Consistent deformation Method
(Trusses)

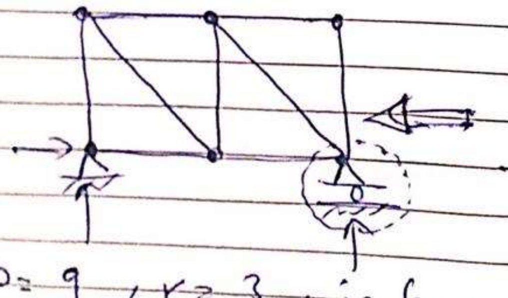
This method can be used to analyse indeterminate trusses using similar steps used to analyse indeterminate beams and frames but with some changes

Use similar principles including the removal of redundants and stability checking.

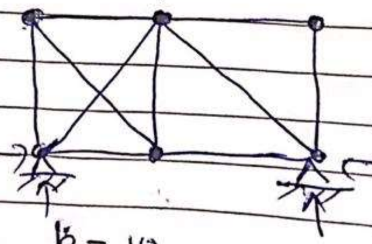
Use virtual work method to determine the deflections required.

Remember that (b+r) should be compared with (2j) to test the stability & Det. of a truss

It is posible to use reaction or member as a redundent

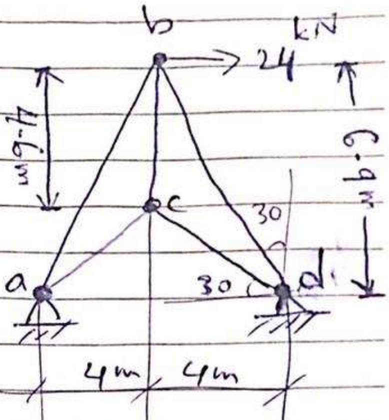


b = 9 , r = 3 , j = 6
9+3 = 12
b+r = 2j O.K



b = 10 , r = 4
j = 6
b+r = 14 > 2j = 12
2 redundents

Ex1 Analyse the truss shown, EA: const.



① Stability & Determinacy

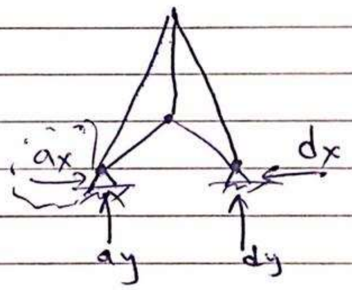
$b + r > 2j$

$9 > 8$ ∴ Indet. to 1^{st} deg.

∴ One redundant should be removed

It is better here to remove one of the reactions

∴ Remove a_x



② Compatibility equations

One redundant → One equation

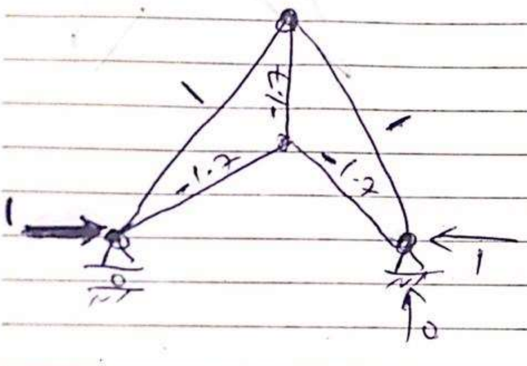
$$\Delta_{10} + \delta_{11} X_1 = 0$$

③ Analysis of determinate truss to find

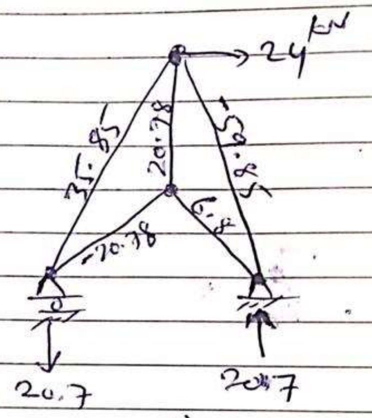
S' & U

S' ↓ due to loading

U ↓ due to unit load



(U_1)



(S')

④ Table of calculations

Mem	L(m)	S'	U_1	$S U_1 L$	$U_1^2 L$
ab	8	35.85	1	268.8	8
ac	4.6	-20.7	-1.7	161.8	13.294
bc	4.6	20.7	-1.7	-162.5	13.294
bd	8	-59.85	1	-408.8	8
cd	4.6	6.8	-1.7	-53.14	13.294
			Σ	-246.3	55.88

Use the compatibility equation to find the magnitude of the redundant X_1

$$\Delta_{10} + \delta_{11} X_1 = 0$$

$$\frac{-246.3}{AE} + \frac{55.9}{AE} X_1 = 0$$

$$X_1 = 4.4 \text{ kN} \longrightarrow$$

(5) Forces in members & reactions

$$F = S + U_1 X_1$$

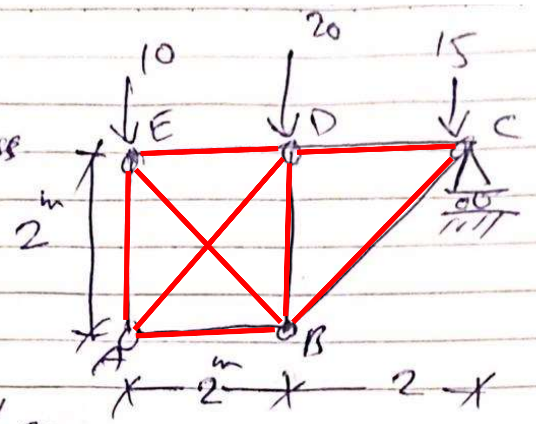
$$F = S + U_1 X_1 + U_2 X_2$$

$$F = S + U_1 X_1 + U_2 X_2 \cdots + U_n X_n$$

member	S	U_1	F
ab	-35.85	+1.0	40.25
ac	-20.7	-1.7	-28.18
bc	20.7	-1.7	13.3
bd	-59.85	+1.0	-55.45
cd	6.8	-1.7	-0.68
ax	0	+1.0	4.4 →
ay	↓20.7	0	20.7 ↓
dx	← 24	←+1.0	28.4 ←
dy	↑ 20.7	0	20.7 ↑

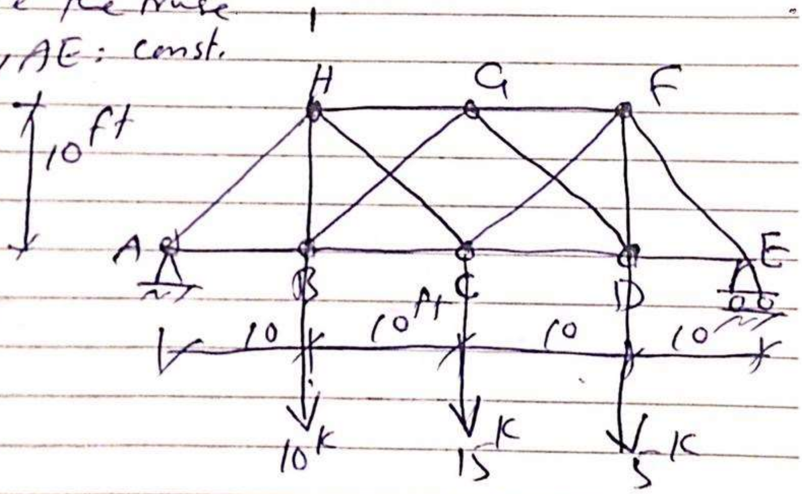
H.W/

(1) Analyse the truss shown, AE: const.



ans. $F_{AD} = 8.54^{kN}$ comp.

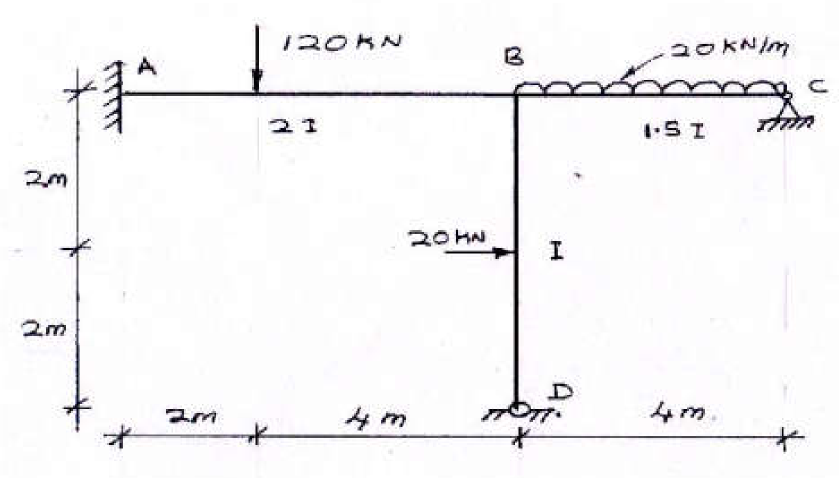
(2) Analyse the truss shown, AE: const.



ans. $F_{GB} = 1.19^{k}$ tension

Slope Deflection Method-Frames

Example: Analyse the simple frame shown in figure. End A is fixed and ends B & C are hinged. Draw the bending moment diagram.



Solution:
In this problem $\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D \neq 0,$

FEMS:-

$$F_{AB} = -\frac{Wab^2}{L^2} = -\frac{120 \times 2 \times 4^2}{6^2} = -106.67 \text{ KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +\frac{120 \times 2^2 \times 4}{6^2} = +53.33 \text{ KNM}$$

$$F_{BC} = -\frac{wl^2}{12} = -\frac{20 \times 4^2}{12} = -26.67 \text{ KNM}$$

$$F_{CB} = +\frac{wl^2}{12} = +\frac{20 \times 4^2}{12} = +26.67 \text{ KNM}$$

$$F_{CD} = +\frac{WL}{8} = +\frac{20 \times 4}{8} = +10 \text{ KNM}$$

$$F_{DB} = -\frac{WL}{8} = -10 \text{ KNM}$$

Slope deflections are

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= -106.67 + \frac{2E2I}{6}(\theta_B) = -106.67 + \frac{2}{3}EI\theta_B \qquad ---->(1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_B)$$
$$= +53.33 + \frac{2E2I}{6}(2\theta_B) = +53.33 + \frac{4}{3}EI\theta_B \qquad ---- > (2)$$
$$M_{BC} = F_{CB} + \frac{2EI}{L}(2\theta_B + \theta_C)$$
$$= -26.67 + \frac{2E}{4} \times \frac{3I}{2}(2\theta_B + \theta_C) = -26.67 + \frac{3}{2}EI\theta_B + \frac{3}{4}EI\theta_C \qquad ---- > (3)$$
$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$
$$= +26.67 + \frac{2E}{4} \times \frac{3I}{2}(2\theta_C + \theta_B) = +26.67 + \frac{3}{2}EI\theta_C + \frac{3}{4}EI\theta_B \qquad ---- > (4)$$

$$M_{BD} = F_{BD} + \frac{2EI}{L}(2\theta_B + \theta_D)$$
$$= +10 + \frac{2EI}{4}(2\theta_B + \theta_D) = +10 + EI\theta_B + \frac{1}{2}EI\theta_D \qquad ---- > (5)$$
$$M_{DB} = F_{DB} + \frac{2EI}{L}(2\theta_D + \theta_B)$$
$$= -10 + \frac{2EI}{4}(2\theta_D + \theta_B) = -10 + EI\theta_D + \frac{1}{2}EI\theta_B \qquad ---- > (6)$$

In the above equations we have three unknown rotations $\theta_B, \theta_C, \theta_D$ accordingly we have three boundary conditions.

$$M_{BA} + M_{BC} + M_{BD} = 0$$
$$M_{CB} = 0 \qquad \text{Since C and D are hinged}$$
$$M_{DB} = 0$$

Now

$$M_{BA} + M_{BC} + M_{BD} = 53.33 + \frac{4}{3}EI\theta_B - 26.67 + \frac{3}{2}EI\theta_B + \frac{3}{4}EI\theta_C + 10 + EI\theta_B + \frac{1}{2}EI\theta_D$$
$$= 36.66 + \frac{23}{6}EI\theta_B + \frac{3}{4}EI\theta_C + \frac{1}{2}EI\theta_D = 0 \qquad ----- > (7)$$
$$M_{CB} = 26.67 + \frac{3}{4}EI\theta_B + \frac{3}{2}EI\theta_C = 0 \qquad ----- > (8)$$
$$M_{DB} = -10 + \frac{1}{2}EI\theta_B + EI\theta_D = 0 \qquad ---- > (9)$$

Solving equations 7, 8, & 9 we get

$$EI\theta_B = -8.83$$
$$EI\theta_C = -13.36$$
$$EI\theta_D = +14.414$$

Substituting these values in slope equations

$$M_{AB} = -106.67 + \frac{2}{3}(-8.83) = -112.56 \text{ KNM}$$

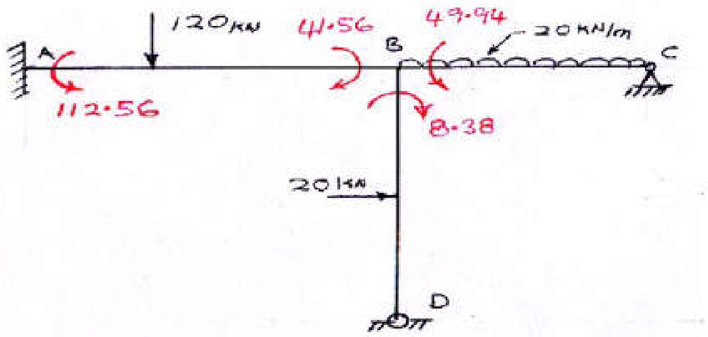
$$M_{BA} = 53.33 + \frac{4}{3}(-8.83) = 41.56 \text{ KNM}$$

$$M_{BC} = -26.67 + \frac{3}{2}(-8.3) + \frac{3}{4}(-13.36) = -49.94 \text{ KNM}$$

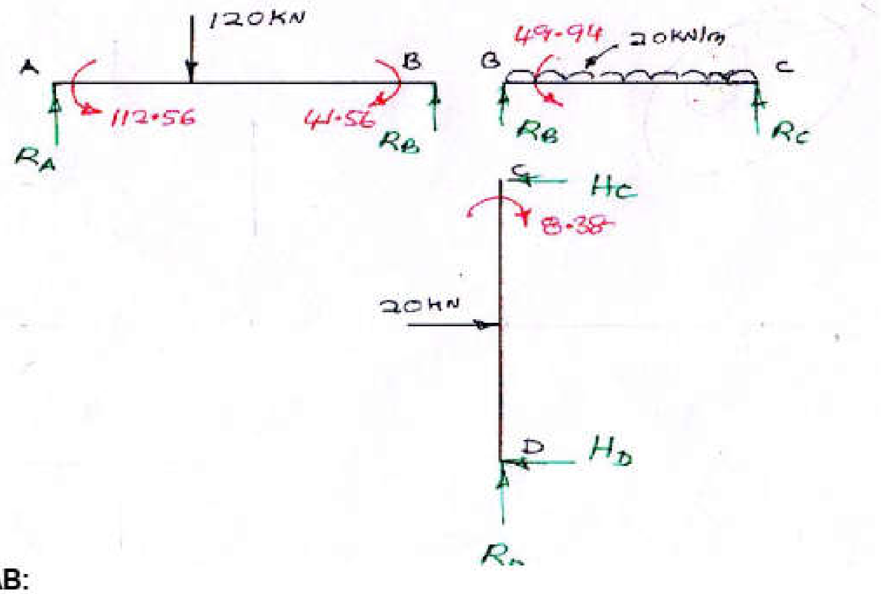
$$M_{CB} = +26.67 + \frac{3}{2}(-13.36) + \frac{3}{4}(-8.83) = 0$$

$$M_{BD} = 10 + (-8.83) + \frac{1}{2}(+14.414) = 8.38 \text{ KNM}$$

$$M_{DB} = -10 + (14.414) + \frac{1}{2}(-8.83) = 0$$



Reactions: Consider free body diagram of each members



Span AB:

$$R_B = \frac{41.56 - 112.56 + 120 \times 2}{6} = 28.17 \text{ KN}$$

$$\therefore R_A = 120 - R_B = 91.83 \text{ KN}$$

Span BC:

$$R_B = \frac{49.94 + 20 \times 4 \times 2}{4} = 52.485 \text{ KN}$$

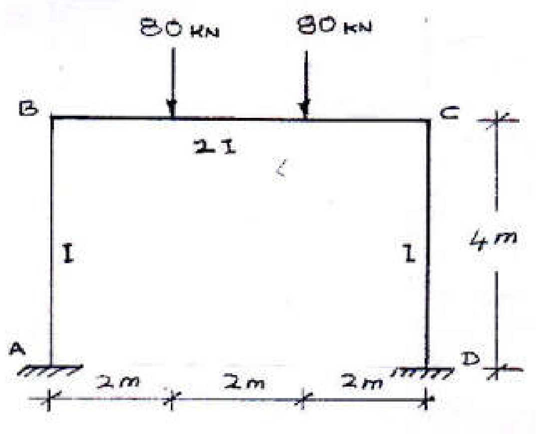
$$\therefore R_C = 20 \times 4 - R_B = 27.515 \text{ KN}$$

Column BD:

$$H_D = \frac{20 \times 2 - 8.33}{4} = 7.92 \text{ KN}$$

$$\therefore H_B = 12.78 \text{ KN} \qquad [\because H_A + H_D = 20]$$

Example: Analyse the portal frame shown in figure and also drawn bending moment and shear force diagram



Solution:

Symmetrical problem
- Sym frame + Sym loading

$\theta_A = 0,\ \ \theta_B \neq 0,\ \ \theta_C \neq 0,\ \ \theta_D = 0$

FEMS

$$F_{BC} = -\frac{W_1ab^2}{L^2} - \frac{W_2cd^2}{L^2}$$

$$= \frac{80 \times 2 \times 4^2}{6^2} - \frac{80 \times 4 \times 2^2}{6^2} = -106.67 \text{ KNM}$$

$$F_{CB} = +\frac{Wa^2b}{L^2} + \frac{W_2c^2d}{L^2} = +106.67 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) = 0 + \frac{2EI}{4}(0 + \theta_B) = \frac{1}{2}EI\theta_B \quad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) = 0 + \frac{2EI}{4}(2\theta_B + 0) = EI\theta_B \quad \text{-------} > (2)$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}(2\theta_B+\theta_C)$$

$$=-106.67+\frac{2E2I}{6}(2\theta_B+\theta_C)=-106.67+\frac{4}{3}EI\theta_B+\frac{2}{3}EI\theta_C \text{ ------}>(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}(2\theta_C+\theta_B)$$

$$=+106.67+\frac{2E2I}{6}(2\theta_C+\theta_B)=+106.67+\frac{4}{3}EI\theta_C+\frac{2}{3}EI\theta_B \text{ ------}>(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}(2\theta_C+\theta_D)$$

$$=0+\frac{2EI}{4}(2\theta_C+0)=EI\theta_C \text{ -------}>(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}(2\theta_D+\theta_C)$$

$$=0+\frac{2EI}{4}(0+\theta_C)=\frac{1}{2}EI\theta_C \text{ -------}>(6)$$

In the above equation there are two unknown rotations. Accordingly the boundary conditions are

$$M_{BA}+M_{BC}=0$$

$$M_{CB}+M_{CD}=0$$

$$\text{Now } M_{BA}+M_{BC}=-106.67+\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_C=0 \text{ -------}>(7)$$

$$M_{CB}+M_{CD}=+106.67+\frac{2}{3}EI\theta_B+\frac{7}{3}EI\theta_C=0 \text{ -------}>(8)$$

Multiply by (7) and (8) by 2

$$\left.\begin{array}{l} -746.69+\frac{49}{3}EI\theta_B+\frac{14}{3}EI\theta_C=0 \\ +213.34+\frac{4}{3}EI\theta_B+\frac{14}{3}EI\theta_C=0 \end{array}\right\}\text{subtracts}$$

$$-960.03+\frac{45}{3}EI\theta_B=0$$

$$EI\theta_B=+960.03\times\frac{3}{45}=+\ 64 \quad \text{Clockwise}$$

Using equation (7)

$$EI\theta_C = -\frac{3}{2}\left[-106.67 + \frac{7}{3}EI\theta_B\right]$$

$$= -\frac{3}{2}\left[-106.67 + \frac{7}{3}\times 64\right] = -\ 64 \quad \text{Anticlockwise}$$

Here we find $\theta_B = -\ \theta_C$. It is obvious because the problem is symmetrical.

$\therefore$ Final moments are

$$M_{AB} = +\frac{64}{2} = +32 \quad KNM$$

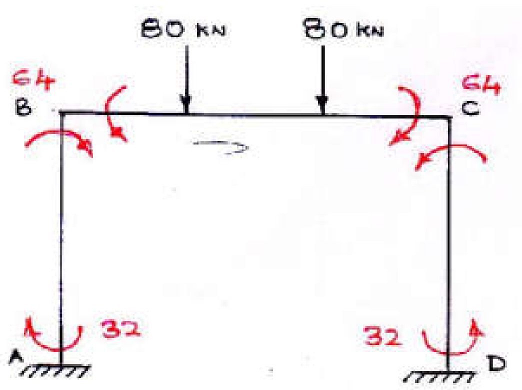
$$M_{BA} = 64 \quad KNM$$

$$M_{BC} = -106.67 + \frac{4}{3}64 + \frac{2}{3}(-64) = -\ 64 \quad KNM$$

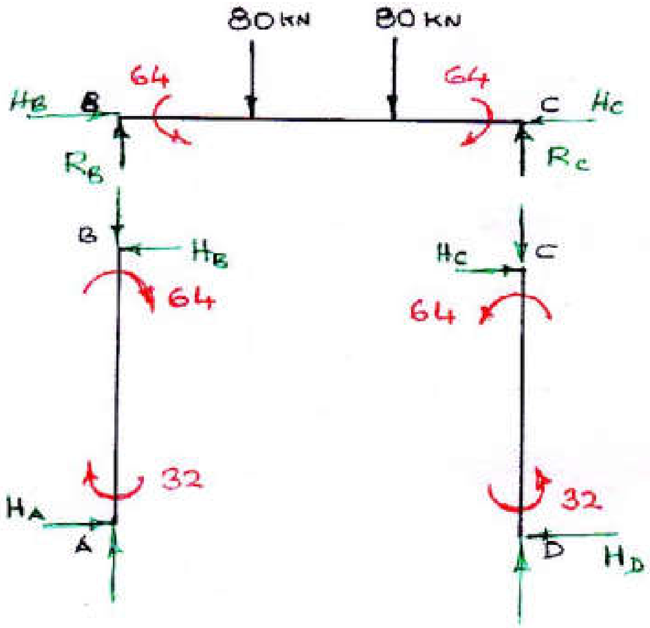
$$M_{CB} = +106.67 + \frac{4}{3}(-64) + \frac{2}{3}(64) = +\ 64 \quad KNM$$

$$M_{CD} = -\ 64 \quad KNM$$

$$M_{DC} = -\ \frac{1}{2}\ 64 = -32 \quad KNM$$



Consider free body diagram's of beam and columns as shown



By symmetrical we can write

$R_A = R_B = 60$ KNM

$R_D = R_C = 80$ KNM

Now consider free body diagram of column AB

Apply

$$\sum M_B = 0$$

$$H_A \times 4 = 64 + 32$$

$$\therefore H_A = 24 \quad KN$$

Similarly from free body diagram of column CD

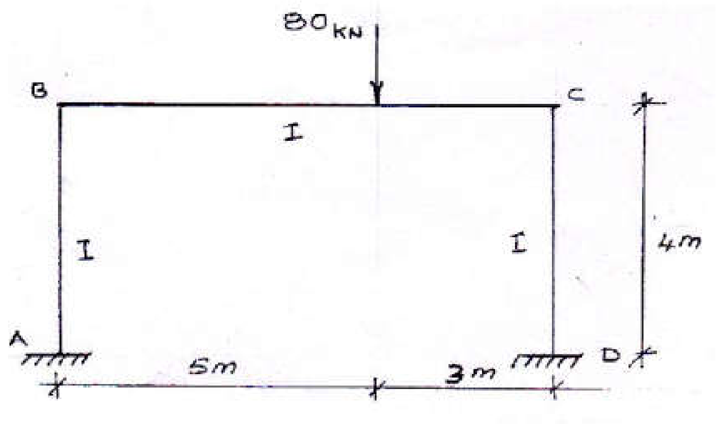
Apply

$$\sum M_C = 0$$

$$H_A \times 4 = 64 + 32$$

$$\therefore H_D = 24 \quad KN$$

Example: Analyse the portal frame and then draw the bending moment diagram



Solution:

This is a symmetrical frame and unsymmetrically loaded, thus it is an unsymmetrical problem and there is a sway

Assume sway to right.

Here $\theta_A = 0, \theta_D = 0, \theta_B \neq 0, \theta_D = 0$

<u>FEMS:</u>

$$F_{BC} = -\frac{Wab^2}{L^2} = -\frac{80 \times 5 \times 3^2}{8^2} = -56.25 \text{ KNM}$$

$$F_{CB} = +\frac{Wa^2b}{L^2} = +\frac{80 \times 5^2 \times 3}{8^2} = +93.75 \text{ KNM}$$

Slope deflection equations

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right) = \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------> (1)}$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right) = EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------> (2)}$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}(2\theta_B+\theta_C)$$

$$=-56.25+\frac{2EI}{8}(2\theta_B+\theta_C)=-56.25+\frac{1}{2}EI\theta_B+\frac{1}{4}EI\theta_C \quad \text{---------}>(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}(2\theta_C+\theta_B)$$

$$=+93.75+\frac{2EI}{8}(2\theta_C+\theta_B)=93.75+\frac{1}{2}EI\theta_C+\frac{1}{4}EI\theta_B \quad \text{---------}>(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}\left(2\theta_C+\theta_D-\frac{3\delta}{L}\right)$$

$$=0+\frac{2EI}{4}\left(2\theta_C+0-\frac{3\delta}{4}\right)=EI\theta_C-\frac{3}{8}EI\delta \quad \text{---------}>(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}\left(2\theta_D+\theta_C-\frac{3\delta}{L}\right)$$

$$=0+\frac{2EI}{4}\left(0+\theta_C-\frac{3\delta}{4}\right)=\frac{1}{2}EI\theta_C-\frac{3}{8}EI\delta \quad \text{---------}>(6)$$

In the above equation there are three unknowns θ_B, θ_C and δ, accordingly the boundary conditions are,

$$M_{BA}+M_{BC}=0 \quad \text{--}>\text{Joint conditions}$$

$$M_{CB}+M_{CD}=0$$

$$H_A+H_D+\sum P_H=0 \quad \text{---}>\text{Shear condition}$$

$$\text{i.e,}\quad \frac{M_{AB}+M_{BA}}{4}+\frac{M_{CD}+M_{DC}}{4}=0$$

$$\therefore\quad M_{AB}+M_{BA}+M_{CD}+M_{DC}=0$$

$$\text{Now,}\quad M_{BA}+M_{BC}=EI\theta_B-\frac{3}{8}EI\delta-56.25+\frac{1}{2}EI\theta_B+\frac{1}{4}EI\theta_C=0$$

$$=-56.25+\frac{3}{2}EI\theta_B+\frac{1}{4}EI\theta_C-\frac{3}{8}EI\delta=0 \quad \text{------}>(7)$$

$$\text{And,}\quad M_{CB}+M_{CD}=93.75+\frac{1}{2}EI\theta_C+\frac{1}{4}EI\theta_B+EI\theta_C-\frac{3}{8}EI\delta=0$$

$$=93.75+\frac{1}{4}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{8}EI\delta=0 \quad \text{-------}>(8)$$

And, $M_{AB}+M_{BA}+M_{CD}+M_{DC}=\frac{1}{2}EI\theta_B-\frac{3}{8}EI\delta+EI\theta_B-\frac{3}{8}EI\delta+EI\theta_C-\frac{3}{8}EI\delta$

$$+\frac{1}{2}EI\theta_C-\frac{3}{8}EI\delta$$

$$=\frac{3}{2}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{2}EI\delta=0 \qquad \text{-------}>(9)$$

From (9) $\quad EI\delta=EI\theta_B+EI\theta_C$

Substitute in (7) & (8)

Eqn (7)

$$-56.25+\frac{3}{2}EI\theta_B+\frac{1}{4}EI\theta_C-\frac{3}{8}\left[EI\theta_B+EI\theta_C\right]=0$$

$$-56.25+\frac{9}{8}EI\theta_B-\frac{1}{8}EI\theta_C=0 \qquad \text{---------}>(10)$$

Eqn(8)

$$+93.75+\frac{1}{4}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{8}\left[EI\theta_B+EI\theta_C\right]=0$$

$$+93.75-\frac{1}{8}EI\theta_B+\frac{9}{8}EI\theta_C=0 \qquad \text{----------}>(11)$$

Solving equations (10) & (11) we get $EI\theta_B=41.25$

By Equation (10)

$$EI\theta_C=8\left[-56.25+\frac{9}{8}EI\theta_B\right]$$

$$=8\left[-56.25+\frac{9}{8}41.25\right]=-78.75$$

$$\therefore EI\delta=EI\theta_B+EI\theta_C=41.25-78.75=-37.5$$

Hence

$$EI\theta_B=41.25,\ EI\theta_C=-78.75,\ EI\delta=-37.5$$

Substituting these values in slope deflection equations, we have

$$M_{AB} = \frac{1}{2}(41.25) - \frac{3}{8}(-37.5) = +34.69 \text{ KNM}$$

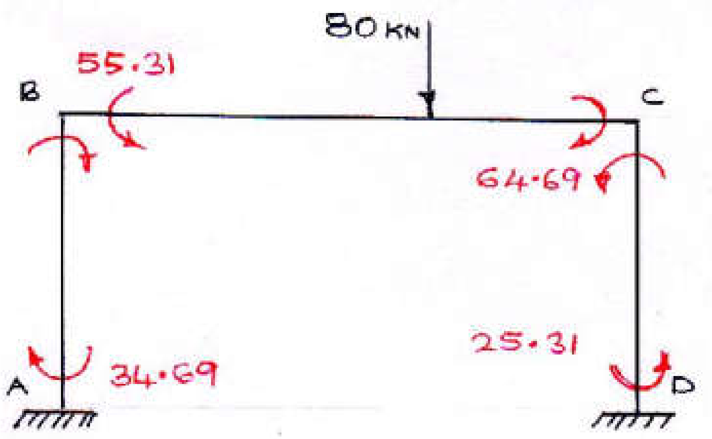
$$M_{BA} = 41.25 - \frac{3}{8}(-37.5) = +55.31 \text{ KNM}$$

$$M_{BC} = -56.25 + \frac{1}{2}(41.25) + \frac{1}{4}(-78.75) = -55.31 \text{ KNM}$$

$$M_{CB} = 93.75 + \frac{1}{2}(-78.75) + \frac{1}{4}(41.75) = +64.69 \text{ KNM}$$

$$M_{CD} = -78.75 - \frac{3}{8}(-37.5) = -64.69 \text{ KNM}$$

$$M_{DC} = \frac{1}{2}(-78.75) - \frac{3}{8}(-37.5) = -25.31 \text{ KNM}$$



Reactions: consider the free body diagram of beam and columns

Column AB:

$$H_A = \frac{34.69 + 55.31}{4} = 22.5 \text{ KN}$$

Span BC:

$$R_B = \frac{55.31 - 64.69 + 80 \times 3}{8} = 28.83 \text{ KN}$$

$$\therefore R_C = 80 - R_B = 51.17$$

Column CD:

$$H_D = \frac{64.69 + 25.31}{4} = 22.5$$

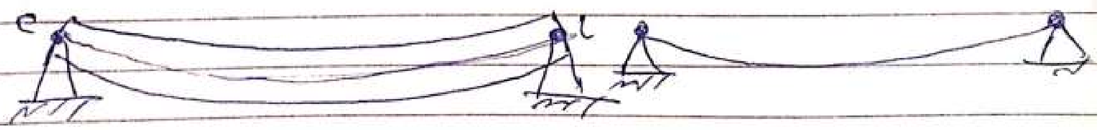
Deflections

Deflection of structures may occur from various sources, such as loads, temperature, fabrication errors, or settlement.

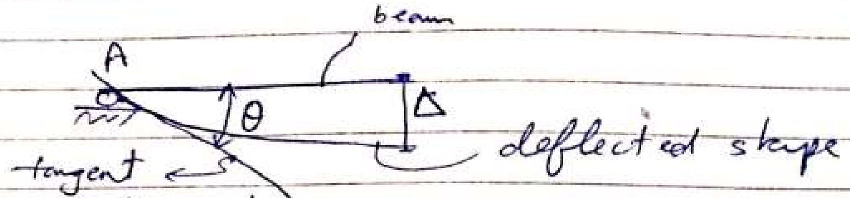
In design, deflection must be limited in order to provide integrity and stability of roofs, and prevent cracking of attached brittle materials such as concrete

The deflection to be considered throughout this text apply only to structures having linear elastic material response.

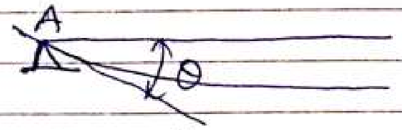
The deflection diagram represents the elastic curve for the points at the centroids of the cross-sectional areas along each of the members



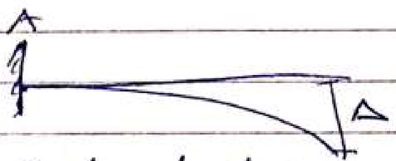
real structure Idealized st.



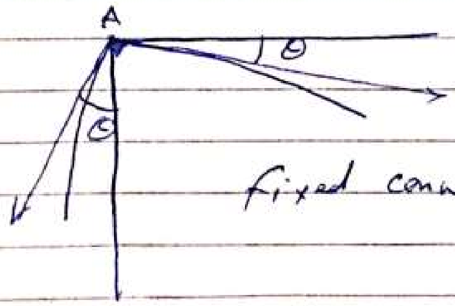
roller at A → $\Delta_A = 0$



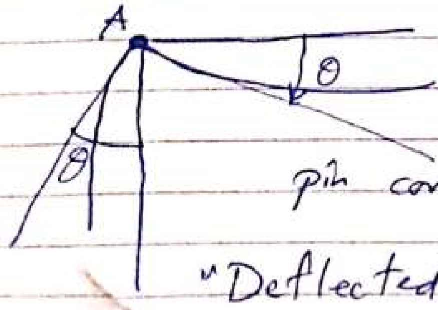
Hinge at A → $\Delta_A = 0$



Fixed end at A → $\Delta_A = 0$, $\theta_A = 0$



fixed connected joint at A



pin connected joint at A

"Deflected shapes for various supports"

Deflection Methods

- Elastic beam theory
- Double integration method
- Moment area theorems
- Conjugate-beam method
- Energy Method
- Virtual work method ✓✓

Note The students need to prepare a presentation for the first five method (5 Marks)

Virtual work method

It is also called the method of virtual force.

$$\Delta = \int_0^L \frac{mM}{EI}\,dx$$

$$\theta = \int_0^L \frac{m_\theta M}{EI}\,dx$$

where;

m : internal virtual moment in the beam or frame, expressed as a function of x and caused by the external virtual unit load

Δ : external displacement of the point caused by the real loads acting on the beam or frame

M : internal moment in the beam or frame, expressed as a function of x and caused by the real loads

E : Modulus of elasticity of the material

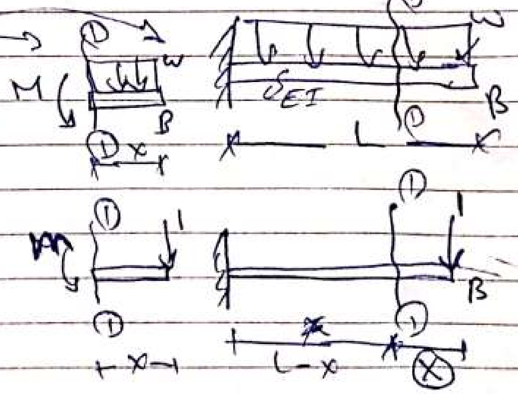
I : Moment of inertia of cross sectional area, computed about the neutral axis

Example 1// find the deflection at the free end (B) of the distributed loaded cantilever shown.

Sol

$M = \frac{-w x^2}{2}$

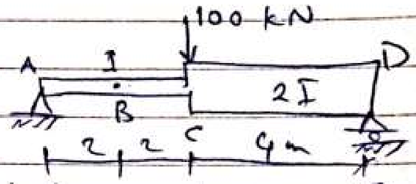
$m = -x$



$\therefore \Delta = \int_0^L \frac{M \cdot m}{E \cdot I} \, dx$

$= \int_0^L \frac{\left(\frac{-w x^2}{2}\right)(-x)\,dx}{EI} = \frac{w}{2EI} \cdot \frac{x^4}{4}\Big|_0^L = \frac{wL^4}{8EI}$

Example 2 // for the simply supported beam shown, find the deflection at B and C

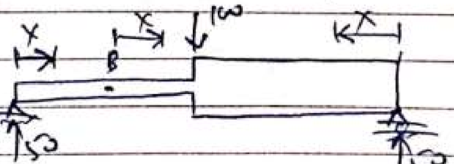


Sol prepare a table as below contains the summary of calculations

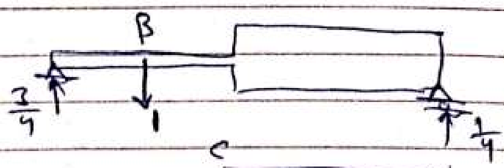
member	origin	limit	EI	M	m_1	m_2
AB	A	$0 \to 2$	EI	$50x$	$\frac{3}{4}x$	$\frac{x}{2}$
BC	B	$0 \to 2$	EI	$50(2+x)$	$\frac{-x}{4}+\frac{3}{2}$	$\frac{1}{2}(2+x)$
CD	D	$0 \to 4$	$2EI$	$50x$	$\frac{x}{4}$	$\frac{x}{2}$

Divide the structures to members according to the variation in
- loading
- Sections (I)
- orientation (in frame)
- function required

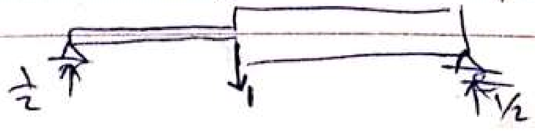
(M)



(m_1)



(m_2)



Member	M	m_1	m_2
AB	$M = 50x$	$m_1 = \frac{3}{4}x$	$m_2 = \frac{x}{2}$
BC	$M = 50(2+x)$	$m_1 = -\frac{x}{4} + \frac{3}{2}$	$m_2 = \frac{1}{2}(2+x)$
CD	$M = 50x$	$m_1 = \frac{x}{4}$	$m_2 = \frac{x}{2}$

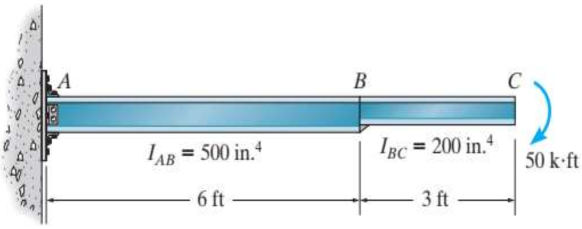
$$\Delta B = \int_0^2 \frac{37.5x^2}{EI}dx + \int_0^2 \frac{150 + 50x - 12.5x^2}{EI}dx + \int_0^4 \frac{6.25x^2}{EI}dx$$

$$\Delta B = 600/EI$$

$$\Delta C = \int_0^2 \frac{25x^2}{EI}dx + \int_0^2 \frac{100 + 100x + 25x^2}{EI}dx + \int_0^4 \frac{12.5x^2}{EI}dx$$

$$\Delta C = 800/EI$$

9–31. Determine the displacement and slope at point C of the cantilever beam. The moment of inertia of each segment is indicated in the figure. Take $E = 29(10^3)$ ksi. Use the principle of virtual work.



Referring to the virtual moment functions indicated in Fig. *a* and *b* and the real moment function in Fig. *c*, we have

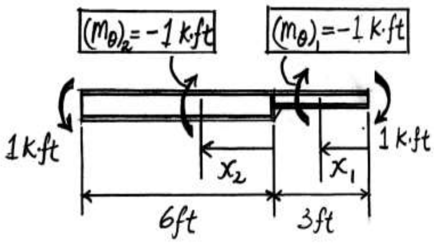
$$1\text{k}\cdot\text{ft}\cdot\theta_c = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^{3\text{ ft}} \frac{(-1)(-50)}{EI_{BC}}dx_1 + \int_0^{6\text{ ft}} \frac{(-1)(-50)}{EI_{AB}}dx_2$$

$$1\text{k}\cdot\text{ft}\cdot\theta_c = \frac{150\text{ k}^2\cdot\text{ft}^3}{EI_{BC}} + \frac{300\text{ k}^2\cdot\text{ft}^3}{EI_{AB}}$$

$$\theta_c = \frac{150\text{ k}\cdot\text{ft}^2}{EI_{BC}} + \frac{300\text{ k}\cdot\text{ft}^2}{EI_{AB}}$$

$$= \frac{150(12^2)\text{ k}\cdot\text{in}^2}{[29(10^3)\text{ k/in}^2](200\text{ in}^4)} + \frac{300(12^2)\text{ k}\cdot\text{in}^2}{[29(10^3)\text{ k/in}^2](500\text{ in}^4)}$$

$$= 0.00670\text{ rad}$$ **Ans.**



(a)

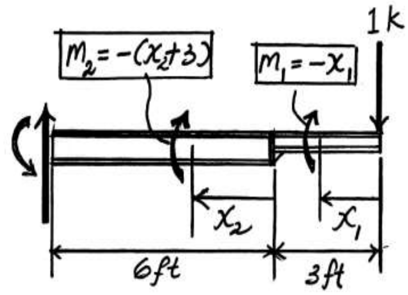
And

$$1\text{ k}\cdot\Delta_C = \int_0^L \frac{mM}{EI}dx = \int_0^{3\text{ ft}} \frac{-x_1(-50)}{EI_{BC}}dx_1 + \int_0^{6\text{ ft}} \frac{-(x_2+3)(-50)}{EI_{AB}}dx_2$$

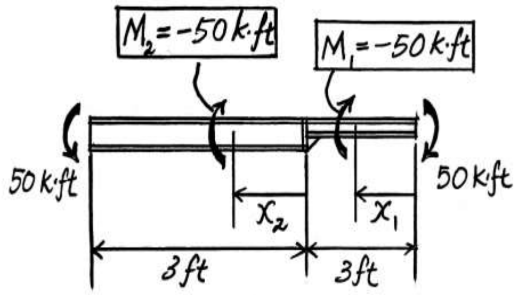
$$1\text{ k}\cdot\Delta_C = \frac{225\text{ k}^2\cdot\text{ft}^3}{EI_{BC}} + \frac{1800\text{ k}^2\cdot\text{ft}^3}{EI_{AB}}$$

$$\Delta_C = \frac{225\text{ k}\cdot\text{ft}^3}{EI_{BC}} + \frac{1800\text{ k}^2\cdot\text{ft}^3}{EI_{AB}}$$

$$= \frac{225(12^3)\text{ k}\cdot\text{in}^3}{[29(10^3)\text{ k/in}^2](200\text{ in}^4)} + \frac{1800(12^3)\text{ k}\cdot\text{in}^3}{[29(10^3)\text{ k/in}^2](500\text{ in}^4)} = 0.282\text{ in}\downarrow$$ **Ans.**



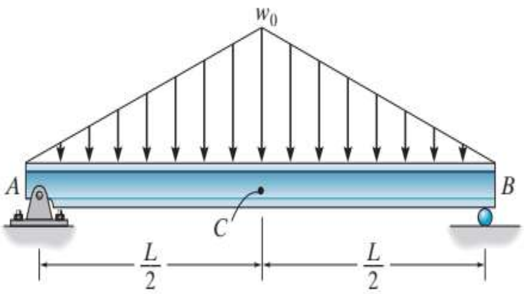
(b)



(c)

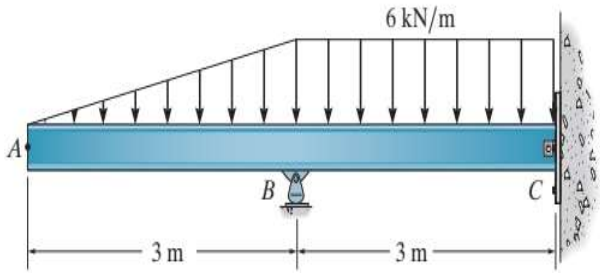
H.W

9–38. Determine the displacement of point C. Use the method of virtual work. EI is constant.



Ans. $$\Delta_C = \frac{w_0 L^4}{120EI} \downarrow$$

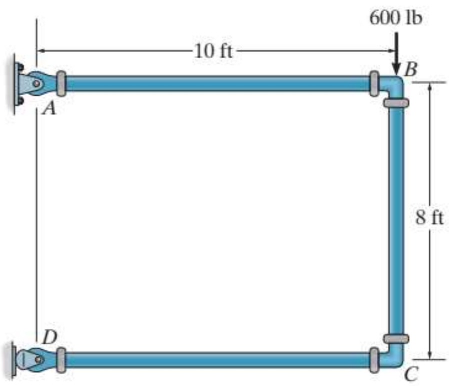
***9–40.** Determine the slope and displacement at point A. Assume C is pinned. Use the principle of virtual work. EI is constant.



Ans.

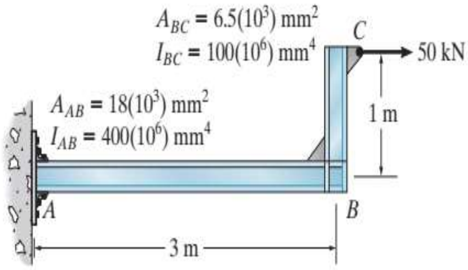
$$\theta_A = \frac{9\ \text{kN}\cdot\text{m}^2}{EI} \quad \Delta_A = \frac{22.95\ \text{kN}\cdot\text{m}^3}{EI} \downarrow$$

***9–44.** Use the method of virtual work and determine the vertical deflection at the rocker support D. EI is constant.



Ans. $$(\Delta_D) = \frac{440\ \text{k}\cdot\text{ft}^3}{EI} \downarrow$$

9–51. Determine the vertical deflection at C. The cross-sectional area and moment of inertia of each segment is shown in the figure. Take $E = 200$ GPa. Assume A is a fixed support. Use the method of virtual work.



Ans. 2.81 mm ↓

Deflection of Trusses

We can use the method of virtual work to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.

External Loading. For the purpose of explanation let us consider the vertical displacement Δ of joint B of the truss in Fig. 9–7. Here a typical element of the truss would be one of its *members* having a length L. If the applied loadings $\mathbf{P}_1$ and $\mathbf{P}_2$ cause a *linear elastic material response*, then this element deforms an amount $\Delta L = NL/AE$, where N is the normal or axial force in the member, caused by the loads. Applying Eq. 9–13, the virtual-work equation for the truss is therefore

$$\boxed{1 \cdot \Delta = \sum \frac{nNL}{AE}}$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

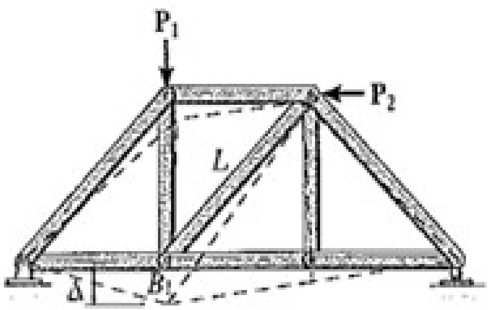
Δ = external joint displacement caused by the real loads on the truss.

N = internal normal force in a truss member caused by the real loads.

L = length of a member.

A = cross-sectional area of a member.

E = modulus of elasticity of a member.



PROCEDURE FOR ANALYSIS

The following procedure may be used to determine a specific displacement of any joint on a truss using the method of virtual work.

Virtual Forces **n**

- Place the unit load on the truss at the joint where the desired displacement is to be determined. The load should be in the same direction as the specified displacement, e.g., horizontal or vertical.
- With the unit load so placed, and all the real loads *removed* from the truss, use the method of joints or the method of sections and calculate the internal **n** force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

Real Forces **N**

- Use the method of sections or the method of joints to determine the **N** force in each member. These forces are caused only by the real loads acting on the truss. Again, assume tensile forces are positive and compressive forces are negative.

Virtual-Work Equation

- Apply the equation of virtual work, to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding **n** and **N** forces when substituting these terms into the equation.
- If the resultant sum $\Sigma nNL/AE$ is positive, the displacement Δ is in the same direction as the unit load. If a negative value results, Δ is opposite to the unit load.

- When applying any formula, attention should be paid to the units of each numerical quantity. In particular, the virtual unit load can be assigned any arbitrary unit (N, kN, etc.), since the **n** forces will have these *same units*, and as a result the units for both the virtual unit load and the **n** forces will cancel from both sides of the equation.

Temperature. In some cases, truss members may change their length due to temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in its temperature, the change in length of a member is $\Delta L = \alpha\, \Delta T\, L$. Hence, we can determine the displacement of a selected truss joint due to this temperature change from Eq. 9–13, written as

$$\boxed{1 \cdot \Delta = \Sigma n \alpha\, \Delta T\, L} \qquad (9\text{–}16)$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

Δ = external joint displacement caused by the temperature change.

α = coefficient of thermal expansion of member.

ΔT = change in temperature of member.

L = length of member.

Fabrication Errors and Camber. Occasionally, errors in fabricating the lengths of the members of a truss may occur. Also, in some cases truss members must be made slightly longer or shorter in order to give the truss a camber. Camber is often built into a bridge truss so that the bottom cord will curve upward by an amount equivalent to the downward deflection of the cord when subjected to the bridge's full dead weight. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from direct application of Eq. 9–13, written as

$$\boxed{1 \cdot \Delta = \Sigma n\, \Delta L} \qquad (9\text{–}17)$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

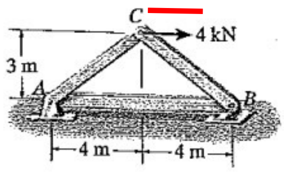
Δ = external joint displacement caused by the fabrication errors.

ΔL = difference in length of the member from its intended size as caused by a fabrication error.

A combination of the right sides of Eqs. 9–15 through 9–17 will be necessary if both external loads act on the truss and some of the members undergo a thermal change or have been fabricated with the wrong dimensions.

EXAMPLE 9-2

The cross-sectional area of each member of the truss shown in Fig. 9–9*a* is $A = 400\ \text{mm}^2$ and $E = 200$ GPa. (a) Determine the vertical displacement of joint C if a 4-kN force is applied to the truss at C.

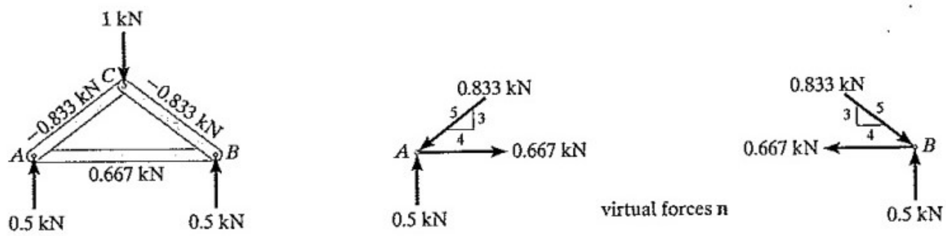


(a)

Fig. 9–9

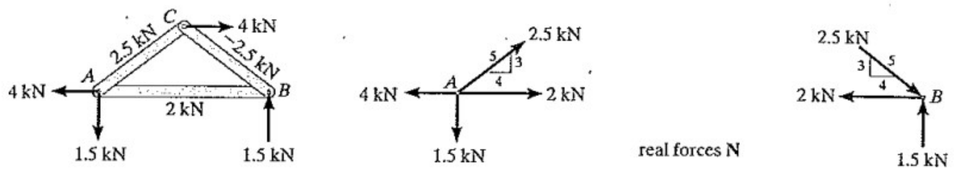
Solution

Virtual Forces n. Since the *vertical displacement* of joint C is to be determined, a virtual force of 1 kN is applied at C in the vertical direction. The units of this force are the *same* as those of the real loading. The support reactions at A and B are calculated and the **n** force in each member is determined by the method of joints as shown on the free-body diagrams of joints A and B, Fig. 9–9*b*.



(b)

Real Forces N. The joint analysis of A and B when the real load of 4 kN is applied to the truss is given in Fig. 9–9*c*.



(c)

Virtual-Work Equation. Since AE is constant, each of the terms nNL can be arranged in tabular form and computed. Here positive numbers indicate tensile forces and negative numbers indicate compressive forces.

Member	n (kN)	N (kN)	L (m)	$n\,NL$ (kN$^2\cdot$m)
AB	0.667	2	8	10.67
AC	−0.833	2.5	5	−10.41
CB	−0.833	−2.5	5	10.41
				$\Sigma 10.67$

Thus,

$$1\text{ kN}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE} = \frac{10.67\text{ kN}^2\cdot\text{m}}{AE}$$

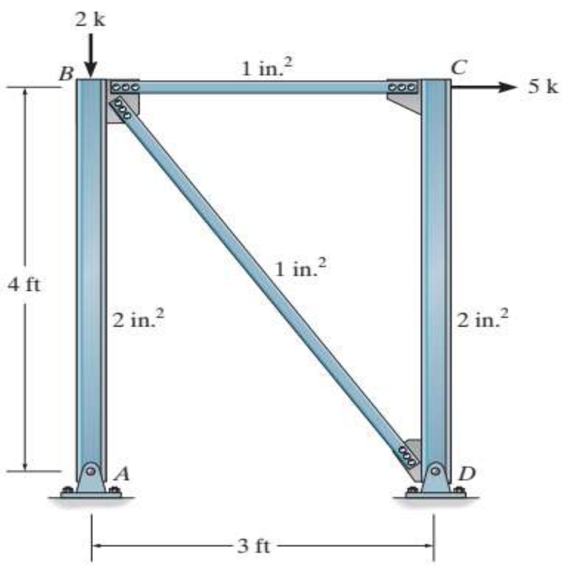
Substituting the values $A = 400\text{ mm}^2 = 400(10^{-6})\text{ m}^2$, $E = 200$ GPa $= 200(10^6)\text{ kN/m}^2$, we have

$$1\text{ kN}\cdot\Delta_{C_v} = \frac{10.67\text{ kN}^2\cdot\text{m}}{400(10^{-6})\text{ m}^2(200(10^6)\text{ kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133\text{ m} = 0.133\text{ mm} \qquad \textit{Ans.}$$

H.W.

***9–56.** Use the method of virtual work and determine the horizontal deflection at C. The cross-sectional area of each member is indicated in the figure. Assume the members are pin connected at their end points. $E = 29(10^3)$ ksi.

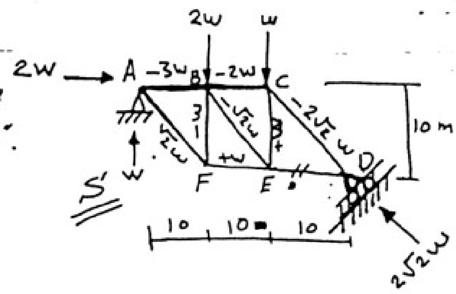


Ans. 0.041 in

Ex:— Area in all members = 2700 mm^2

$E = 200 \frac{kN}{mm^2}$, if movement at "D" is limited to ∝ 50 mm, Find the value of "W".

Solution:—



$\Sigma M_A = 0 \Rightarrow R_D = 2\sqrt{2}\,w$ ↖

$\Sigma F_x = 0 \Rightarrow A_x = 2w$ →

$\Sigma F_y = 0 \Rightarrow A_y = w$ ↑

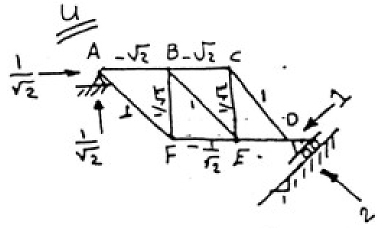
joint A $\Rightarrow \Sigma F_y = 0 \Rightarrow AF = \sqrt{2}\,w$

$\Sigma F_x = 0 \Rightarrow AB = -3W$

joint F $\Rightarrow \Sigma F_x = 0 \Rightarrow FE = +w$

$\Sigma F_y = 0 \Rightarrow FB = 1/\sqrt{2}$

⋮ ⋮ ⋮



mem.	L (m)	A (mm^2)	S'	u	S'.u.L kN.m
AB	10		$-3W$	$-\sqrt{2}$	$30\sqrt{2}\,w$
BC	10		$-2W$	$\frac{-1}{\sqrt{2}}$	$20\,w/\sqrt{2}$
CD	$10\sqrt{2}$		$-2\sqrt{2}\,W$	-1	$40w$
CE	10		W	$\frac{1}{\sqrt{2}}$	$10\,w/\sqrt{2}$
BE	$10\sqrt{2}$		$-\sqrt{2}\,W$	-1	$20W$
FE	10		W	$\frac{-1}{\sqrt{2}}$	$-10W/\sqrt{2}$
BF	10		$-W$	$1/\sqrt{2}$	$-10W/\sqrt{2}$
AF	$10\sqrt{2}$		$\sqrt{2}\,W$	-1	$-20W$
DE	—		—	—	—
					$\Sigma = 89.497W$

$$\Delta = \Sigma \frac{S.u.L}{A.E} = \frac{1}{A.E}\Sigma S.u.L$$

ثابتة لكل الأضلاع

من السؤال

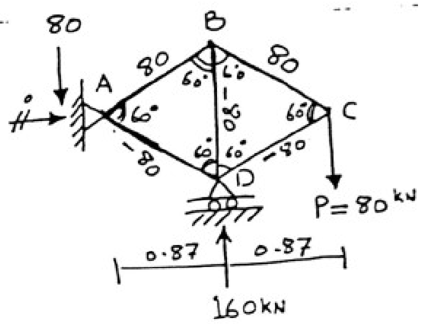
$$\Delta = \frac{89.497\,W\,kN.m * 1000\,mm/m}{2700 * 200\,\frac{kN}{mm^2}} = 50\,mm$$

$\therefore\ w = 301.68$ KN

Ex:- IF length of each bar = 1000 mm

$$EA \begin{cases} K = 40000 \text{ kN} & \text{for tension bar} \\ 2K = 80000 \text{ kN} & \text{for Compression bar} \end{cases}$$

Calculate Vertical deflection at C.



Solution :-

∴ المنشأ الأصلي محمل بقوة واحدة في نفس موقع واتجاه الـ (u.L) ⟸ $u = \frac{S}{80}$

$\Sigma M_A = 0$

$80(0.87 + 0.87) = RD(0.87)$

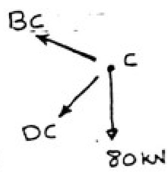
$\therefore RD = 160 \text{ kN}$

$\Sigma F_y = 0$

$80 + A_y = 160 \Rightarrow A_y = 80 \text{ kN}$

$\Sigma F_x = 0 \Rightarrow A_x = 0$

Joint C



$\Sigma F_x = 0$

$BC\left(\frac{\sqrt{3}}{2}\right) = -DC\left(\frac{\sqrt{3}}{2}\right)$

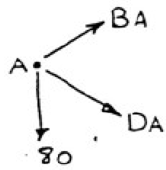
$\therefore BC = -DC$

$\Sigma F_y = 0$

$80 + DC\left(\frac{1}{2}\right) = BC\left(\frac{1}{2}\right)$

$\therefore BC = 80 \Rightarrow DC = -80$

Joint A



$\Sigma F_x = 0$

$BA\left(\frac{\sqrt{3}}{2}\right) + DA\left(\frac{\sqrt{3}}{2}\right) = 0$

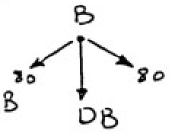
$\therefore BA = -DA$

$\Sigma F_y = 0$

$BA\left(\frac{1}{2}\right) - DA\left(\frac{1}{2}\right) = 80$

$\therefore BA = 80 \Rightarrow DA = -80$

Joint B



$\Sigma F_y = 0$

$80\left(\frac{1}{2}\right) + 80\left(\frac{1}{2}\right) = -DB$

$\therefore DB = -80$

mem.	EA	S	u	$\frac{Su}{AE}$
AB	40000	80	1	0.002
BC	40000	80	1	0.002
CD	80000	-80	-1	0.001
DA	80000	-80	-1	0.001
BD	80000	-80	-1	0.001

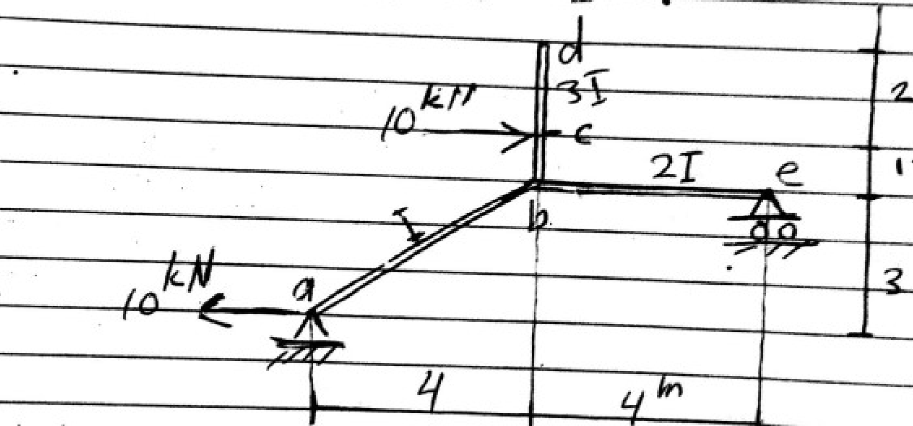
$$\therefore \Delta V)_c = \Sigma \frac{SuL}{A.E} = 0.007 * 1000 = 7 \text{ mm}$$

Examples on frames' deflection

Ex1: For the frame shown in the figure below, a is a pin connection, e is a roller connection, b is a rigid connection and d is a free end. Find:

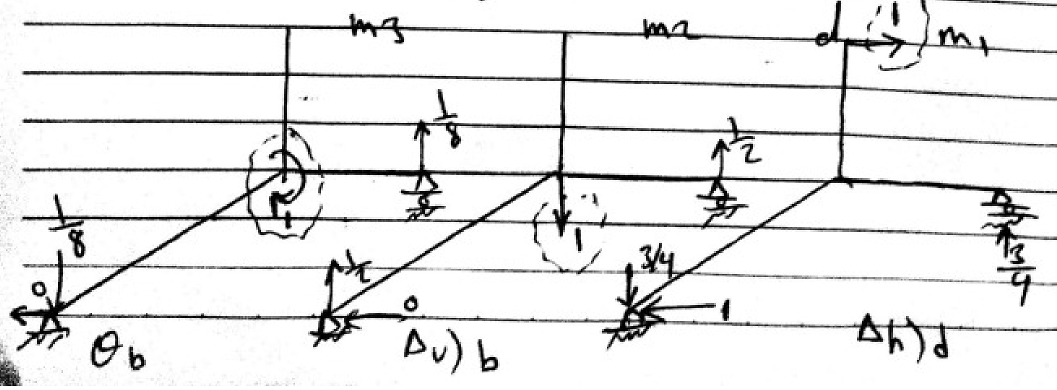
E is constant

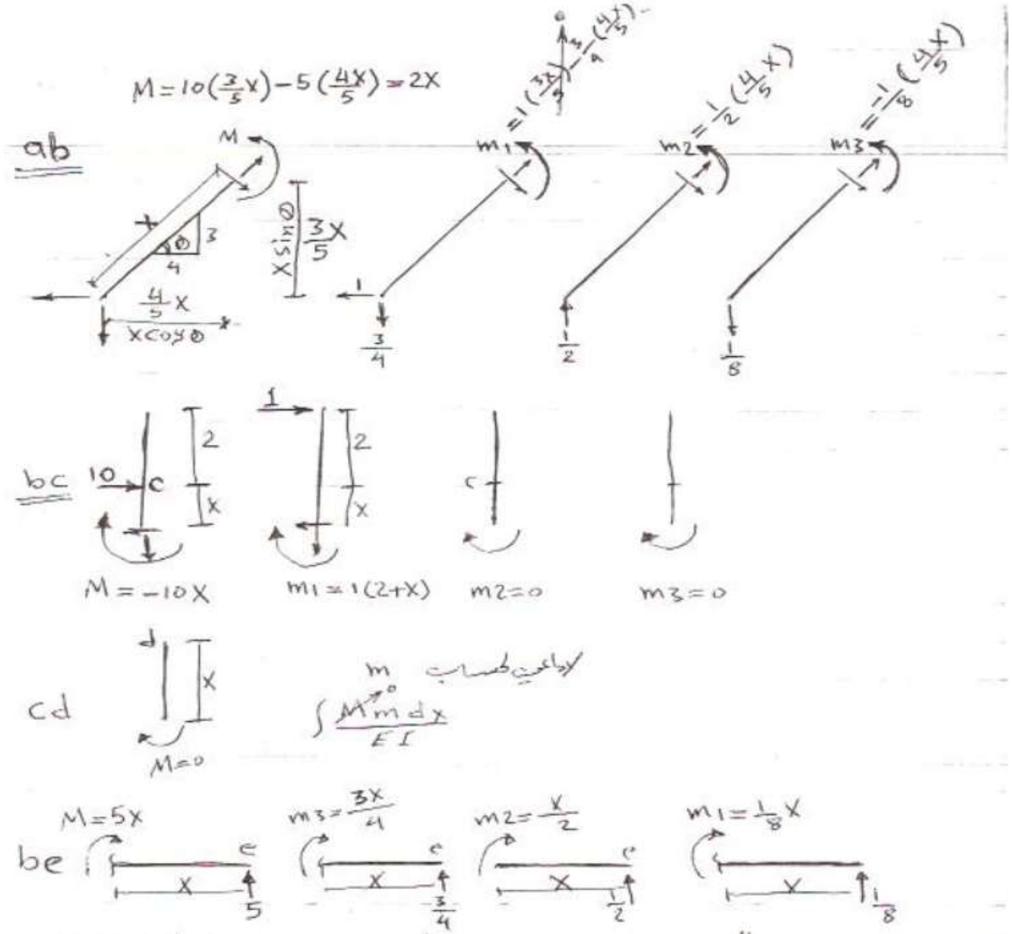
1- The horizontal deflection at 'd'.
2- The vertical deflection at 'b'.
3- The rotation at 'b'.



Solution

Three functions required $\Delta_h)_d$, $\Delta_v)_b$ and θ_b





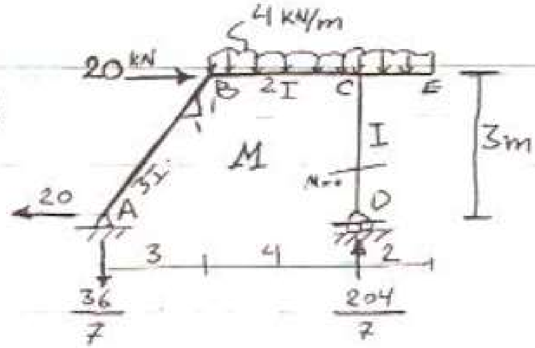
$$\Delta h)_d = \int \frac{M m_1 dx}{EI} = \frac{1}{EI}\int_0^1 \frac{(-10x)(-(2+x))\,dx}{3} + 0 + \int_0^4 \frac{5x(\frac{3}{4}x)\,dx}{2} = \frac{44.4}{EI}$$

$$\Delta v)_b = \int \frac{M m_2 dx}{EI} = \frac{1}{EI}\int_0^5 \frac{(2x)(\frac{2}{5}x)\,dx}{1} + 0 + 0 + \int_0^4 \frac{(5x)(\frac{x}{2})\,dx}{2} = \frac{60}{EI}$$

$$\theta_b = \int \frac{M m_3 dx}{EI} = \frac{1}{EI}\int_0^5 \frac{2x(\frac{-x}{10})\,dx}{1} + 0 + 0 + \int_0^4 \frac{(5x)(-\frac{x}{8})\,dx}{2} = -\frac{1.67}{EI}$$

الإشارة السالبة تعني أن الـ (Rotation) عكس اتجاه دوران الـ (unit moment) .

member	origin	limit	EI	M	m_1	m_2	m_3
ab	a	0–5	1	$2X$	0	$\frac{2x}{5}$	$\frac{-x}{10}$
bc	c	0–1	3	$-10X$	$-2-x$	0	0
cd	d	0–2	3	0	صفر		
be	e	0–4	2	$5X$	$\frac{3x}{4}$	$\frac{x}{2}$	$\frac{x}{8}$



Find) θA

لإيجاد θA نضع وحدة عزم على نقطة A على المنشأ الوهمي

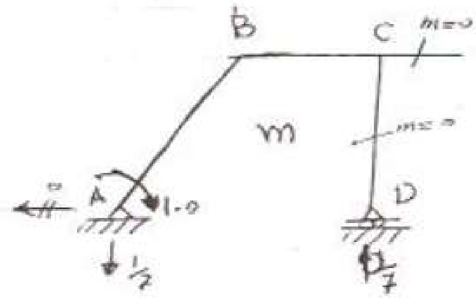
$\Sigma M_A = 0 \Rightarrow D_y = \frac{204}{7} \uparrow$

$\Sigma F_y = 0 \Rightarrow A_y = \frac{36}{7} \downarrow$

$\Sigma F_x = 0 \Rightarrow A_x = 20 \leftarrow$

$\Sigma M_A = 0 \Rightarrow D_y = \frac{1}{7} \downarrow$

$\Sigma F_y = 0 \Rightarrow A_y = \frac{1}{7} \uparrow$



mem.	origin	limit	EI	M	m
AB	A	$0-3\sqrt{2}$	3	$\frac{104x}{7\sqrt{2}}$	$1-\frac{x}{7\sqrt{2}}$
BC	C	$0\sim4$	2	$-2(x+2)^2+\frac{204x}{7}$	$\frac{x}{7}$
CD	—	—	—	—	—
DE	—	—	—	—	—

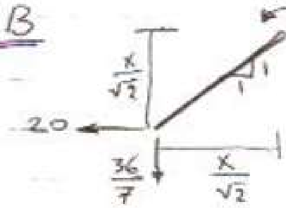
$$EI\theta_A = \int_0^{3\sqrt{2}} \frac{\left(\frac{104x}{7\sqrt{2}}\right)\left(1-\frac{x}{7\sqrt{2}}\right)dx}{3} + \int_0^4 \frac{\left[-2(x+2)^2+\frac{204}{7}x\right]\frac{x}{7}\,dx}{2}$$

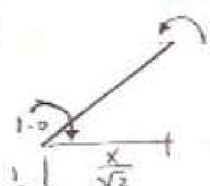
$$\therefore \theta A = \frac{47.9}{EI} \text{ rad.}$$

AB

$M = 20\left(\frac{x}{\sqrt{2}}\right) - \frac{36}{7}\left(\frac{x}{\sqrt{2}}\right) = \frac{104x}{7\sqrt{2}}$

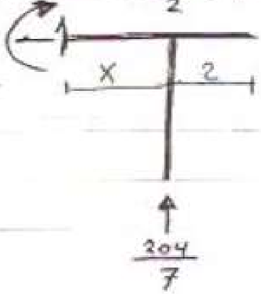
$m = 1 - \frac{1}{7}\left(\frac{x}{\sqrt{2}}\right)$

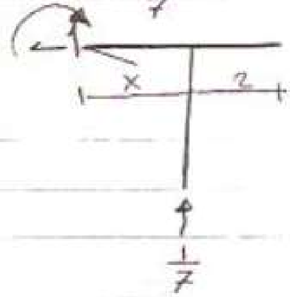




BC $M = -4\frac{(x+2)^2}{2} + \frac{204x}{7}$

$m = \frac{x}{7}$





1

Analysis of Statically Indeterminate structures

Four methods will be presented in this course as followed:

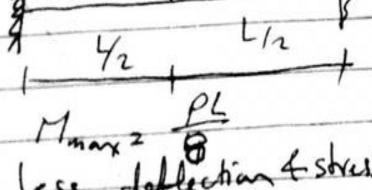
1. Method of Consitent Deformation
2. Slope Deflection Method
3. Moment Distribution Method
4. Stiffness Matrix Method

Let's Start with the first method

1. Method of consistent Deformation

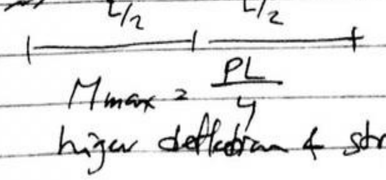
Why indeterminate structures? compared with determinate structures

1. For a given loading the maximum stress and deflection of an indeterminate structure are generally <u>smaller</u> than those of its statically determinate counterpart.



$M_{max} = \frac{PL}{8}$

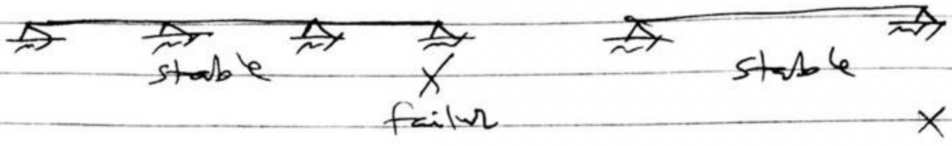
less deflection & stress

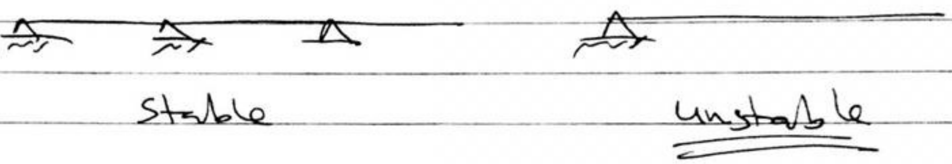


$M_{max} = \frac{PL}{4}$

higer deflection & stress

2. Statically indeterminate structure has the tendency to redistribute its load to its redundant supports in cases where faulty design or overloading occures.



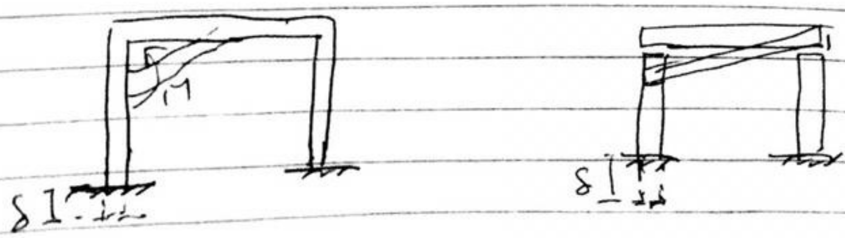


3. Thinner members needed for indet. structures compared with those required for det. ones.

But, in some cases the above advantages may instead become disadvantages.

1. Indeterminate structures → more supports → more connections → higher cost

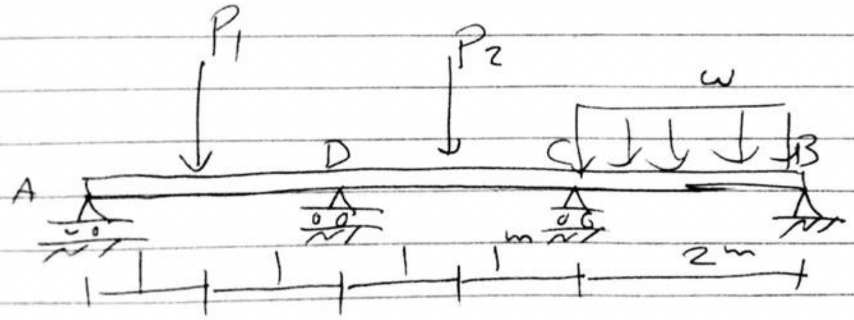
2. Indet. structures may be exposed to differential displacement more than det. structure



Method of consistent deformation also called "Force method", "Compatibility method" or Method of consistent displacements. It is originally developed by James Clerk Maxwell in 1864 and later refined by Otto Mohr and Heinrich Müller-Breslau.

- Analysis of Indet. beams and frames with the Method of Consistent Deformation

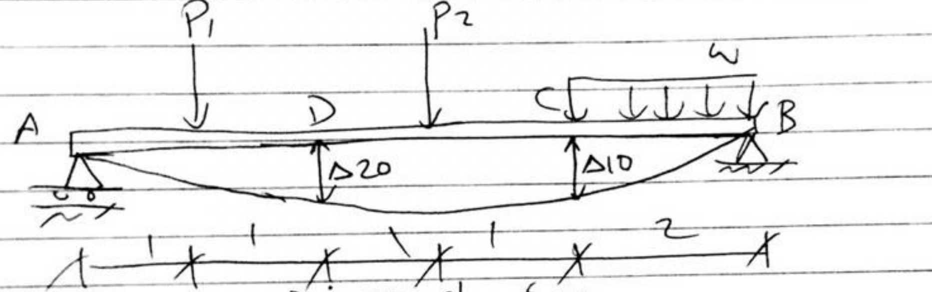
① The beam shown in figure below is stable and indet. to the 2nd degree; therefore, two redundants restrains should be removed in order to make it stable and determinate. This structure is called "original structure" ~~or "primary structure"~~



original structure

② The two redundants to be removed in order to make the beam stable and determinate are the rollers at 'C' and 'D', reffered to as (X_1) and (X_2), respectively.

The structure after the removal of the redundants is called the "primary structure"



primary structure

$\boxed{\Delta_{i0}}$

Δ10 : the deformation at C due to the external applied load

Δ20 : " " " D " " " " " "

③ Remove the applied loads and apply a unit load at 'C' and determine the deformation at 'C' and 'D' due to the unit load, (δ_{11}) and (δ_{22}), respectively

$\boxed{\delta_{ij}}$

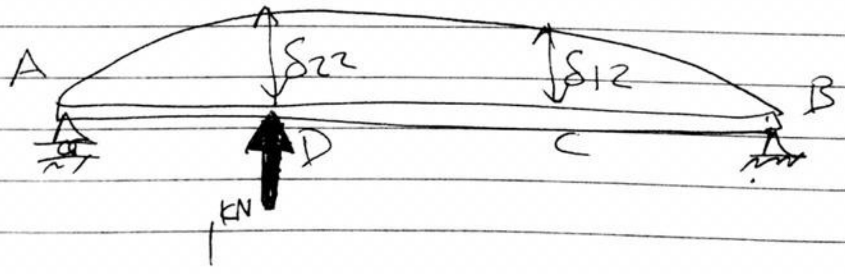
Note that

$\Delta i0$: represents the deformation of point 'i', at the location where the redundant was removed, in the primary structure

δij : represents the deformation of point 'i' due to a unit load applied at point 'j'

$\delta i j$ → location of U.L.

(4) Remove the applied loads and apply a unit load at 'D' and determine the deformations at 'C' and 'D', due to the unit load, ($\delta 21$) and ($\delta 22$), respectively.



(5) Apply
$$\Delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} = \Delta_{S1}$$
$$\Delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} = \Delta_{S2}$$

where

$$\Delta_{10} = \int \frac{M\, m_1}{EI}\, dx \quad , \quad \Delta_{20} = \int \frac{M\, m_2}{EI}\, dx$$

$$\delta_{11} = \int \frac{m_1 \cdot m_1}{EI}\, dx \quad , \quad \delta_{22} = \int \frac{m_2\, m_2}{EI}\, dx$$

$$\delta_{12} = \delta_{21} = \int \frac{m_1\, m_2}{EI}\, dx$$

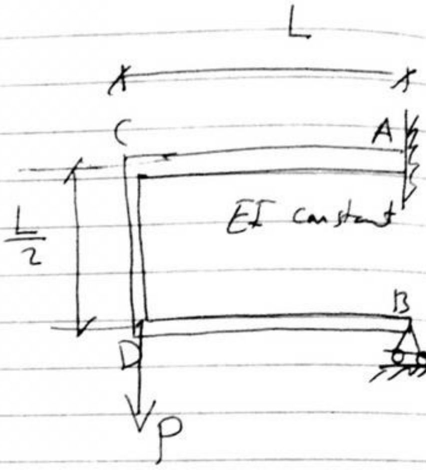
Δ_{si} : represents the deformation for the structure at point 'i', the location where the redundant was removed. Usually at support, $\Delta_{si} = 0$ unless it was mentioned else.

(6) Solve the equations obtained in (5) to get the value of X_1 and X_2

(7) By satisfying equilibrium requirements, the remaining support reactions on the structure can be determined

Example (1)

For the frame shown, use the consistent deformation method to:



1. find the reaction of the roller at B
2. Draw S.F.D and B.M.D for all members

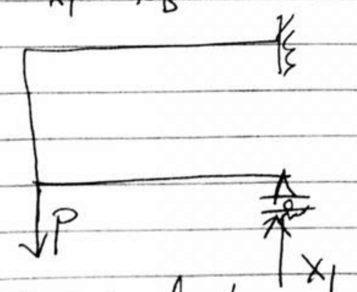
Solution

1. Check the determinacy of the frame

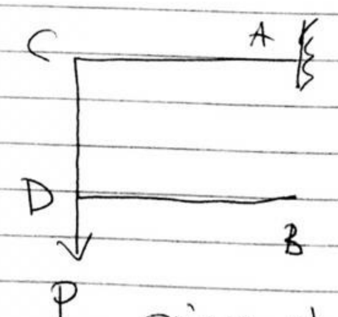
   indet. 1^{st} degree

2. Choose the redundant that by removing it, the frame remains stable and det.

   Let $X_1 = R_B$

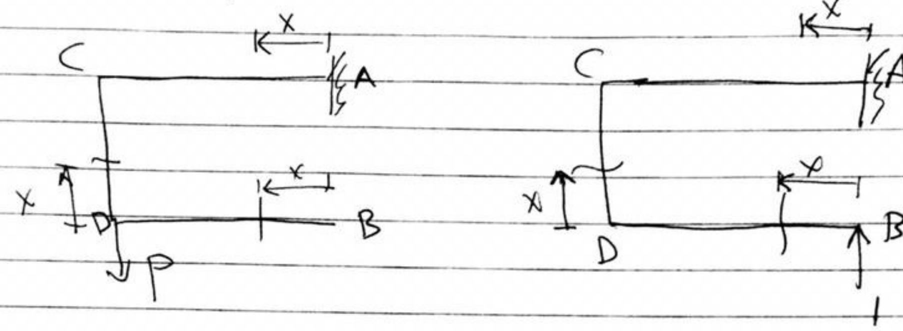


Original st.



primary st

③ M & m



M m₁

member	Origin	Limit	M	m_1
AC	A	$0 \to L$	$Px - PL$	$-x$
CD	D	$0 \to \frac{L}{2}$	0	$-L$
DB	B	$0 \to L$	0	x

④ Write the compatibility equation

$$\Delta_{10} + X_1 \delta_{11} = \Delta S_1$$

$$\Delta_{10} = \int \frac{M m_1}{EI} dx = \int_0^L \frac{(Px - PL)(-x)\,dx}{EI} = \frac{PL^3}{6EI}$$

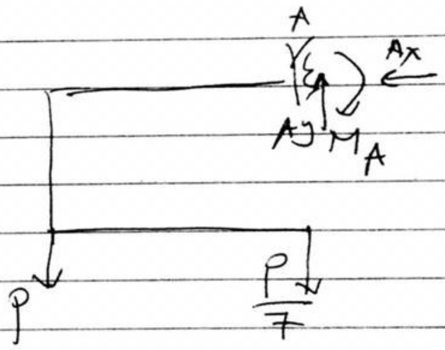
$$\delta_{11} = \int \frac{m_1 m_1}{EI} dx = \frac{1}{EI}\left[\int_0^L (-x)(-x)\,dx + \int_0^{L/2} (-L)^2\,dx + \int_0^L x^2\,dx\right] = \frac{7L^3}{6EI}$$

ئ

$$\frac{PL^3}{6EI} + X_1 \frac{7L^3}{6EI} = 0$$

$$X_1 = \frac{-P}{7} = R_B$$

(2) To draw S.F.D and B.M.D, find the reactions



$\Sigma M_A = 0$

$M_A - PL = 0 \rightarrow M_A = PL$

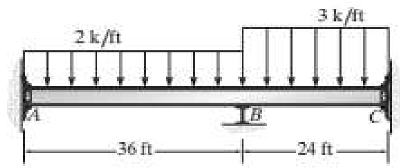
$\Sigma F_x = 0 \quad A_x = 0$

$\Sigma F_y = 0$

$$A_y - P - \frac{P}{7} = 0 \rightarrow A_y = \frac{8}{7}P$$

After that you can draw S.F.D and B.M.D

12–2. Determine the moments at A, B, and C. Assume the support at B is a roller and A and C are fixed. EI is constant.



$$(\mathrm{DF})_{AB} = 0 \qquad (\mathrm{DF})_{BA} = \frac{I/36}{I/36 + I/24} = 0.4$$

$$(\mathrm{DF})_{BC} = 0.6 \qquad (\mathrm{DF})_{CB} = 0$$

$$(\mathrm{FEM})_{AB} = \frac{-2(36)^2}{12} = -216 \text{ k}\cdot\text{ft}$$

$$(\mathrm{FEM})_{BA} = 216 \text{ k}\cdot\text{ft}$$

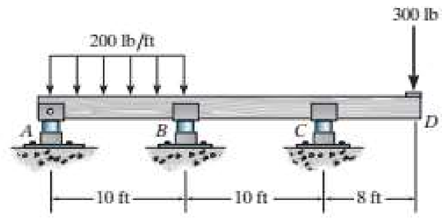
$$(\mathrm{FEM})_{BC} = \frac{-3(24)^2}{12} = -144 \text{ k}\cdot\text{ft}$$

$$(\mathrm{FEM})_{CB} = 144 \text{ k}\cdot\text{ft}$$

Joint	A	B		C
Mem.	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM	–216	216	–144	144
		–28.8	–43.2	
	–14.4			–21.6
$\sum M$	–230	187	–187	–122 k · ft

Ans.

12–9. Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A is a pin. EI is constant.



Member Stiffness Factor and Distribution Factor.

$$K_{AB} = \frac{3EI}{L_{AB}} = \frac{3EI}{10} = 0.3EI \qquad K_{BC} = \frac{4EI}{L_{BC}} = \frac{4EI}{10} = 0.4EI.$$

$$(\text{DF})_{BA} = \frac{0.3EI}{0.3EI + 0.4EI} = \frac{3}{7} \qquad (\text{DF})_{BC} = \frac{0.4EI}{0.3EI + 0.4EI} = \frac{4}{7}$$

$$(\text{DF})_{CB} = 1 \qquad (\text{DF})_{CD} = 0$$

Fixed End Moments. Referring to the table on the inside back cover,

$$(\text{FEM})_{CD} = -300(8) = 2400\ \text{lb}\cdot\text{ft} \qquad (\text{FEM})_{BC} = (\text{FEM})_{CB} = 0$$

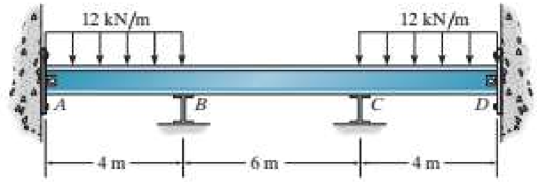
$$(\text{FEM})_{BA} = \frac{wL_{AB}^2}{8} = \frac{200(10^2)}{8} = 2500\ \text{lb}\cdot\text{ft}$$

Moment Distribution. Tabulating the above data,

Joint	A	B		C	
Member	AB	BA	BC	CB	CD
DF	1	3/7	4/7	1	0
FEM	0	2500	0	0	–2400
Dist.		–1071.43	–1428.57	2400	
CO			1200	–714.29	
Dist.		–514.29	–685.71	714.29	
CO			357.15	–342.86	
Dist.		–153.06	–204.09	342.86	
CO			171.43	–102.05	
Dist.		–73.47	–97.96	102.05	
CO			51.03	–48.98	
Dist.		–21.87	–29.16	48.98	
CO			24.99	–14.58	
Dist.		–10.50	–13.99	14.58	
CO			7.29	–7.00	
Dist.		–3.12	–4.17	7.00	
CO			3.50	–2.08	
Dist.		–1.50	–2.00	2.08	
CO			1.04	–1.00	
Dist.		–0.45	–0.59	1.00	
CO			0.500	–0.30	
Dist.		–0.21	–0.29	0.30	
CO			0.15	–0.15	
Dist.		–0.06	–0.09	0.15	
CO			0.07	–0.04	
Dist.		–0.03	–0.04	0.04	
$\sum M$	0	650.01	–650.01	2400	–2400

Using these results, the shear at both ends of members AB, BC, and CD are computed and shown in Fig. a. Subsequently, the shear and moment diagrams can be plotted, Fig. b and c, respectively.

***12–8.** Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A and D are pins. EI is constant.



Member Stiffness Factor and Distribution Factor.

$$K_{AB} = \frac{3EI}{L_{AB}} = \frac{3EI}{4} \qquad K_{BC} = \frac{2EI}{L_{BC}} = \frac{2EI}{6} = \frac{EI}{3}$$

$$(\mathrm{DF})_{AB} = 1 \quad (\mathrm{DF})_{BA} = \frac{3EI/4}{3EI/4 + 3EI/3} = \frac{9}{13} \quad (\mathrm{DF})_{BC} = \frac{EI/3}{3EI/4 + EI/3} = \frac{4}{13}$$

Fixed End Moments. Referring to the table on the inside back cover,

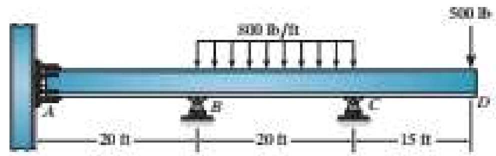
$$(\mathrm{FEM})_{AB} = (\mathrm{FEM})_{BC} = 0 \qquad (\mathrm{FEM})_{BA} = \frac{wL^2}{8} = \frac{12(4^2)}{8} = 24\ \mathrm{kN \cdot m}$$

Moment Distribution. Tabulating the above data,

Joint	A	B	
Member	AB	BA	BC
DF	1	$\frac{9}{13}$	$\frac{4}{13}$
FEM	0	24	0
Dist.		–16.62	–7.385
$\sum M$	0	7.385	–7.385

Using these results, the shear at both ends of members AB, BC, and CD are computed and shown in Fig. *a*. Subsequently, the shear and moment diagram can be plotted, Fig. *b* and *c*, respectively.

*12–4. Determine the reactions at the supports and then draw the moment diagram. Assume A is fixed. EI is constant.



$$\text{FEM}_{BC} = -\frac{wL^2}{12} = -26.67, \quad \text{FEM}_{CB} = \frac{wL^2}{12} = 26.67$$

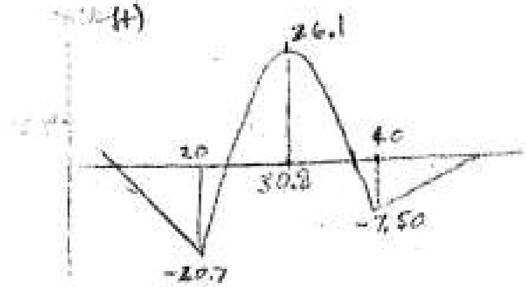
$$M_{CD} = 0.5(15) = 7.5 \text{ k}\cdot\text{ft}$$

$$K_{AB} = \frac{4EI}{20}, \quad K_{BC} = \frac{4EI}{20}$$

$$\text{DF}_{AB} = 0$$

$$\text{DF}_{BA} = \text{DF}_{BC} = \frac{\frac{4EI}{20}}{\frac{4EI}{20} + \frac{4EI}{20}} = 0.5$$

$$\text{DF}_{CB} = 1$$



Joint	A	B		C	
Member	AB	BA	BC	CB	CD
DF	0	0.5	0.5	1	0
FEM			–26.67	26.67	–7.5
		13.33	13.33	–19.167	
	6.667		–9.583	6.667	
		4.7917	4.7917	–6.667	
	2.396		–3.333	2.396	
		1.667	1.667	–2.396	
	0.8333		–1.1979	0.8333	
		0.5990	0.5990	–0.8333	
	0.2994		–0.4167	0.2994	
		0.2083	0.2083	–0.2994	
	0.1042		–0.1497	0.1042	
		0.07485	0.07485	–0.1042	
	10.4	20.7	–20.7	7.5	–7.5 k · ft

Displacement Method of Analysis: Moment Distribution

12

The moment-distribution method is a displacement method of analysis that is easy to apply once certain elastic constants have been determined. In this chapter we will first state the important definitions and concepts for moment distribution and then apply the method to solve problems involving statically indeterminate beams and frames. Application to multistory frames is discussed in the last part of the chapter.

12.1 General Principles and Definitions

The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930. At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

As will be explained in detail later, moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply. Before explaining the techniques of moment distribution, however, certain definitions and concepts must be presented.

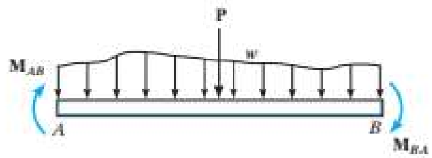


Fig. 12–1

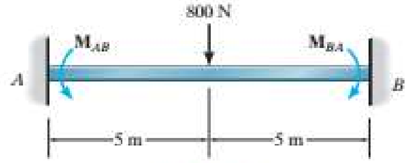


Fig. 12–2

Sign Convention. We will establish the same sign convention as that established for the slope-deflection equations: *Clockwise moments that act on the member* are considered *positive,* whereas *counterclockwise moments* are *negative,* Fig. 12–1.

Fixed-End Moments (FEMs). The moments at the "walls" or fixed joints of a loaded member are called *fixed-end moments.* These moments can be determined from the table given on the inside back cover, depending upon the type of loading on the member. For example, the beam loaded as shown in Fig. 12–2 has fixed-end moments of $\text{FEM} = PL/8 = 800(10)/8 = 1000\ \text{N}\cdot\text{m}$. Noting the action of these moments *on the beam* and applying our sign convention, it is seen that $M_{AB} = -1000\ \text{N}\cdot\text{m}$ and $M_{BA} = +1000\ \text{N}\cdot\text{m}$.

Member Stiffness Factor. Consider the beam in Fig. 12–3, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end A to rotate through an angle θ_A. In Chapter 11 we related M to θ_A using the conjugate-beam method. This resulted in Eq. 11–1, that is, $M = (4EI/L)\,\theta_A$. The term in parentheses

$$\boxed{K = \frac{4EI}{L}} \qquad (12\text{–}1)$$

Far End Fixed

is referred to as the *stiffness factor* at A and can be defined as the amount of moment M required to rotate the end A of the beam $\theta_A = 1$ rad.

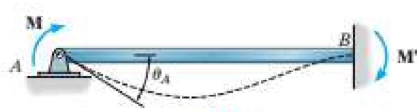


Fig. 12–3

Joint Stiffness Factor. If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the *total stiffness factor* at the joint is the sum of the member stiffness factors at the joint, that is, $K_T = \Sigma K$. For example, consider the frame joint A in Fig. 12–4*a*. The numerical value of each member stiffness factor is determined from Eq. 12–1 and listed in the figure. Using these values, the total stiffness factor of joint A is $K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000$. This value represents the amount of moment needed to rotate the joint through an angle of 1 rad.

Distribution Factor (DF). If a moment **M** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the *distribution factor* (DF). To obtain its value, imagine the joint is fixed connected to n members. If an applied moment **M** causes the joint to rotate an amount θ, then each member i rotates by this same amount. If the stiffness factor of the ith member is K_i, then the moment contributed by the member is $M_i = K_i\theta$. Since equilibrium requires $M = M_1 + M_n = K_1\theta + K_n\theta = \theta\Sigma K_i$ then the distribution factor for the ith member is

$$\mathrm{DF}_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i}$$

Canceling the common term θ, it is seen that the distribution factor for a member is equal to the stiffness factor of the member divided by the total stiffness factor for the joint; that is, in general,

$$\boxed{\mathrm{DF} = \frac{K}{\Sigma K}} \qquad (12\text{–}2)$$

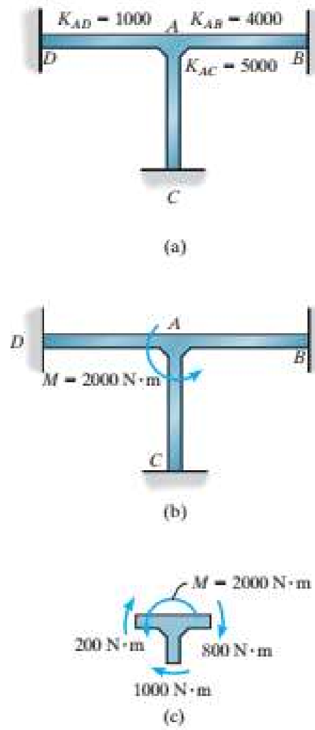


Fig. 12–4

For example, the distribution factors for members AB, AC, and AD at joint A in Fig. 12–4*a* are

$$\mathrm{DF}_{AB} = 4000/10\,000 = 0.4$$
$$\mathrm{DF}_{AC} = 5000/10\,000 = 0.5$$
$$\mathrm{DF}_{AD} = 1000/10\,000 = 0.1$$

As a result, if $M = 2000$ N·m acts at joint A, Fig. 12–4*b*, the equilibrium moments exerted by the members on the joint, Fig. 12–4*c*, are

$$M_{AB} = 0.4(2000) = 800 \text{ N}\cdot\text{m}$$
$$M_{AC} = 0.5(2000) = 1000 \text{ N}\cdot\text{m}$$
$$M_{AD} = 0.1(2000) = 200 \text{ N}\cdot\text{m}$$

Carry-Over Factor. Consider again the beam in Fig. 12–3. It was shown in Chapter 11 that $M_{AB} = (4EI/L)\,\theta_A$ (Eq. 11–1) and $M_{BA} = (2EI/L)\,\theta_A$ (Eq. 11–2). Solving for θ_A and equating these equations we get $M_{BA} = M_{AB}/2$. In other words, the moment **M** at the pin induces a moment of $\mathbf{M}' = \frac{1}{2}\mathbf{M}$ at the wall. The carry-over factor represents the fraction of **M** that is "carried over" from the pin to the wall. Hence, in the case of a beam with *the far end fixed*, the carry-over factor is $+\frac{1}{2}$. The plus sign indicates both moments act in the same direction.

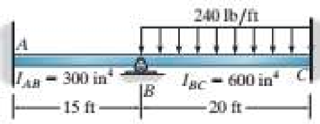
Fig. 12–3

12.2 Moment Distribution for Beams

Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced. The best way to explain the method is by examples.

Consider the beam with a constant modulus of elasticity E and having the dimensions and loading shown in Fig. 12–5*a*. Before we begin, we must first determine the distribution factors at the two ends of each span. Using Eq. 12–1, $K = 4EI/L$, the stiffness factors on either side of B are

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\ \text{in}^4/\text{ft} \qquad K_{BC} = \frac{4E(600)}{20} = 4E(30)\ \text{in}^4/\text{ft}$$



(a)

Fig. 12–5

Thus, using Eq. 12–2, $\text{DF} = K/\Sigma K$, for the ends connected to joint B, we have

$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4$$

$$\text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$

At the walls, joint A and joint C, the distribution factor depends on the member stiffness factor and the "stiffness factor" of the wall. Since in theory it would take an "infinite" size moment to rotate the wall one radian, the wall stiffness factor is infinite. Thus for joints A and C we have

$$\text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0$$

$$\text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

Note that the above results could also have been obtained if the relative stiffness factor $K_R = I/L$ (Eq. 12–3) had been used for the calculations. Furthermore, as long as a *consistent* set of units is used for the stiffness factor, the DF will always be dimensionless, and at a joint, except where it is located at a fixed wall, the sum of the DFs will always equal 1.

Having computed the DFs, we will now determine the FEMs. Only span BC is loaded, and using the table on the inside back cover for a uniform load, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000\ \text{lb}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000\ \text{lb}\cdot\text{ft}$$

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM			−8000	8000
Dist,CO	1600	3200	4800	2400
ΣM	1600	3200	−3200	10 400

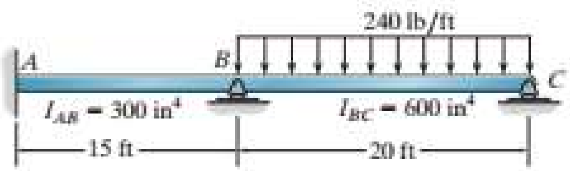
(c)

(f)

Fig. 12–5

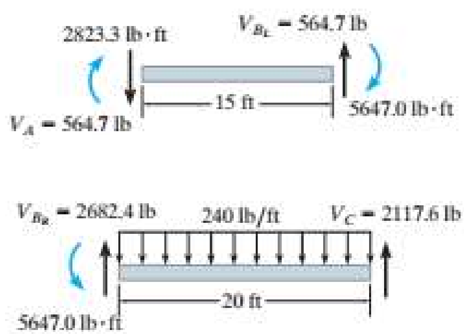
Consider now the same beam, except the support at C is a rocker, Fig. 12–6a. In this case only *one member* is at joint C, so the distribution factor for member CB at joint C is

$$\text{DF}_{CB} = \frac{4E(30)}{4E(30)} = 1$$



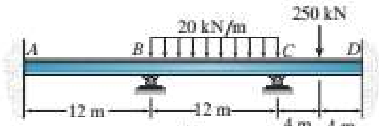
Joint	A	B		C	
Member	AB	BA	BC	CB	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
Dist.		3200	4800	−8000	3
CO	1600		−4000	2400	4
Dist.		1600	2400	−2400	5
CO	800		−1200	1200	6
Dist.		480	720	−1200	7
CO	240		−600	360	8
Dist.		240	360	−360	9
CO	120		−180	180	10
Dist.		72	108	−180	11
CO	36		−90	54	12
Dist.		36	54	−54	13
CO	18		−27	27	14
Dist.		10.8	16.2	−27	15
CO	5.4		−13.5	8.1	16
Dist.		5.4	8.1	−8.1	17
CO	2.7		−4.05	4.05	18
Dist.		1.62	2.43	−4.05	19
CO	0.81		−2.02	1.22	20
Dist.		0.80	1.22	−1.22	21
CO	0.40		−0.61	0.61	22
Dist.		0.24	0.37	−0.61	23
ΣM	2823	5647	−5647	0	24

(c)



(d)

Determine the internal moments at each support of the beam shown in Fig. 12–7*a*. *EI* is constant.



(a)

Fig. 12–7

SOLUTION

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\mathrm{DF}_{AB} = \mathrm{DF}_{DC} = 0 \qquad \mathrm{DF}_{BA} = \mathrm{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\mathrm{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \mathrm{DF}_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$

The fixed-end moments are

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240\ \mathrm{kN\cdot m}$$

$$(\mathrm{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250\ \mathrm{kN\cdot m}$$

Starting with the FEMs, line 4, Fig. 12–7*b*, the moments at joints *B* and *C* are distributed *simultaneously*, line 5. These moments are then carried over *simultaneously* to the respective ends of each span, line 6. The resulting moments are again simultaneously distributed and carried over, lines 7 and 8. The process is continued until the resulting moments are diminished an appropriate amount, line 13. The resulting moments are found by summation, line 14.

Placing the moments on each beam span and applying the equations of equilibrium yields the end shears shown in Fig. 12–7*c* and the bending-moment diagram for the entire beam, Fig. 12–7*d*.

*Here we have used the stiffness factor $4EI/L$; however, the relative stiffness factor I/L could also have been used.

Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

(b)