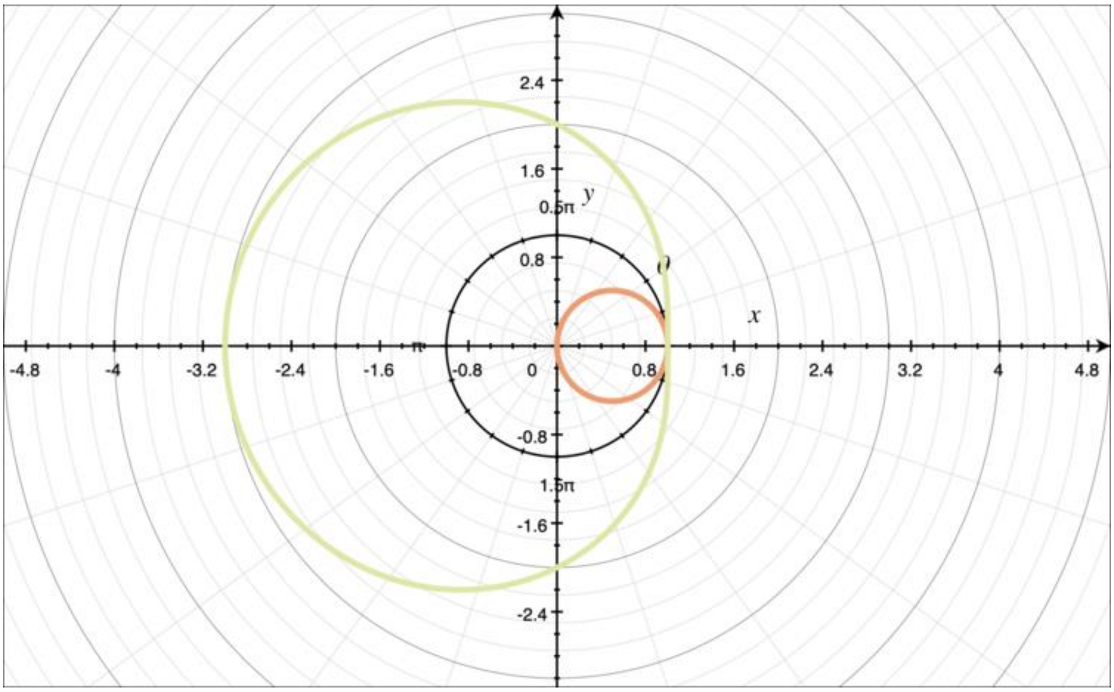
Since the equation for \theta is undefined, the rectangular point (0,0) can’t be defined in polar coordinates and therefore isn’t a polar point of intersection.

For (1,0)

$r=\sqrt{(1)2 + (0)2}$ so r=1

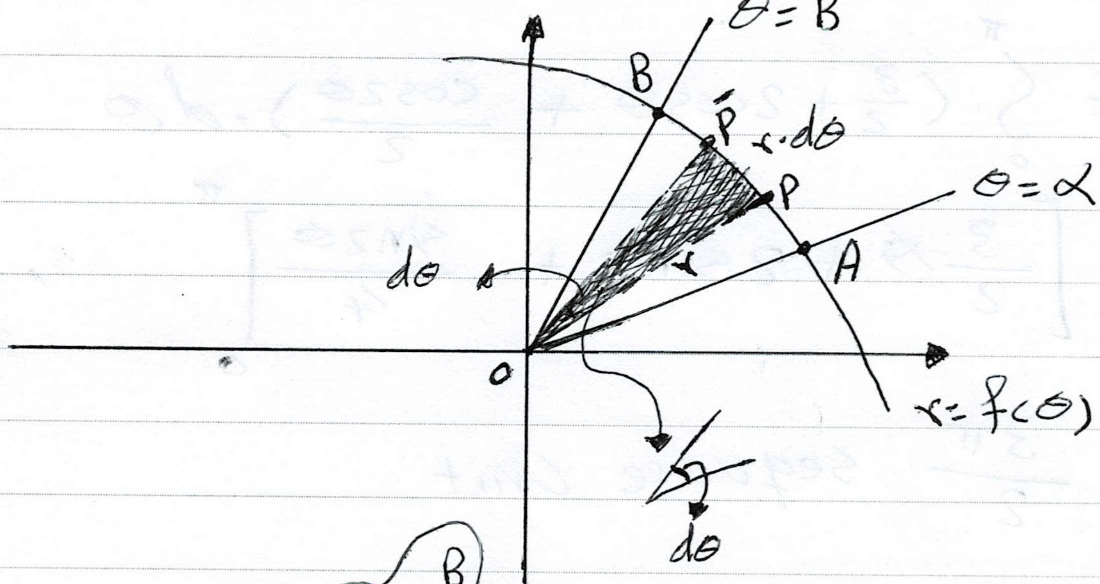
$\theta=\tan-1(0/1)$ so $\theta=0$

The only point of intersection of the given polar curves is the polar point (1,0).If we want to double-check ourselves, we can sketch the polar curves and confirm this point of intersection.



*Plane Areas of Polar Coordinates

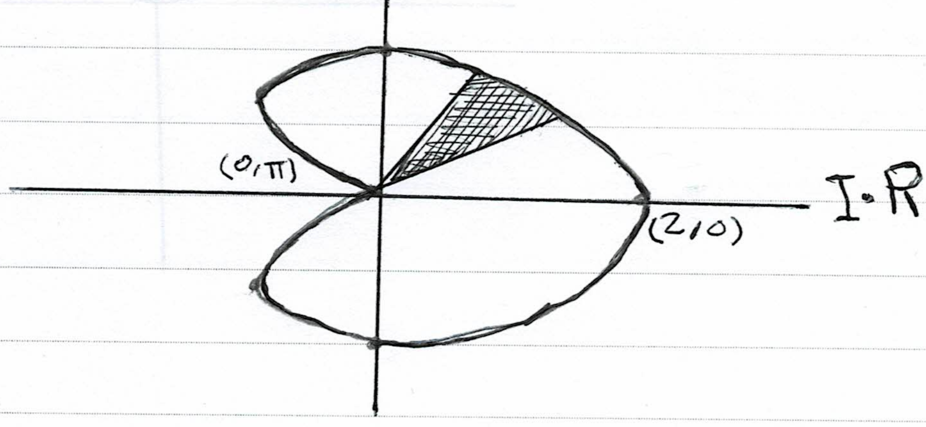
The area of (AoB) shown in figure is bounded by the rays $\theta = \alpha$, $\theta = B$ and $r = f(\theta)$. To find the area of this section, the strip $(Po\bar{P})$ is taken and integrated.



$$\text{Area (AoB)} = \int_{\alpha}^{B} \frac{1}{2} r^2 . d\theta$$ *

Ex:- Find the area of the region that bounded by the curve $r = 1 + \cos\theta$

Sol



$$A = \int_{\alpha}^{B} \frac{1}{2} r^2 \, d\theta$$

$$= 2 \int_{0}^{\pi} \frac{1}{2}(1+\cos\theta)^2 . d\theta = \int_{0}^{\pi} (1+2\cos\theta+\cos^2\theta) . d\theta$$

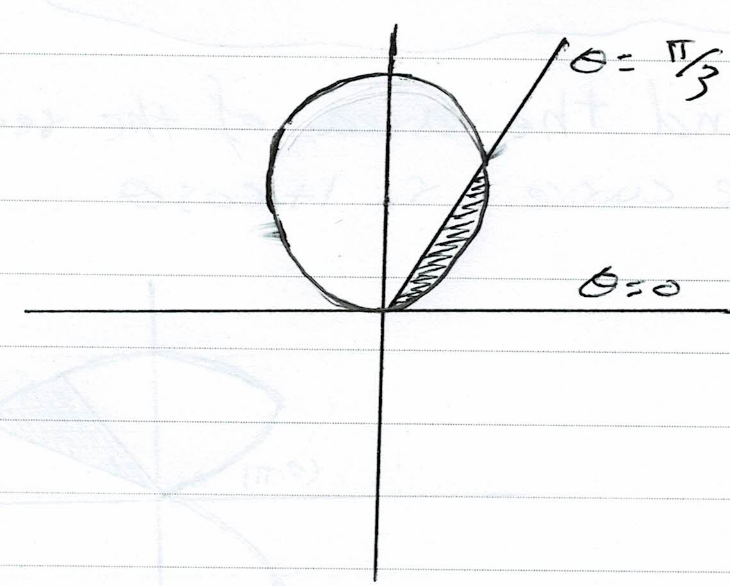
$$= \int_{0}^{\pi} \left(1+2\cos\theta + \frac{1+\cos 2\theta}{2}\right) . d\theta$$

$$= \int_{0}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right) . d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{0}^{\pi}$$

$$= \frac{3\pi}{2} \text{ sequare Unit}$$

Ex:- Find the area that enclosed by the curve $r = 5\sin\theta$ and the radius $\theta = 0$ and $\theta = \frac{\pi}{3}$



Sol

$$A = \int_{\alpha}^{B} \frac{1}{2} r^2 . d\theta$$

$$= \int_{0}^{\pi/3} \frac{1}{2} (5 \sin\theta)^2 . d\theta$$

$$= \int_{0}^{\pi/3} \frac{1}{2} * 25 * \sin^2\theta . d\theta$$

$$= \frac{25}{2} \int_{0}^{\pi/3} \left(\frac{1-\cos 2\theta}{2}\right) . d\theta$$

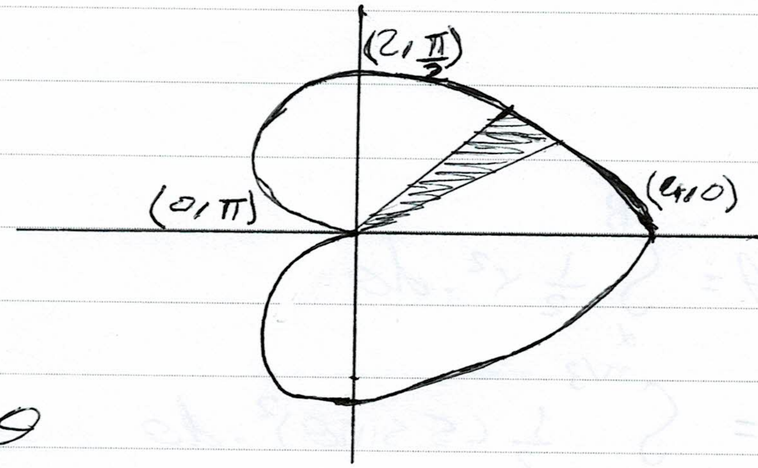
$$= \frac{25}{4} \int_{0}^{\pi/3} (1-\cos 2\theta) . d\theta$$

$$= \frac{25}{4} \left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\pi/3}$$

$$= \frac{25}{4} \left[\left(\frac{\pi}{3} - \frac{\sin 2 * \pi/3}{2}\right) - \left(0 - \frac{\sin 2 * 0}{2}\right)\right]$$

$$= \frac{25}{4} \left(\frac{3.14}{3} - 0.433\right) = 3.84 \text{ square unit}$$

EX:- Find the area of the region that enclosed by the curve $r = 2(1+\cos\theta)$.



Sol

$$A = \int_{\alpha}^{B} \frac{1}{2} r^2 \cdot d\theta$$

$$A = \int_{0}^{2\pi} \frac{1}{2} * 4 * (1 + 2\cos\theta + \cos^2\theta) \cdot d\theta$$

$$= \int_{0}^{2\pi} 2\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) \cdot d\theta$$

$$= \int_{0}^{2\pi} 2\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right) \cdot d\theta$$

$$= 2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{0}^{2\pi}$$

$$= 2\left[\frac{3}{\cancel{2}}\cancel{2}\pi + \cancel{2\sin 2\pi + \frac{\sin 4\pi}{4}}\right] - [0]$$

$$= 6\pi \text{ square unit}$$