Ordinary Differential Equations (ODEs)

There are Several method to solve Ordinary Differential Equations like:

1 -First-Order equations

2 -Second Order Linear Equations

3- Power Series Solutions

4- The Laplace Transform Method

5- Systems of Linear Differential Equations

1- First Order Ordinary Differential Equations (ODE)

Types of first order ODE :

1-1- Separable equation

1-2- Homogenous equation

1-3- Exact equation

1-4- Linear equation

1-5- Bernoulli Equation

1-1 First Order Separable Equation

Sometimes the function $f(x, y)$ in the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be written as the product of a function $F(x)$ depending only on x and a function $G(y)$ depending only on y, so that $f(x, y) = F(x)G(y)$, be written

$$\frac{dy}{dx} = F(x)G(y).$$

Then we can integrate on both sides with respect to *x*, obtaining

$$\int F(x)dx = \int \frac{dy}{G(y)} + c$$

Where c is an arbitrary constant

Example : solve

$$x(2y-3)dx + (x^2+1)\,dy = 0 \qquad \div (2y-3)(x^2+1)$$

$$\therefore \frac{x}{x^2+1}dx + \frac{1}{(2y-3)}dy = 0 \qquad \text{by inty.}$$

$$\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(2y-3) = c$$

Example : Find the general Solution of the following differential equations

$$\frac{dy}{dx} - \frac{x^2}{1-y^2} = 0$$

Solution : $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

$(1-y^2)\,dy = x^2\,dx$ it is seperable equation

integrate both sides

$y - \frac{y^3}{3} = \frac{x^3}{3} + c$ multibly by [3]

$3y - y^3 - x^3 + c = 0$ Note: $3c = c$ and $-3c = c$

Example : $e^{x-y}\,dy + dx = 0$

Solution $(e^{x} \cdot e^{-y})\,dy + dx = 0$ $\div [e^{x}]$

$$e^{-y}\,dy + \frac{dx}{e^{x}} = 0$$

$$e^{-y}\,dy + e^{-x}\,dx = 0$$

integrate both side

$-e^{-y} - e^{-x} = c$ $[* -1]$

$$e^{-y} + e^{-x} = c$$

$$\frac{1}{e^{y}} + \frac{1}{e^{x}} = c$$

Example : $y' \sqrt{2xy} = 1$

Solution :

$$\frac{dy}{dx} \sqrt{2xy} = 1$$

$$\sqrt{2} \cdot \sqrt{x} \cdot \sqrt{y} \, \frac{dy}{dx} = 1$$

$$\sqrt{2} \cdot \sqrt{y} \, dy = \frac{dx}{\sqrt{x}}$$

$$\sqrt{2} \, (y)^{\frac{1}{2}} \, dy - x^{-\frac{1}{2}} \, dx = 0$$

3/2

$$\frac{2}{3} \sqrt{2} \, y^{2/3} - 2x^{\frac{1}{2}} = c \qquad [\div 2]$$

$$\frac{\sqrt{2}}{3} \, y^{2/3} - \sqrt{x} = c$$

Example : $\frac{dx}{dy} \ln x = \frac{x}{y}$

Solution : $\ln x \, dx = \frac{x}{y} \, dy$

$$\frac{\ln x}{x} \, dx = \frac{dy}{y}$$

integrate both side

(2) First order Homogenous D.E

If the differential equation (D.E) has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

then it's called " homogenous

How to identify Homogenous Equation ?

Suppose $f(x,y) = \frac{dy}{dx}$, $f(x,y)$ is homogenous if

$$f(\lambda x, \lambda y) = f(x,y)$$

for every real value of λ

methode of Solution :

i) Determine whether the equation homogenes or not

ii) if its homogenous then use $\boxed{V = \frac{y}{x}}$ (1) ort

$$\frac{1}{V} = \frac{x}{y} \Longrightarrow y = Vx \Longrightarrow \boxed{\frac{dy}{dx} = V + x\frac{dv}{dx}}$$ (2)

in the original D.E

iii) seperate the variable x and V

iv) integrate both sides and substitute $V = \frac{y}{x}$

v) use initial condition (if given) to find the constant value

seperable equation method

Example: Determine whether the DE is homogeneos or not:

$\boxed{a}$ $$\frac{dy}{dx} = \frac{x^2+y^2}{(x-y)(x+y)}$$

Solution: $$f(x,y) = \frac{dy}{dx} = \frac{x^2+y^2}{(x-y)(x+y)}$$

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x - \lambda y)(\lambda x + \lambda y)}$$

$$= \frac{\cancel{\lambda^2}(x^2+y^2)}{\cancel{\lambda^2}(x-y)(x+y)} = f(x,y)$$

∴ It is homogenous equation

$\boxed{b}$ $$x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$$

$$f(x,y) = \frac{dy}{dx} = \frac{y}{x} + \sqrt{x^2+y^2}$$

$$f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} + \sqrt{(\lambda x)^2 + (\lambda y)^2}$$

$$= \frac{y}{x} + \lambda\sqrt{x^2+y^2} \neq f(x,y)$$

∴ the differential equation is non-homogenous

First Order Linear Differential Equations :

The general linear differential equation of order (n) is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

where $a_n(x)$, $a_{n-1}(x)$, ---, $a_1(x)$, $a_0(x)$ and $f(x)$ are function depending on x only.

- for first order $(n=1)$ the general eq. become:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

- if $a_0(x) = 0$, then the equation become seperable

$$a_1(x)\frac{dy}{dx} = 0$$

- the $a_1(x) \neq 0$, $a_0(x)$ and $f(x)$ are a function depending on the Variable (x) only

Example 1: $x\frac{dy}{dx} + x^2y = \frac{1}{x}$ is a Linear D.E of first order

where :

$$a_1(x) = x$$

$$a_0(x) = x^2$$

$$f(x) = \frac{1}{x}$$

Example : Solve the D.E $\dfrac{dy}{dx} = \dfrac{1}{x+y^2}$

Solution :

$$\frac{dx}{dy} = x + y^2$$

$$\frac{dx}{dy} - x = y^2$$

$$\mu(y) = e^{\int -1\, dy} = e^{-y}$$

$$e^{-y} \cdot x = \int e^{-y} \cdot y^2 \cdot dy$$

$$e^{-y} \cdot x = -y^2 \cdot e^{-y} - \int -e^{-y} \cdot 2y\, dy$$

$$e^{-y} \cdot x = -y^2 e^{-y} + 2\left[-y \cdot e^{-y} - \int -e^{-y} \cdot 1 \cdot dy\right]$$

$$e^{-y} \cdot x = -y^2 e^{-y} + 2\left[-y e^{-y} - e^{-y} + c\right]$$

$$e^{-y} \cdot x = -y^2 e^{-y} - 2y e^{-y} - 2e^{-y} + 2c$$

$$x = -y^2 - 2y - 2 + \frac{c}{e^{-y}}$$

First Order Exact Equation

The general first order differential equation of the form

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

is said to be an **exact equation** if the expression on the left side is an exact differential.

Let $M(x,y)$ and $N(x,y)$ be continous and have continouse first partial derivative in a rectangular region R defined by $a < x < b$, $c < y < d$.

Then a necessary and sufficient condition that $M(x,y)\,dx + N(x,y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example: Check the following D.E is exact or not

$$2xy\,dx + (x^2-1)\,dy = 0$$

$$M = 2xy \qquad N = +(x^2-1)$$

$$\frac{\partial M}{\partial y} = 2x \quad , \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$

$\therefore$ Exact equation

$$\frac{\partial u}{\partial x} = x^2 + \frac{y}{x} \quad \text{-----------} \ (1)$$

$$\frac{\partial u}{\partial y} = \ln x + 2y \quad \text{-----------} \ (2)$$

integrate eq. (1)

$$\int \partial u = \int \left(x^2 + \frac{y}{x} \right) \partial x$$

$$u = \frac{x^3}{3} + y \ln x + Ay \quad \text{------} \ (3)$$

Derive eq. (3) with respect to y

$$\frac{\partial u}{\partial y} = 0 + \ln x + A'y \quad \text{-------} \ (4)$$

eq (4) = eq (2)

$$\cancel{\ln x} + A'y = \cancel{\ln x} + \cancel{2}y$$

$$A'y = 2y \qquad \text{integrate wit y}$$

$$Ay = y^2 + B \qquad \text{sub. in eq (3)}$$

$$u = \frac{x^3}{3} + y \ln x + y^2 + B$$

$$\boxed{\frac{x^3}{3} + y \ln x + y^2 = B}$$

general eq.

Ex Solve: $y\,dx + (3+3x-y)\,dy = 0$

$M = y \Longrightarrow \frac{\partial M}{\partial y} = 1$ — ①

$N = 3+3x-y \Longrightarrow \frac{\partial N}{\partial x} = 0+3-0 = 3$ — ②

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Not exact

$f = \frac{1-2}{N} = \frac{1-3}{3+3x-y} = \frac{-2}{3+3x-y}$

f is not function of (x) only

then use $f = \frac{2-1}{M} = \frac{3-1}{y} = \frac{2}{y}$

f is a function of y only

$\mu = e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$

$y^2(y\,dx) + y^2(3+3x-y)\,dy = 0$

$y^3\,dx + (3y^2 + 3y^2x - y^3)\,dy = 0$

$M = y^3 \Rightarrow \frac{\partial M}{\partial y} = 3y^2$

$N = 3y^2 + 3y^2x - y^3 \Longrightarrow \frac{\partial N}{\partial x} = 0 + 3y^2 - 0 = 3y^2$

Ex Solve : $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$

$$2xy\,dx = (x^2-y^2)\,dy$$

$$2xy\,dx - (x^2-y^2)\,dy = 0$$

$$2xy\,dx + (y^2-x^2)\,dy = 0$$

$$M = 2xy \Rightarrow \frac{\partial M}{\partial y} = 2x \quad \text{—— (1)}$$

$$N = y^2-x^2 \Rightarrow \frac{\partial N}{\partial x} = -2x \quad \text{—— (2)}$$

-- Not exact --

$$f = \frac{(1)-(2)}{N} = \frac{2x+2x}{y^2-x^2} = \frac{4x}{y^2-x^2}$$

It is not a function of x only

then $f = \frac{(2)-(1)}{M} = \frac{-2x-2x}{2xy} = \frac{-4x}{2xy} = \frac{-2}{y}$

$$\therefore \mu = e^{\int \frac{-2}{y} dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \frac{1}{y^2}$$

$$\frac{1}{y^2}(2xy)\,dx + (y^2-x^2)\,dy\,\frac{1}{y^2} = 0$$

$$\frac{2x}{y}\,dx + \left(1-\frac{x^2}{y^2}\right)dy = 0$$

Example

$$(x + x^3 \sin 2y)dy - 2y\,dx = 0$$

$$M = -2y \Rightarrow \frac{\partial M}{\partial y} = -2 \quad —①$$

$$N = x + x^3 \sin 2y \Rightarrow \frac{\partial N}{\partial x} = 1 + 3x^2 \sin 2y \quad —②$$

not exact

$$f = \frac{1-2}{N}$$

$$f = \frac{-2 - 1 - 3x^2 \sin 2y}{x + x^3 \sin 2y} = \frac{-3 - 3x^2 \sin 2y}{x(1 + x^2 \sin 2y)}$$

$$= \frac{-3(1 + x^2 \sin 2y)}{x(1 + x^2 \sin 2y)} = \boxed{\frac{-3}{x}}$$

$$\mu = e^{\int \frac{-3}{x} dx} = e^{-3\ln x} = \frac{\ln x^{-3}}{e^{\ln x^3}} = \frac{1}{x^3}$$

$$\frac{1}{x^3}(x + x^3 \sin 2y)dy - (2y\,dx)\frac{1}{x^3} = 0$$

$$\left(\frac{1}{x^2} + \sin 2y\right)dy - \frac{2y}{x^3}dx = 0$$

$$M = \frac{-2y}{x^3} \Longrightarrow \frac{\partial M}{\partial y} = \frac{(-2x^3) - 0}{x^6} = \frac{-2}{x^3}$$

$$N = \frac{1}{x^2} + \sin 2y \Longrightarrow \frac{\partial N}{\partial x} = \frac{0 - 2x}{x^4} = \frac{-2}{x^3}$$

Ex

$$(y^2\cos x - y)\,dx + (x + y^2)\,dy = 0$$

$$M = y^2\cos x - y \Longrightarrow \frac{\partial M}{\partial y} = 2y\cos x - 1 \quad \text{--- } 1$$

$$N = x + y^2 \Longrightarrow \frac{\partial N}{\partial x} = 1 \quad \text{--- } 2$$

not exact

$$f = \frac{2-1}{M} = \frac{1 - 2y\cos x + 1}{y^2\cos x - y} = \frac{2 - 2y\cos x}{y^2\cos x - y} = \frac{-2(-1+\cancel{y\cos x})}{y(\cancel{y\cos x} - 1)}$$

$$= \frac{-2}{y}$$

$$\mu = e^{\int \frac{-2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = \boxed{\frac{1}{y^2}} \quad \text{Integration factor}$$

$$* \; \frac{1}{y^2}(y^2\cos x - y)\,dx + \frac{1}{y^2}(x + y^2)\,dy = 0$$

$$\left(\cos x - \frac{1}{y}\right)dx + \left(\frac{x}{y^2} + 1\right)dy = 0$$

$$M = \cos x - \frac{1}{y} \Longrightarrow \frac{\partial M}{\partial y} = 0 - \left(\frac{-1}{y^2}\right) = \frac{1}{y^2}$$

$$N = \frac{x}{y^2} + 1 \Longrightarrow \frac{\partial N}{\partial x} = \frac{y^2}{y^4} = \frac{1}{y^2}$$

Example

$$(x + x^3 \sin 2y)dy - 2y\,dx = 0$$

$$M = -2y \Longrightarrow \frac{\partial M}{\partial y} = -2 \quad -(1)$$

$$N = x + x^3 \sin 2y \Longrightarrow \frac{\partial N}{\partial x} = 1 + 3x^2 \sin 2y \quad -(2)$$

not exact

$$f = \frac{1-2}{N}$$

$$f = \frac{-2 - 1 - 3x^2 \sin 2y}{x + x^3 \sin 2y} = \frac{-3 - 3x^2 \sin 2y}{x(1 + x^2 \sin 2y)}$$

$$= \frac{-3(1 + x^2 \sin 2y)}{x(1 + x^2 \sin 2y)} = \boxed{\frac{-3}{x}}$$

$$\mu = e^{\int \frac{-3}{x} dx} = e^{-3\ln x} = \frac{1}{e^{\ln x^3}} = \frac{1}{x^3}$$

$$\frac{1}{x^3}(x + x^3 \sin 2y)dy - (2y\,dx)\frac{1}{x^3} = 0$$

$$\left(\frac{1}{x^2} + \sin 2y\right)dy - \frac{2y}{x^3}dx = 0$$

$$M = -\frac{2y}{x^3} \Longrightarrow \frac{\partial M}{\partial y} = \frac{(-2x^3) - 0}{x^6} = \frac{-2}{x^3}$$

$$N = \frac{1}{x^2} + \sin 2y \Longrightarrow \frac{\partial M}{\partial x} = \frac{0 - 2x}{x^4} = \frac{-2}{x^3}$$

Ex

$$(y^2 \cos x - y)\,dx + (x + y^2)\,dy = 0$$

$$M = y^2 \cos x - y \Longrightarrow \frac{\partial M}{\partial y} = 2y \cos x - 1 \quad \text{--- } 1$$

$$N = x + y^2 \Longrightarrow \frac{\partial N}{\partial x} = 1 \quad \text{--- } 2$$

not exact

$$f = \frac{2-1}{M} = \frac{1 - 2y\cos x + 1}{y^2 \cos x - y} = \frac{2 - 2y\cos x}{y^2 \cos x - y} = \frac{-2(-1 + \cancel{y\cos x})}{y(\cancel{y\cos x - 1})}$$

$$= \frac{-2}{y}$$

$$\mu = e^{\int \frac{-2}{y} dy} = e^{-2\ln y} = e^{\ln y^{-2}} = \boxed{\frac{1}{y^2}} \quad \text{Integration factor}$$

$$* \; \frac{1}{y^2}(y^2 \cos x - y)\,dx + \frac{1}{y^2}(x + y^2)\,dy = 0$$

$$\left(\cos x - \frac{1}{y}\right) dx + \left(\frac{x}{y^2} + 1\right) dy = 0$$

$$M = \cos x - \frac{1}{y} \Longrightarrow \frac{\partial M}{\partial y} = 0 - \left(\frac{-1}{y^2}\right) = \frac{1}{y^2}$$

$$N = \frac{x}{y^2} + 1 \Longrightarrow \frac{\partial N}{\partial x} = \frac{y^2}{y^4} = \frac{1}{y^2}$$

Bernoulli Equation

In som differential equation, the independent variable was in degree greter than one ($y^2, y^3, \ldots, y^n$), thease equation can be reduced to linear D.E.

Bernoulli equation is one of this example.

The form of Birnoulli equation is

$$\frac{dy}{dx} + P(x,y) = q(x)\, y^n$$

where $n \neq 0$ or 1

Example 1 : Solve the following D.E

$$y(6y^2 - x - 1)\,dx + 2x\,dy$$

Solution

$$(6y^3 - yx - y)\,dx + 2x\,dy = 0 \quad [\div dx]$$

$$(6y^3 - yx - y) + 2x\frac{dy}{dx} = 0 \quad [\div 2x]$$

$$\frac{3y^3}{x} - \frac{y}{2} - \frac{y}{2x} + \frac{dy}{dx} = 0$$

$$p(x) = \left(\frac{x+1}{x}\right) = 1 + \frac{1}{x}$$

$$q(x) = \frac{6}{x}$$

integration factor ($\mu(x) = e^{\int p(x)dx}$

$$\mu(x) = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = e^{x} \cdot e^{\ln x} = x e^{x}$$

$$(x e^{x} \cdot v) = \int \cancel{x} e^{x} \cdot \frac{6}{\cancel{x}}\, dx$$

$$x e^{x} \cdot v = 6 e^{x} + c \qquad [\div x e^{x}]$$

$$v = \frac{6}{x} + \frac{c}{x e^{x}}$$

but $y = v^{-\frac{1}{2}}$

$$y = \left(\frac{6}{x} + \frac{c}{x e^{x}}\right)^{-\frac{1}{2}}$$

general eq.