

DIFFERENTIATION

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

EXAMPLE 1

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

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EXAMPLE 2 Interpreting Rule 2

f	x	x^2	x^3	x^4	\dots
f'	1	$2x$	$3x^2$	$4x^3$	\dots

RULE 3 Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

EXAMPLE 3

- (a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y -coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.9).

- (b) A useful special case

The derivative of the negative of a differentiable function u is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

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RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 4 Derivative of a Sum

$$\begin{aligned}y &= x^4 + 12x \\ \frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\ &= 4x^3 + 12\end{aligned}$$

EXAMPLE 5 Derivative of a Polynomial

$$\begin{aligned}y &= x^3 + \frac{4}{3}x^2 - 5x + 1 \\ \frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\ &= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 \\ &= 3x^2 + \frac{8}{3}x - 5\end{aligned}$$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

RULE 5 Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x} \left(x^2 + \frac{1}{x} \right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\begin{aligned}\frac{d}{dx} \left[\frac{1}{x} \left(x^2 + \frac{1}{x} \right) \right] &= \frac{1}{x} \left(2x - \frac{1}{x^2} \right) + \left(x^2 + \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) \\&= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3} \\&= 1 - \frac{2}{x^3}.\end{aligned}$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}, \text{ and}$$

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \text{ by}$$

Example 3, Section 2.7.

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EXAMPLE 8 Derivative from Numerical Values

Let $y = uv$ be the product of the functions u and v . Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

$$\begin{aligned}y'(2) &= u(2)v'(2) + v(2)u'(2) \\&= (3)(2) + (1)(-4) = 6 - 4 = 2.\end{aligned}$$

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EXAMPLE 9 Differentiating a Product in Two Ways

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

- (a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2 + 1)(x^3 + 3)] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

- (b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2 + 1) \cdot 2t - (t^2 - 1) \cdot 2t}{(t^2 + 1)^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2}$$

$$= \frac{4t}{(t^2 + 1)^2}.$$

$$\frac{d}{dt} \left(\frac{u}{v} \right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$



Negative Integer Powers of x

The Power Rule for negative integers is the same as the rule for positive integers.

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

(a) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$ Agrees with Example 3,

(b) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$

EXAMPLE 12 Tangent to a Curve

Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point $(1, 3)$ (Figure 3.11).

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2 \frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through $(1, 3)$ with slope $m = -1$ is

$$y - 3 = (-1)(x - 1) \quad \text{Point-slope equation}$$

$$y = -x + 1 + 3$$

$$y = -x + 4.$$

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .

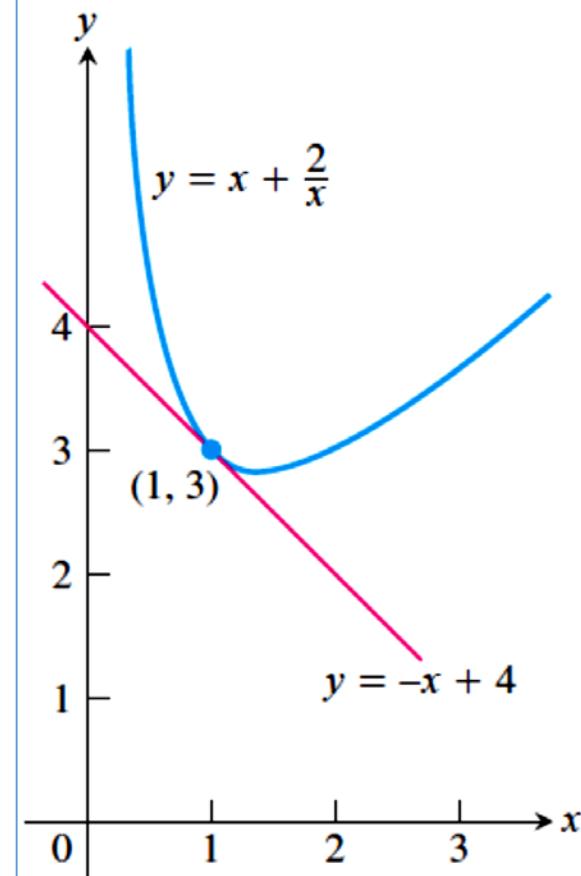


FIGURE 3.11 The tangent to the curve: $y = x + (2/x)$ at $(1, 3)$ in Example 12.

EXAMPLE 13 Choosing Which Rule to Use

Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4},$$

expand the numerator and divide by x^4 :

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function.

So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx} (6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third derivative** of y with respect to x . The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\text{First derivative: } y' = 3x^2 - 6x$$

$$\text{Second derivative: } y'' = 6x - 6$$

$$\text{Third derivative: } y''' = 6$$

$$\text{Fourth derivative: } y^{(4)} = 0.$$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ■

Derivatives of Trigonometric Functions

$$\frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} (\csc x) = -\csc x \cot x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

$$\cos^2 \theta + \sin^2 \theta = 1.$$

$$1 + \tan^2 \theta = \sec^2 \theta.$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B\end{aligned}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta} \qquad \csc \theta = \frac{1}{\sin \theta}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\begin{aligned}\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\ &= 2x - \cos x.\end{aligned}$$

(b) $y = x^2 \sin x$:

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\ &= x^2 \cos x + 2x \sin x.\end{aligned}$$

(c) $y = \frac{\sin x}{x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\ &= \frac{x \cos x - \sin x}{x^2}.\end{aligned}$$



The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 3.23 shows a way to visualize this result.

EXAMPLE 2 Derivatives Involving the Cosine

(a) $y = 5x + \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5 - \sin x.\end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule} \\ &= \sin x(-\sin x) + \cos x(\cos x) \\ &= \cos^2 x - \sin^2 x.\end{aligned}$$

(c) $y = \frac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x}. && \blacksquare\end{aligned}$$

EXAMPLE 5

Find $d(\tan x)/dx$.

Solution

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\&= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} \\&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Quotient Rule



EXAMPLE 6

Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \quad \text{Product Rule}$$

$$= \sec x (\sec^2 x) + \tan x (\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$



The Chain Rule

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

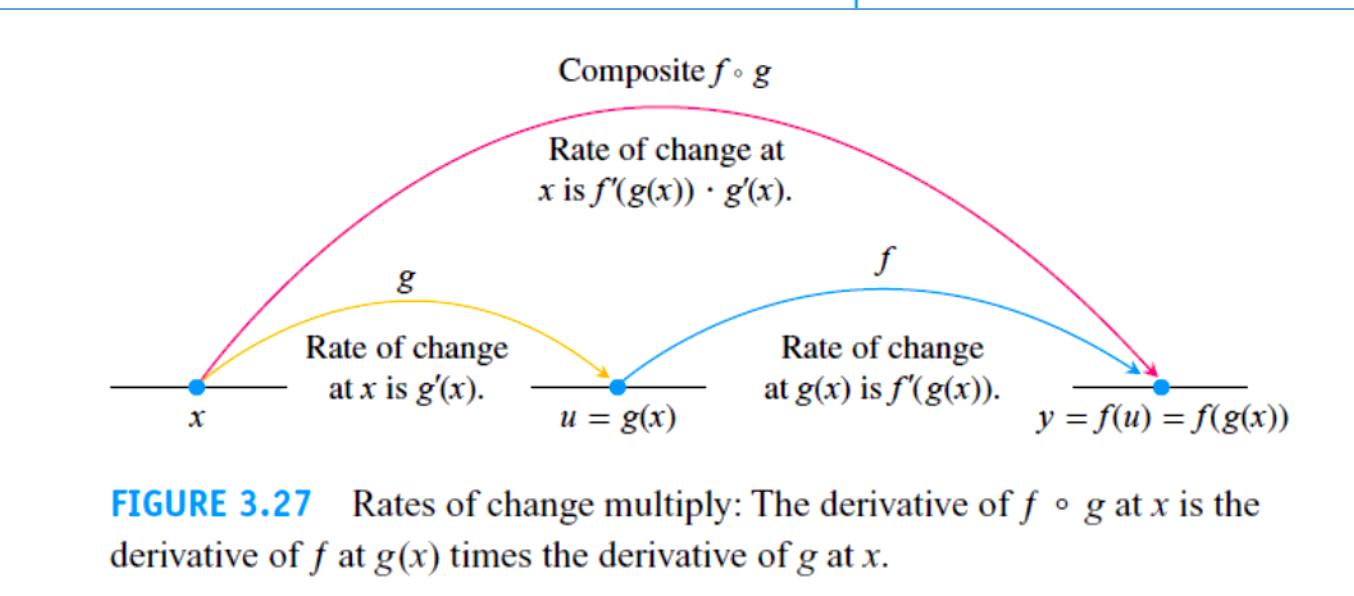


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x .

EXAMPLE 1 Relating Derivatives

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$.

How are the derivatives of these functions related?

Solution We have

$$\frac{dy}{dx} = \frac{3}{2}, \quad \frac{dy}{du} = \frac{1}{2}, \quad \text{and} \quad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

EXAMPLE 2

The function

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Calculating the derivative from the expanded formula, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

EXAMPLE 3 Applying the Chain Rule

An object moves along the x -axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t .

Solution We know that the velocity is dx/dt . In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \quad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \quad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\&= -\sin(u) \cdot 2t \quad \frac{dx}{du} \text{ evaluated at } u \\&= -\sin(t^2 + 1) \cdot 2t \\&= -2t \sin(t^2 + 1).\end{aligned}$$

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“Outside-Inside” Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate it at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside function.”

EXAMPLE 4 Differentiating from the Outside In

Differentiate $\sin(x^2 + x)$ with respect to x .

Solution

$$\frac{d}{dx} \sin \underbrace{(x^2 + x)}_{\text{inside}} = \cos \underbrace{(x^2 + x)}_{\text{inside left alone}} \cdot \underbrace{(2x + 1)}_{\text{derivative of the inside}}$$



EXAMPLE 5 A Three-Link “Chain”

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t . Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt} \left(\tan(5 - \sin 2t) \right) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t) \right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t) \sec^2(5 - \sin 2t). \end{aligned}$$



The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x , then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

Here's an example of how it works: If n is a positive or negative integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x , then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx} u^n = mu^{n-1} \frac{du}{dx}. \quad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 6 Applying the Power Chain Rule

$$\begin{aligned}\text{(a)} \quad \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) \\&= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\&= 7(5x^3 - x^4)^6(15x^2 - 4x^3)\end{aligned}$$

Power Chain Rule with
 $u = 5x^3 - x^4, n = 7$

$$\begin{aligned}\text{(b)} \quad \frac{d}{dx}\left(\frac{1}{3x - 2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\&= -1(3x - 2)^{-2} \frac{d}{dx}(3x - 2) \\&= -1(3x - 2)^{-2}(3) \\&= -\frac{3}{(3x - 2)^2}\end{aligned}$$

Power Chain Rule with
 $u = 3x - 2, n = -1$

EXAMPLE 7 Finding Tangent Slopes

- (a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.
(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution

(a) $\frac{dy}{dx} = 5 \sin^4 x \cdot \frac{d}{dx} \sin x$ Power Chain Rule with $u = \sin x, n = 5$
 $= 5 \sin^4 x \cos x$

The tangent line has slope

$$\left. \frac{dy}{dx} \right|_{x=\pi/3} = 5 \left(\frac{\sqrt{3}}{2} \right)^4 \left(\frac{1}{2} \right) = \frac{45}{32}.$$

(b) $\frac{dy}{dx} = \frac{d}{dx} (1 - 2x)^{-3}$
 $= -3(1 - 2x)^{-4} \cdot \frac{d}{dx} (1 - 2x)$ Power Chain Rule with $u = (1 - 2x), n = -3$
 $= -3(1 - 2x)^{-4} \cdot (-2)$
 $= \frac{6}{(1 - 2x)^4}$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

Derivatives of Logarithmic & Exponential Functions

$$\frac{d}{dx}(\mathrm{e}^x) = \mathrm{e}^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Power Rule

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Derivative of an exponential function

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5 \log_9 w$

(b) $f(x) = 3e^x + 10x^3 \ln x$

(c) $y = \frac{5e^x}{3e^x + 1}$

$$R'(w) = 4^w \ln 4 - \frac{5}{w \ln 9}$$

$$\begin{aligned}f'(x) &= 3e^x + 30x^2 \ln x + 10x^3 \left(\frac{1}{x} \right) \\&= 3e^x + 30x^2 \ln x + 10x^2\end{aligned}$$

$$y' = \frac{5e^x(3e^x + 1) - (5e^x)(3e^x)}{(3e^x + 1)^2}$$

$$= \frac{15e^{2x} + 5e^x - 15e^{2x}}{(3e^x + 1)^2}$$

$$= \frac{5e^x}{(3e^x + 1)^2}$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

$$\sin^{-1} x = \arcsin x$$

$$\cos^{-1} x = \arccos x$$

$$\tan^{-1} x = \arctan x$$

$$\cot^{-1} x = \operatorname{arccot} x$$

$$\sec^{-1} x = \operatorname{arcsec} x$$

$$\csc^{-1} x = \operatorname{arccsc} x$$

Example 4 Differentiate the following functions.

(a) $f(t) = 4 \cos^{-1}(t) - 10 \tan^{-1}(t)$

(b) $y = \sqrt{z} \sin^{-1}(z)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2} z^{-\frac{1}{2}} \sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \frac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

Example 1 Find the first four derivatives for each of the following.

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + e^t$

(b) $y = \cos x$

(c) $f(y) = \sin(3y) + e^{-2y} + \ln(7y)$

Solution

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + e^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + e^t$$

$$R''(t) = 6 - 2t^{-\frac{3}{2}} + e^t$$

$$R'''(t) = 3t^{-\frac{5}{2}} + e^t$$

$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + e^t$$

(b) $y = \cos x$

Again, let's just do some derivatives.

$$y = \cos x$$

$$y' = -\sin x$$

$$y'' = -\cos x$$

$$y''' = \sin x$$

$$y^{(4)} = \cos x$$

(c) $f(y) = \sin(3y) + e^{-2y} + \ln(7y)$

$$f'(y) = 3\cos(3y) - 2e^{-2y} + \frac{1}{y} = 3\cos(3y) - 2e^{-2y} + y^{-1}$$

$$f''(y) = -9\sin(3y) + 4e^{-2y} - y^{-2}$$

$$f'''(y) = -27\cos(3y) - 8e^{-2y} + 2y^{-3}$$

$$f^{(4)}(y) = 81\sin(3y) + 16e^{-2y} - 6y^{-4}$$

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

1

$$x^2 + y^2 = 25$$

or

2

$$x^3 + y^3 = 6xy$$

(a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

SOLUTION 1

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \frac{dy}{dx} = 2y \frac{dy}{dx}$$

Thus

$$2x + 2y \frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx :

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) At the point $(3, 4)$ we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at $(3, 4)$ is therefore

$$y - 4 = -\frac{3}{4}(x - 3) \quad \text{or} \quad 3x + 4y = 25$$

Example 1 Find y' for $xy = 1$.

Solution

There are actually two solution methods for this problem.

Solution 1 :

This is the simple way of doing the problem. Just solve for y to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \quad \Rightarrow \quad y' = -\frac{1}{x^2}$$

Solution 2 :

$$\frac{d}{dx}(xy(x)) = \frac{d}{dx}(1) \quad (1)y(x) + x\frac{d}{dx}(y(x)) = 0$$

$$y + xy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = -\frac{1/x}{x} = -\frac{1}{x^2}$$

Example 2 Differentiate each of the following.

(a) $(5x^3 - 7x + 1)^5, [f(x)]^5, [y(x)]^5$

(b) $\sin(3 - 6x), \sin(y(x))$

(c) $e^{x^2 - 9x}, e^{y(x)}$

$$\frac{d}{dx} \left[(5x^3 - 7x + 1)^5 \right] = 5(5x^3 - 7x + 1)^4 (15x^2 - 7)$$

$$\frac{d}{dx} [f(x)]^5 = 5[f(x)]^4 f'(x)$$

$$\frac{d}{dx} [y(x)]^5 = 5[y(x)]^4 y'(x)$$

$$\frac{d}{dx} [\sin(3 - 6x)] = -6 \cos(3 - 6x)$$

$$\frac{d}{dx} [\sin(y(x))] = y'(x) \cos(y(x))$$

$$\frac{d}{dx} (e^{x^2 - 9x}) = (2x - 9)e^{x^2 - 9x}$$

$$\frac{d}{dx} (e^{y(x)}) = y'(x)e^{y(x)}$$

Example 2 Find the second derivative for each of the following functions.

(a) $Q(t) = \sec(5t)$

(b) $g(w) = e^{1-2w^3}$

(c) $f(t) = \ln(1+t^2)$

Solution

(a) $Q(t) = \sec(5t)$

$$\begin{aligned} Q''(t) &= 25 \sec(5t) \tan(5t) \tan(5t) + 25 \sec(5t) \sec^2(5t) \\ &= 25 \sec(5t) \tan^2(5t) + 25 \sec^3(5t) \end{aligned}$$

$$Q'(t) = 5 \sec(5t) \tan(5t)$$

(b) $g(w) = e^{1-2w^3}$

$$g'(w) = -6w^2 e^{1-2w^3}$$

$$\begin{aligned} g''(w) &= -12w e^{1-2w^3} - 6w^2 (-6w^2) e^{1-2w^3} \\ &= -12w e^{1-2w^3} + 36w^4 e^{1-2w^3} \end{aligned}$$

(c) $f(t) = \ln(1+t^2)$

$$\begin{aligned} f''(t) &= \frac{2(1+t^2) - (2t)(2t)}{(1+t^2)^2} \\ &= \frac{2-2t^2}{(1+t^2)^2} \end{aligned}$$

$$f'(t) = \frac{2t}{1+t^2}$$

Example 1 Differentiate the function.

$$y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

$$\ln y = \ln(x^5) - \ln((1-10x)\sqrt{x^2+2})$$

$$\ln y = \ln(x^5) - \ln(1-10x) - \ln(\sqrt{x^2+2})$$

$$y' = y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

$$= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}(x^2+2)^{-\frac{1}{2}}(2x)}{(x^2+2)^{\frac{1}{2}}}$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

Example 2 Differentiate $y = x^x$

Solution

We've seen two functions similar to this at this point.

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

$$y' = y(1 + \ln x)$$

$$= x^x (1 + \ln x)$$

Example 3 Differentiate $y = (1 - 3x)^{\cos(x)}$

$$\ln y = \ln[(1 - 3x)^{\cos(x)}] = \cos(x) \ln(1 - 3x)$$

$$\frac{y'}{y} = -\sin(x) \ln(1 - 3x) + \cos(x) \frac{-3}{1 - 3x} = -\sin(x) \ln(1 - 3x) - \cos(x) \frac{3}{1 - 3x}$$

$$y' = -y \left(\sin(x) \ln(1 - 3x) + \cos(x) \frac{3}{1 - 3x} \right)$$
$$= -(1 - 3x)^{\cos(x)} \left(\sin(x) \ln(1 - 3x) + \cos(x) \frac{3}{1 - 3x} \right)$$

$$\frac{d}{dx}(a^b) = 0$$

This is a constant

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Power Rule

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Derivative of an exponential function

$$\frac{d}{dx}(x^x) = x^x(1 + \ln x)$$

Logarithmic Differentiation

INTEGRATION-Indefinite Integrals

$$\int cf(x) dx = c \int f(x) dx$$

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$$

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.

$$(a) \int 5t^3 - 10t^{-6} + 4 dt$$

$$(b) \int x^8 + x^{-8} dx$$

$$(c) \int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}} dx$$

$$(d) \int dy$$

$$(e) \int (w + \sqrt[3]{w})(4 - w^2) dw$$

$$(f) \int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} dx$$

$$\begin{aligned}\int 5t^3 - 10t^{-6} + 4 dt &= 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c \\ &= \frac{5}{4}t^4 + 2t^{-5} + 4t + c\end{aligned}$$

$$\int x^8 + x^{-8} dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

$$\int dy = \int 1 dy = y + c$$

$$\begin{aligned}\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}} dx &= \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}} dx \\ &= 3\frac{1}{7/4}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}} + c\end{aligned}$$

$$= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c$$

$$\begin{aligned}\int (w + \sqrt[3]{w})(4 - w^2) dw &= \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}} dw \\ &= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c\end{aligned}$$

$$\begin{aligned}\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} dx &= \int \frac{4x^{10}}{x^3} - \frac{2x^4}{x^3} + \frac{15x^2}{x^3} dx \\ &= \int 4x^7 - 2x + \frac{15}{x} dx \\ &= \frac{1}{2}x^8 - x^2 + 15\ln|x| + c\end{aligned}$$

$$\int \frac{15}{x} dx = 15 \int \frac{1}{x} dx = 15 \ln|x| + c$$

$$(a) \int 3e^x + 5 \cos x - 10 \sec^2 x \, dx$$

$$(b) \int 2 \sec w \tan w + \frac{1}{6w} \, dw$$

$$(c) \int \frac{23}{y^2+1} + 6 \csc y \cot y + \frac{9}{y} \, dy$$

$$(d) \int \frac{3}{\sqrt{1-x^2}} + 6 \sin x + 10 \sinh x \, dx$$

$$(e) \int \frac{7-6 \sin^2 \theta}{\sin^2 \theta} \, d\theta$$

$$\int \frac{23}{y^2+1} + 6 \csc y \cot y + \frac{9}{y} \, dy = 23 \tan^{-1} y - 6 \csc y + 9 \ln|y| + c$$

$$\int \frac{3}{\sqrt{1-x^2}} + 6 \sin x + 10 \sinh x \, dx = 3 \sin^{-1} x - 6 \cos x + 10 \cosh x + c$$

$$\int 3e^x + 5 \cos x - 10 \sec^2 x \, dx = 3e^x + 5 \sin x - 10 \tan x + c$$

$$\begin{aligned}\int 2 \sec w \tan w + \frac{1}{6w} \, dw &= \int 2 \sec w \tan w \, dw + \int \frac{1}{6} \frac{1}{w} \, dw \\ &= \int 2 \sec w \tan w \, dw + \frac{1}{6} \int \frac{1}{w} \, dw\end{aligned}$$

$$\int 2 \sec w \tan w + \frac{1}{6w} \, dw = 2 \sec w + \frac{1}{6} \ln|w| + c$$

$$\begin{aligned}\int \frac{7-6 \sin^2 \theta}{\sin^2 \theta} \, d\theta &= \int \frac{7}{\sin^2 \theta} - 6 \, d\theta \\ &= \int 7 \csc^2 \theta - 6 \, d\theta\end{aligned}$$

$$\int \frac{7-6 \sin^2 \theta}{\sin^2 \theta} \, d\theta = -7 \cot \theta - 6\theta + c$$

If u is any differentiable function, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1, n \text{ rational}).$$

EXAMPLE 1 Using the Power Rule

$$\int \sqrt{1+y^2} \cdot 2y dy = \int \sqrt{u} \cdot \left(\frac{du}{dy} \right) dy$$

$$= \int u^{1/2} du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (1+y^2)^{3/2} + C$$

Let $u = 1+y^2$,
 $du/dy = 2y$

Integrate, using Eq. (1)
with $n = 1/2$.

Simpler form

Replace u by $1+y^2$.

EXAMPLE 2 Adjusting the Integrand by a Constant

$$\begin{aligned}\int \sqrt{4t - 1} dt &= \int \frac{1}{4} \cdot \sqrt{4t - 1} \cdot 4 dt \\&= \frac{1}{4} \int \sqrt{u} \cdot \left(\frac{du}{dt} \right) dt \\&= \frac{1}{4} \int u^{1/2} du \\&= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C \\&= \frac{1}{6} u^{3/2} + C \\&= \frac{1}{6} (4t - 1)^{3/2} + C\end{aligned}$$

Replace u by $4t - 1$. ■

Let $u = 4t - 1$,
 $du/dt = 4$.

With the $1/4$ out front,
the integral is now in
standard form.

Integrate, using Eq. (1)
with $n = 1/2$.

Simpler form

Substitution: Running the Chain Rule Backwards

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

EXAMPLE 3 Using Substitution

$$\begin{aligned}\int \cos(7\theta + 5) d\theta &= \int \cos u \cdot \frac{1}{7} du \\&= \frac{1}{7} \int \cos u du \\&= \frac{1}{7} \sin u + C \\&= \frac{1}{7} \sin(7\theta + 5) + C\end{aligned}$$

Let $u = 7\theta + 5$, $du = 7 d\theta$,
 $(1/7) du = d\theta$.

With the $(1/7)$ out front, the integral is now in standard form.

Replace u by $7\theta + 5$.

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$. ■

EXAMPLE 4 Using Substitution

$$\begin{aligned}\int x^2 \sin(x^3) dx &= \int \sin(x^3) \cdot x^2 dx \\&= \int \sin u \cdot \frac{1}{3} du \\&= \frac{1}{3} \int \sin u du \\&= \frac{1}{3}(-\cos u) + C \\&= -\frac{1}{3} \cos(x^3) + C\end{aligned}$$

Let $u = x^3$,
 $du = 3x^2 dx$,
 $(1/3) du = x^2 dx$.

Integrate with respect to u .

Replace u by x^3 .

EXAMPLE 5 Using Identities and Substitution

$$\begin{aligned}\int \frac{1}{\cos^2 2x} dx &= \int \sec^2 2x dx & \frac{1}{\cos 2x} = \sec 2x \\&= \int \sec^2 u \cdot \frac{1}{2} du & u = 2x, \\&= \frac{1}{2} \int \sec^2 u du & du = 2 dx, \\&= \frac{1}{2} \tan u + C & dx = (1/2) du \\&= \frac{1}{2} \tan 2x + C & \frac{d}{du} \tan u = \sec^2 u \\&& u = 2x\end{aligned}$$

EXAMPLE 6 Using Different Substitutions

Evaluate

$$\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}}.$$

Solution 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}}$$

$$= \int u^{-1/3} \, du$$

$$= \frac{u^{2/3}}{2/3} + C$$

$$= \frac{3}{2} u^{2/3} + C$$

$$= \frac{3}{2} (z^2 + 1)^{2/3} + C$$

Let $u = z^2 + 1$,
 $du = 2z \, dz$.

In the form $\int u^n \, du$

Integrate with respect to u .

Replace u by $z^2 + 1$.

Solution 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2 \, du}{u} \\&= 3 \int u \, du \\&= 3 \cdot \frac{u^2}{2} + C \\&= \frac{3}{2}(z^2 + 1)^{2/3} + C\end{aligned}$$

Let $u = \sqrt[3]{z^2 + 1}$,
 $u^3 = z^2 + 1$,
 $3u^2 \, du = 2z \, dz$.

Integrate with respect to u .

Replace u by $(z^2 + 1)^{1/3}$.

The Integrals of $\sin^2 x$ and $\cos^2 x$

EXAMPLE 7

(a) $\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$ $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx$$
$$= \frac{1}{2}x - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx$ $\cos^2 x = \frac{1 + \cos 2x}{2}$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$
 As in part (a), but
with a sign change

$$(a) \int \left(1 - \frac{1}{w}\right) \cos(w - \ln w) dw$$

$$(b) \int 3(8y-1)e^{4y^2-y} dy$$

$$(c) \int x^2 (3-10x^3)^4 dx$$

$$(d) \int \frac{x}{\sqrt{1-4x^2}} dx$$

$$u = w - \ln w$$

$$\int \left(1 - \frac{1}{w}\right) \cos(w - \ln w) dw = \int \cos(u) du$$

$$= \sin(u) + c$$

$$= \sin(w - \ln w) + c$$

$$u = 4y^2 - y$$

$$du = (8y-1)dy$$

$$\int 3(8y-1)e^{4y^2-y} dy = 3 \int e^u du$$

$$= 3e^u + c$$

$$= 3e^{4y^2-y} + c$$

$$u = 3 - 10x^3$$

$$du = -30x^2 dx \quad x^2 dx = -\frac{1}{30} du$$

$$\int x^2 (3-10x^3)^4 dx = \int (3-10x^3)^4 x^2 dx$$

$$= \int u^4 \left(-\frac{1}{30}\right) du$$

$$= -\frac{1}{30} \left(\frac{1}{5}\right) u^5 + c$$

$$= -\frac{1}{150} (3-10x^3)^5 + c$$

$$u = 1 - 4x^2$$

$$du = -8x dx \Rightarrow x dx = -\frac{1}{8} du$$

$$\int \frac{x}{\sqrt{1-4x^2}} dx = \int x (1-4x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{1}{8} \int u^{-\frac{1}{2}} du$$

$$= -\frac{1}{4} u^{\frac{1}{2}} + c$$

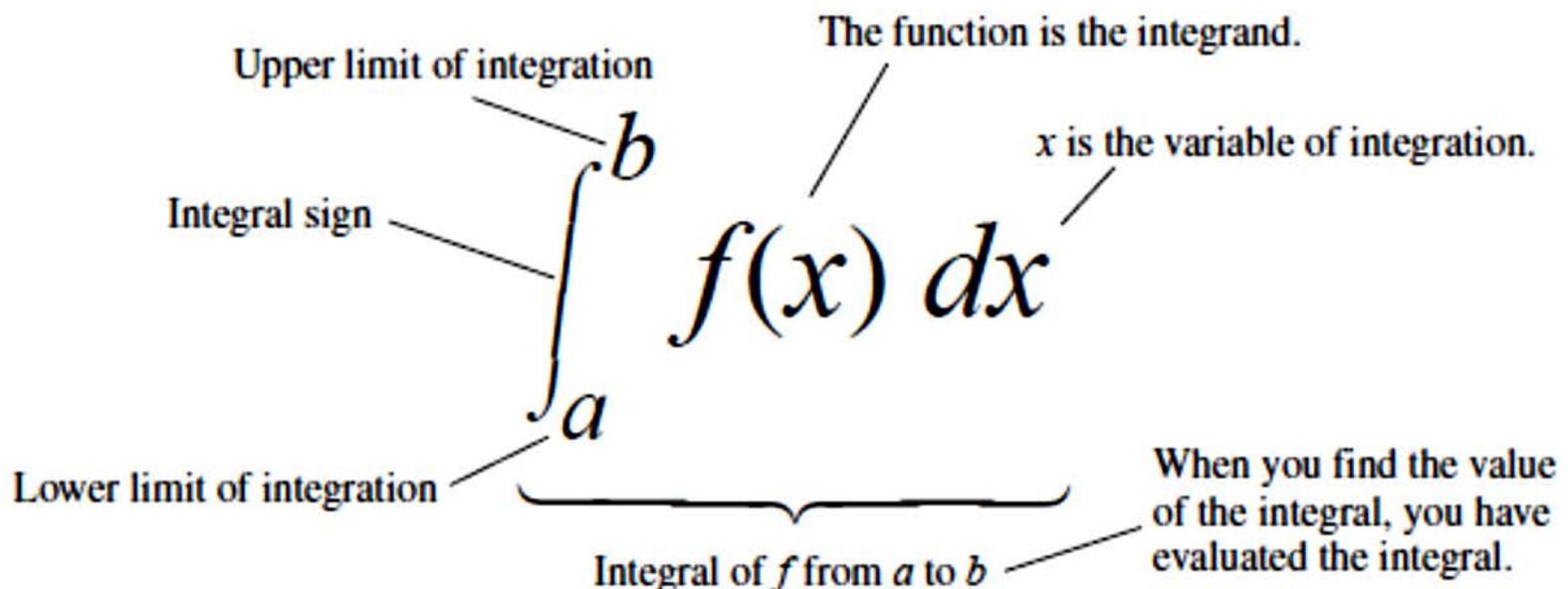
$$= -\frac{1}{4} (1-4x^2)^{\frac{1}{2}} + c$$

Notation and Existence of the Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x) dx$$

which is read as “the integral from a to b of f of x dee x ” or sometimes as “the integral from a to b of f of x with respect to x .” The component parts in the integral symbol also have names:



1. *Order of Integration:* $\int_b^a f(x) dx = - \int_a^b f(x) dx$ A Definition
2. *Zero Width Interval:* $\int_a^a f(x) dx = 0$ Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ Any Number k
 $\int_a^b -f(x) dx = - \int_a^b f(x) dx$ $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
5. *Additivity:* $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Example 1 Evaluate each of the following.

(a) $\int y^2 + y^{-2} dy$

(b) $\int_1^2 y^2 + y^{-2} dy$

$$\int y^2 + y^{-2} dy = \frac{1}{3}y^3 - y^{-1} + c$$

$$\begin{aligned}\int_1^2 y^2 + y^{-2} dy &= \left(\frac{1}{3}y^3 - \frac{1}{y} \right) \Big|_1^2 \\ &= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1} \right) \\ &= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\ &= \frac{17}{6}\end{aligned}$$

Example 2 Evaluate each of the following.

(a) $\int_{-3}^1 6x^2 - 5x + 2 \, dx$

(b) $\int_4^0 \sqrt{t}(t-2) \, dt$

(c) $\int_1^2 \frac{2w^5 - w + 3}{w^2} \, dw$

(d) $\int_{25}^{-10} dR$

Solution

(a) $\int_{-3}^1 6x^2 - 5x + 2 \, dx$

There isn't a lot to this one other than simply doing the work.

$$\int_{-3}^1 6x^2 - 5x + 2 \, dx = \left(2x^3 - \frac{5}{2}x^2 + 2x \right) \Big|_{-3}^1$$

$$= \left(2 - \frac{5}{2} + 2 \right) - \left(-54 - \frac{45}{2} - 6 \right)$$

$$= 84$$

(b) $\int_4^0 \sqrt{t}(t-2) \, dt$

$$\int_4^0 \sqrt{t}(t-2) \, dt = \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}} \, dt$$

$$= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} \right) \Big|_4^0$$

$$= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}} \right)$$

$$= -\frac{32}{15}$$

(c) $\int_1^2 \frac{2w^5 - w + 3}{w^2} dw$

$$\begin{aligned}\int_1^2 \frac{2w^5 - w + 3}{w^2} dw &= \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2} dw \\&= \left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w} \right) \Big|_1^2 \\&= \left(8 - \ln 2 - \frac{3}{2} \right) - \left(\frac{1}{2} - \ln 1 - 3 \right) \\&= 9 - \ln 2\end{aligned}$$

(d) $\int_{25}^{-10} dR$

$$\begin{aligned}\int_{25}^{-10} dR &= R \Big|_{25}^{-10} \\&= -10 - 25 \\&= -35\end{aligned}$$

Example 3 Evaluate each of the following.

(a) $\int_0^1 4x - 6\sqrt[3]{x^2} dx$

(b) $\int_0^{\frac{\pi}{3}} 2 \sin \theta - 5 \cos \theta d\theta$

(c) $\int_{\pi/6}^{\pi/4} 5 - 2 \sec z \tan z dz$

(d) $\int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} dz$

$$\int_0^1 4x - 6\sqrt[3]{x^2} dx = \int_0^1 4x - 6x^{\frac{2}{3}} dx$$

$$= \left(2x^2 - \frac{18}{5}x^{\frac{5}{3}} \right) \Big|_0^1$$

$$= 2 - \frac{18}{5} - (0)$$

$$= -\frac{8}{5}$$

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2 \sin \theta - 5 \cos \theta d\theta &= (-2 \cos \theta - 5 \sin \theta) \Big|_0^{\pi/3} \\&= -2 \cos\left(\frac{\pi}{3}\right) - 5 \sin\left(\frac{\pi}{3}\right) - (-2 \cos 0 - 5 \sin 0) \\&= -1 - \frac{5\sqrt{3}}{2} + 2 \\&= 1 - \frac{5\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}
\int_{\pi/6}^{\pi/4} 5 - 2 \sec z \tan z \, dz &= (5z - 2 \sec z) \Big|_{\pi/6}^{\pi/4} \\
&= 5\left(\frac{\pi}{4}\right) - 2 \sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2 \sec\left(\frac{\pi}{6}\right)\right) \\
&= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
\int_{-20}^{-1} \frac{3}{e^{-z}} - \frac{1}{3z} \, dz &= \left(3e^z - \frac{1}{3} \ln|z| \right) \Big|_{-20}^{-1} \\
&= 3e^{-1} - \frac{1}{3} \ln|-1| - \left(3e^{-20} - \frac{1}{3} \ln|-20| \right) \\
&= 3e^{-1} - 3e^{-20} + \frac{1}{3} \ln|20|
\end{aligned}$$

Using the Rules for Definite Integrals

Suppose that

$$\int_{-1}^1 f(x) \, dx = 5, \quad \int_1^4 f(x) \, dx = -2, \quad \int_{-1}^1 h(x) \, dx = 7.$$

Then

$$1. \quad \int_4^1 f(x) \, dx = - \int_1^4 f(x) \, dx = -(-2) = 2 \qquad \text{Rule 1}$$

$$2. \quad \begin{aligned} \int_{-1}^1 [2f(x) + 3h(x)] \, dx &= 2 \int_{-1}^1 f(x) \, dx + 3 \int_{-1}^1 h(x) \, dx \\ &= 2(5) + 3(7) = 31 \end{aligned} \qquad \text{Rules 3 and 4}$$

$$3. \quad \int_{-1}^4 f(x) \, dx = \int_{-1}^1 f(x) \, dx + \int_1^4 f(x) \, dx = 5 + (-2) = 3 \qquad \text{Rule 5}$$

Example

(a) $\int_2^0 x^2 + 1 dx$

(b) $\int_0^2 10x^2 + 10 dx$

(c) $\int_0^2 t^2 + 1 dt$

$$\int_0^2 x^2 + 1 dx = \left(\frac{1}{3}x^3 + x \right) \Big|_0^2$$

$$= \frac{1}{3}(2)^3 + 2 - \left(\frac{1}{3}(0)^3 + 0 \right)$$

$$= \frac{14}{3}$$

$$\int_2^0 x^2 + 1 dx = - \int_0^2 x^2 + 1 dx$$

$$= -\frac{14}{3}$$

$$\int_0^2 10x^2 + 10 dx = \int_0^2 10(x^2 + 1) dx$$

$$= 10 \int_0^2 x^2 + 1 dx$$

$$= 10 \left(\frac{14}{3} \right)$$

$$= \frac{140}{3}$$

$$\int_0^2 t^2 + 1 dt = \int_0^2 x^2 + 1 dx = \frac{14}{3}$$

Example 3 Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x \sin(x) + \cos(x)}{x^2 + 1} dx$$

$$\int_{130}^{130} \frac{x^3 - x \sin(x) + \cos(x)}{x^2 + 1} dx = 0$$

Example 4 Given that $\int_6^{-10} f(x) dx = 23$ and $\int_{-10}^6 g(x) dx = -9$ determine the value of

$$\int_{-10}^6 2f(x) - 10g(x) dx$$

$$\int_{-10}^6 2f(x) - 10g(x) dx = \int_{-10}^6 2f(x) dx - \int_{-10}^6 10g(x) dx$$

$$= 2 \int_{-10}^6 f(x) dx - 10 \int_{-10}^6 g(x) dx$$

$$\int_{-10}^6 2f(x) - 10g(x) dx = -2 \int_6^{-10} f(x) dx - 10 \int_{-10}^6 g(x) dx$$

$$= -2(23) - 10(-9)$$

$$= 44$$

Substitution Rule for Definite Integrals

Example 1 Evaluate the following definite integral.

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt$$

In this case the substitution is,

$$u = 1 - 4t^3 \quad du = -12t^2 dt \quad \Rightarrow \quad t^2 dt = -\frac{1}{12} du$$

Plugging this into the integral gives,

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt = -\frac{1}{6} \int_{-2}^0 u^{\frac{1}{2}} du$$

$$= -\frac{1}{9} u^{\frac{3}{2}} \Big|_{-2}^0$$

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt = -\frac{1}{9} (1-4t^3)^{\frac{3}{2}} \Big|_{-2}^0$$

$$= -\frac{1}{9} \left(-\frac{1}{9} (33)^{\frac{3}{2}} \right)$$

$$= \frac{1}{9} (33\sqrt{33} - 1)$$

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt$$

$$u = 1 - 4t^3 \quad du = -12t^2 dt \quad \Rightarrow \quad t^2 dt = -\frac{1}{12} du$$

$$t = -2 \quad \Rightarrow \quad u = 1 - 4(-2)^3 = 33$$

$$t = 0 \quad \Rightarrow \quad u = 1 - 4(0)^3 = 1$$

The integral is now,

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt = -\frac{1}{6} \int_{33}^1 u^{\frac{1}{2}} du$$

$$= -\frac{1}{9} u^{\frac{3}{2}} \Big|_{33}^1$$

$$\int_{-2}^0 2t^2 \sqrt{1-4t^3} dt = -\frac{1}{9} u^{\frac{3}{2}} \Big|_{33}^1$$

$$= -\frac{1}{9} - \left(-\frac{1}{9} (33)^{\frac{3}{2}} \right) = \frac{1}{9} (33\sqrt{33} - 1)$$

Example 2 Evaluate each of the following.

(a) $\int_{-1}^5 (1+w)(2w+w^2)^5 dw$

(b) $\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x} dx$

(a) $\int_{-1}^5 (1+w)(2w+w^2)^5 dw$

The substitution and converted limits are,

$$\begin{aligned} u &= 2w + w^2 & du &= (2+2w)dw & \Rightarrow & (1+w)dw = \frac{1}{2}du \\ w = -1 & \Rightarrow u = -1 & & & w = 5 & \Rightarrow u = 35 \end{aligned}$$

$$\begin{aligned} \int_{-1}^5 (1+w)(2w+w^2)^5 dw &= \frac{1}{2} \int_{-1}^{35} u^5 du \\ &= \frac{1}{12} u^6 \Big|_{-1}^{35} = 153188802 \end{aligned}$$

$$(b) \int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x} dx$$

Here is the substitution and converted limits for this problem,

$$u = 1 + 2x \quad du = 2dx \quad \Rightarrow \quad dx = \frac{1}{2}du$$

$$x = -2 \quad \Rightarrow \quad u = -3 \quad x = -6 \quad \Rightarrow \quad u = -11$$

The integral is then,

$$\begin{aligned}\int_{-2}^{-6} \frac{4}{(1+2x)^3} - \frac{5}{1+2x} dx &= \frac{1}{2} \int_{-3}^{-11} 4u^{-3} - \frac{5}{u} du \\&= \frac{1}{2} \left(-2u^{-2} - 5 \ln|u| \right) \Big|_{-3}^{-11} \\&= \frac{1}{2} \left(-\frac{2}{121} - 5 \ln 11 \right) - \frac{1}{2} \left(-\frac{2}{9} - 5 \ln 3 \right) \\&= \frac{112}{1089} - \frac{5}{2} \ln 11 + \frac{5}{2} \ln 3\end{aligned}$$

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$

2. $\int k \, du = ku + C \quad (\text{any number } k)$

3. $\int (du + dv) = \int du + \int dv$

4. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$

5. $\int \frac{du}{u} = \ln |u| + C$

6. $\int \sin u \, du = -\cos u + C$

7. $\int \cos u \, du = \sin u + C$

8. $\int \sec^2 u \, du = \tan u + C$

9. $\int \csc^2 u \, du = -\cot u + C$

10. $\int \sec u \tan u \, du = \sec u + C$

11. $\int \csc u \cot u \, du = -\csc u + C$

12. $\int \tan u \, du = -\ln |\cos u| + C$
 $= \ln |\sec u| + C$

13. $\int \cot u \, du = \ln |\sin u| + C$

$= -\ln |\csc u| + C$

14. $\int e^u \, du = e^u + C$

15. $\int a^u \, du = \frac{a^u}{\ln a} + C \quad (a > 0, a \neq 1)$

16. $\int \sinh u \, du = \cosh u + C$

17. $\int \cosh u \, du = \sinh u + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$

20. $\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C$

21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C \quad (a > 0)$

22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left(\frac{u}{a} \right) + C \quad (u > a > 0)$

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx$$

$$\begin{aligned}\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx &= \int \frac{du}{\sqrt{u}} \\&= \int u^{-1/2} du \\&= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C \\&= 2u^{1/2} + C \\&= 2\sqrt{x^2 - 9x + 1} + C\end{aligned}$$

$$\begin{aligned}u &= x^2 - 9x + 1, \\du &= (2x - 9) dx.\end{aligned}$$

Table 8.1 Formula 4,
with $n = -1/2$



EXAMPLE 2 Completing the Square

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

We complete the square to simplify the denominator:

$$\begin{aligned}8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\&= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\&= \int \frac{du}{\sqrt{a^2 - u^2}} \quad a = 4, u = (x - 4), \\&\quad du = dx \\&= \sin^{-1} \left(\frac{u}{a} \right) + C \\&= \sin^{-1} \left(\frac{x - 4}{4} \right) + C.\end{aligned}$$

Table 8.1, Formula 18

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$$

■

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2 dx.$$

$$\tan^2 x + 1 = \sec^2 x, \quad \tan^2 x = \sec^2 x - 1.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2 dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) dx \\&= 2 \int \sec^2 x dx + 2 \int \sec x \tan x dx - \int 1 dx \\&= 2 \tan x + 2 \sec x - x + C.\end{aligned}$$

EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} dx \\&= \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx \quad \sqrt{u^2} = |u| \\&= \sqrt{2} \int_0^{\pi/4} \cos 2x dx \quad \text{On } [0, \pi/4], \cos 2x \geq 0, \\&\quad \text{so } |\cos 2x| = \cos 2x. \\&= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} \quad \text{Table 8.1, Formula 7, with} \\&= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}. \quad u = 2x \text{ and } du = 2 dx\end{aligned}$$

■

$$\begin{array}{r}
 x - 3 \\
 3x + 2) \overline{)3x^2 - 7x} \\
 3x^2 + 2x \\
 \hline
 -9x \\
 \hline
 -9x - 6 \\
 \hline
 + 6
 \end{array}$$

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2} dx.$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \blacksquare$$

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = 3 \int \frac{x dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \quad du = -2x dx, \quad \text{and} \quad x dx = -\frac{1}{2} du.$$

$$\begin{aligned} 3 \int \frac{x dx}{\sqrt{1 - x^2}} &= 3 \int \frac{(-1/2) du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2} du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{u} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.$$

■

EXAMPLE 7 Integral of $y = \sec x$ —Multiplying by a Form of 1

Evaluate

$$\int \sec x \, dx.$$

Solution

$$\begin{aligned}\int \sec x \, dx &= \int (\sec x)(1) \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\&= \int \frac{du}{u} \\&= \ln |u| + C = \ln |\sec x + \tan x| + C.\end{aligned}$$

$$\begin{aligned}u &= \tan x + \sec x, \\du &= (\sec^2 x + \sec x \tan x) \, dx\end{aligned}$$

■

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
2. $\int \csc u \, du = -\ln |\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE

Making a simplifying substitution

Completing the square

Using a trigonometric identity

Eliminating a square root

Reducing an improper fraction

Separating a fraction

Multiplying by a form of 1

EXAMPLE

$$\frac{2x - 9}{\sqrt{x^2 - 9x + 1}} \, dx = \frac{du}{\sqrt{u}}$$

$$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$$

$$\begin{aligned} (\sec x + \tan x)^2 &= \sec^2 x + 2 \sec x \tan x + \tan^2 x \\ &= \sec^2 x + 2 \sec x \tan x \\ &\quad + (\sec^2 x - 1) \\ &= 2 \sec^2 x + 2 \sec x \tan x - 1 \end{aligned}$$

$$\sqrt{1 + \cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2} |\cos 2x|$$

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}$$

$$\frac{3x + 2}{\sqrt{1 - x^2}} = \frac{3x}{\sqrt{1 - x^2}} + \frac{2}{\sqrt{1 - x^2}}$$

$$\begin{aligned} \sec x &= \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \\ &= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \end{aligned}$$

Product Rule in Integral Form

If f and g are differentiable functions of x , the Product Rule says

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx} [f(x)g(x)] dx = \int [f'(x)g(x) + f(x)g'(x)] dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \quad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x) dx$ and $dv = g'(x) dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u dv = uv - \int v du \quad (2)$$

EXAMPLE 1 Using Integration by Parts

Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$\begin{aligned} u &= x, & dv &= \cos x \, dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$

Simplest antiderivative of $\cos x$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$



EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = \ln x \quad \text{Simplifies when differentiated}$$

$$dv = dx \quad \text{Easy to integrate}$$

$$du = \frac{1}{x} dx,$$

$$v = x. \quad \text{Simplest antiderivative}$$

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C. \quad \blacksquare$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x dx.$$

Solution With $u = x^2$, $dv = e^x dx$, $du = 2x dx$, and $v = e^x$, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\&= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$



Average Function Value

The average value of a function $f(x)$ over the interval $[a, b]$ is given by,

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

Example 1 Determine the average value of each of the following functions on the given interval.

$$f(t) = t^2 - 5t + 6 \cos(\pi t) \text{ on } \left[-1, \frac{5}{2}\right]$$

There's really not a whole lot to do in this problem other than just use the formula.

$$f_{avg} = \frac{1}{\frac{5}{2} - (-1)} \int_{-1}^{\frac{5}{2}} t^2 - 5t + 6 \cos(\pi t) dt$$

$$= \frac{2}{7} \left(\frac{1}{3} t^3 - \frac{5}{2} t^2 + \frac{6}{\pi} \sin(\pi t) \right) \Big|_{-1}^{\frac{5}{2}}$$

$$= \frac{12}{7\pi} - \frac{13}{6}$$

$$= -1.620993$$

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x) dx = f(c)(b-a)$$

Note that one way to think of this theorem is the following. First rewrite the result as,

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

and from this we can see that this theorem is telling us that there is a number $a < c < b$ such that $f_{avg} = f(c)$. Or, in other words, if $f(x)$ is a continuous function then somewhere in $[a, b]$ the function will take on its average value.

Example 2 Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

Solution

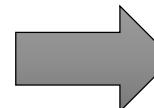
First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. So, let's do that.

$$\int_1^4 x^2 + 3x + 2 \, dx = (c^2 + 3c + 2)(4 - 1)$$

$$\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x \right) \Big|_1^4 = 3(c^2 + 3c + 2)$$

$$\frac{99}{2} = 3c^2 + 9c + 6$$

$$0 = 3c^2 + 9c - \frac{87}{2}$$



$$\frac{1}{b-a} \int_a^b f(x) \, dx = f(c)$$

$$c = \frac{-3 + \sqrt{67}}{2} = 2.593$$

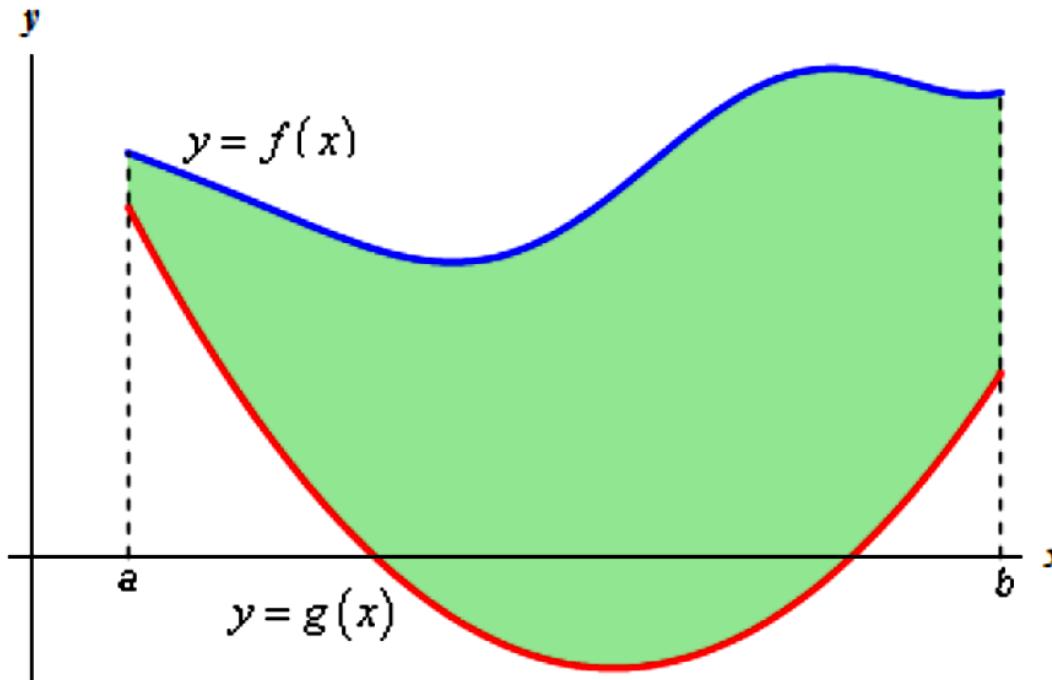
$$c = \frac{-3 - \sqrt{67}}{2} = -5.593$$

The Quadratic Formula If $a \neq 0$ and $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Area Between Curves

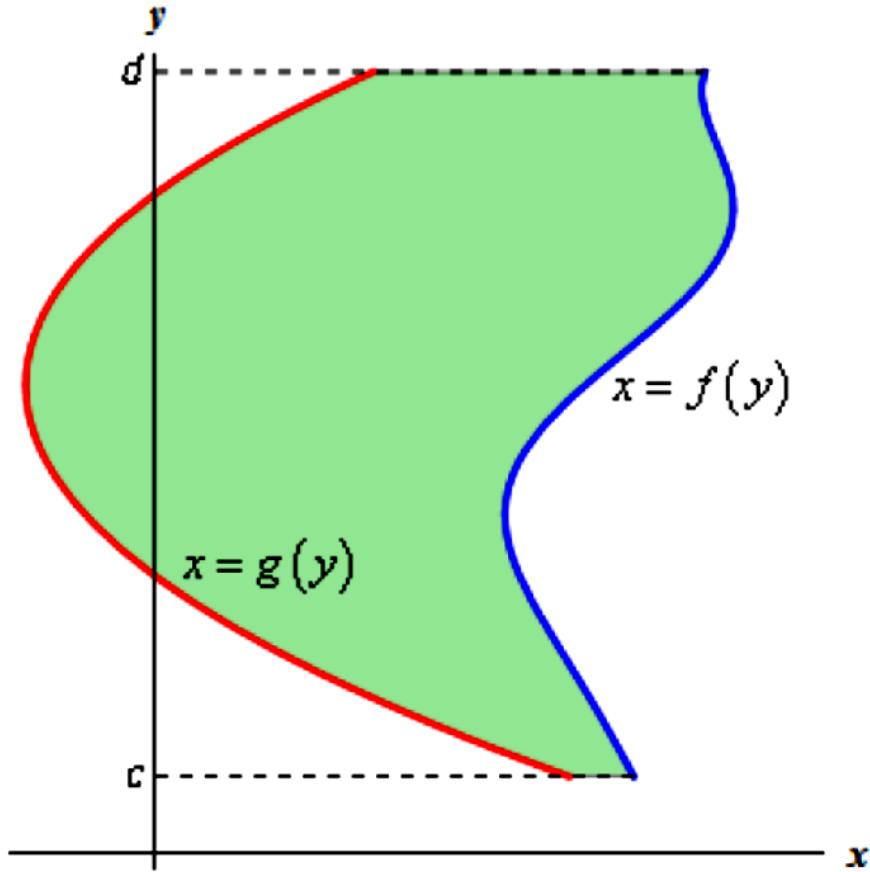
In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



In the [Area and Volume Formulas](#) section of the Extras chapter we derived the following formula for the area in this case.

$$A = \int_a^b f(x) - g(x) dx \quad (3)$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



In this case the formula is,

$$A = \int_c^d f(y) - g(y) dy \quad (4)$$

In the first case we will use,

$$A = \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) dx, \quad a \leq x \leq b \quad (5)$$

In the second case we will use,

$$A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy, \quad c \leq y \leq d \quad (6)$$

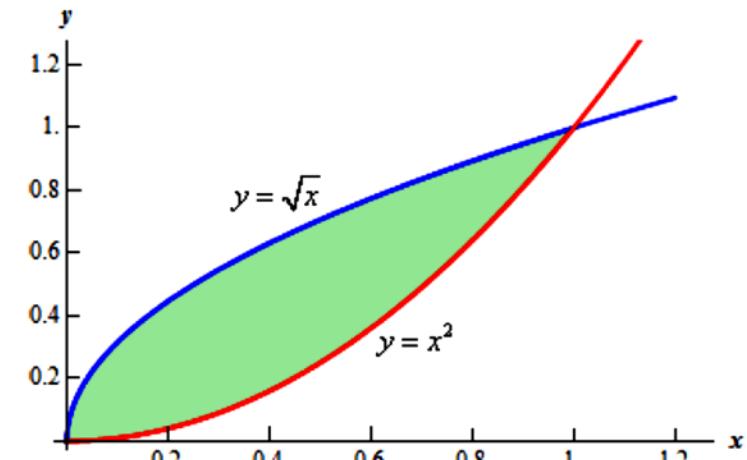
Example 1 Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

$$A = \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) dx$$

$$= \int_0^1 \sqrt{x} - x^2 dx$$

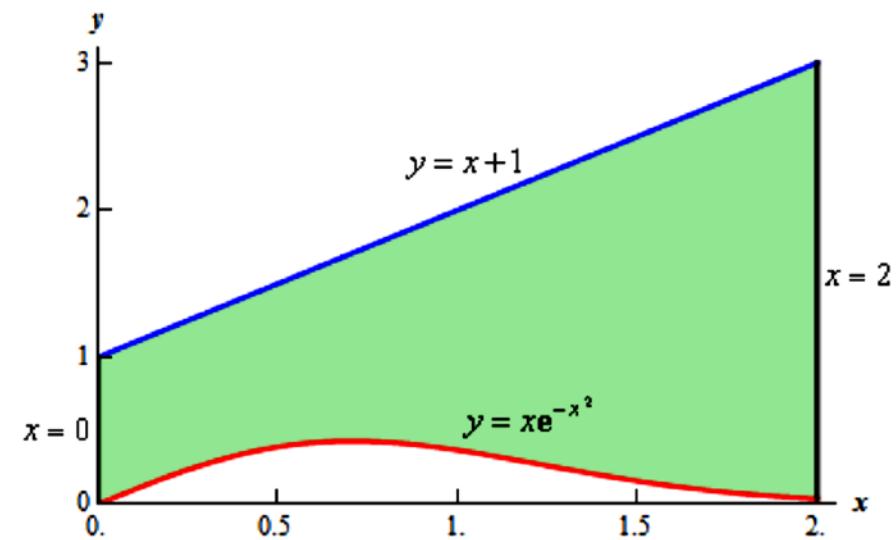
$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right) \Big|_0^1$$

$$= \frac{1}{3}$$



Example 2 Determine the area of the region bounded by $y = xe^{-x^2}$, $y = x + 1$, $x = 2$, and the y -axis.

$$\begin{aligned} A &= \int_a^b \left(\begin{array}{l} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{l} \text{lower} \\ \text{function} \end{array} \right) dx \\ &= \int_0^2 x + 1 - xe^{-x^2} dx \\ &= \left(\frac{1}{2}x^2 + x + \frac{1}{2}e^{-x^2} \right) \Big|_0^2 \\ &= \frac{7}{2} + \frac{e^{-4}}{2} = 3.5092 \end{aligned}$$



Example 3 Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.

$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0$$

$$2(x+1)(x-3) = 0$$

intersect at $x = -1$ and $x = 3$

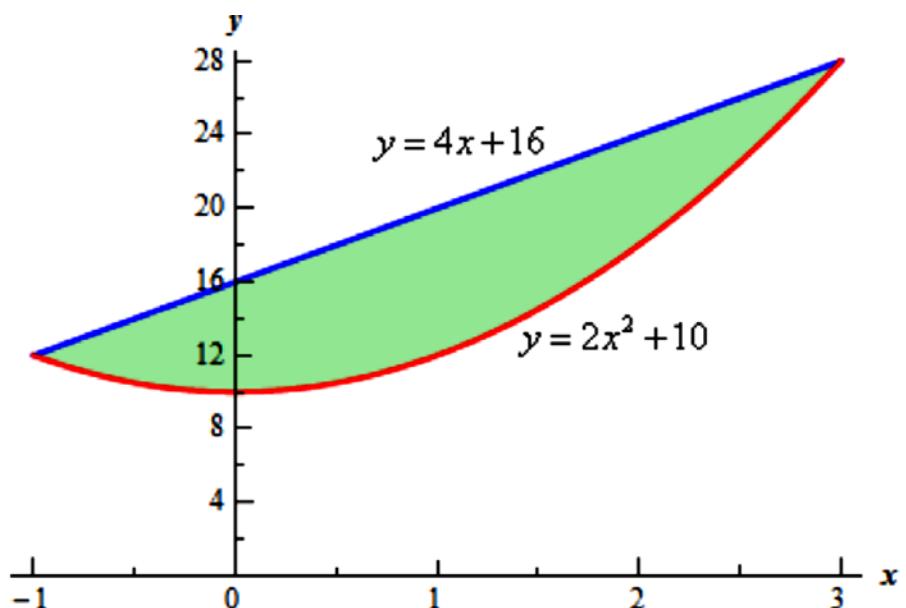
$$A = \int_a^b \left(\begin{array}{l} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{l} \text{lower} \\ \text{function} \end{array} \right) dx$$

$$= \int_{-1}^3 4x + 16 - (2x^2 + 10) dx$$

$$= \int_{-1}^3 -2x^2 + 4x + 6 dx$$

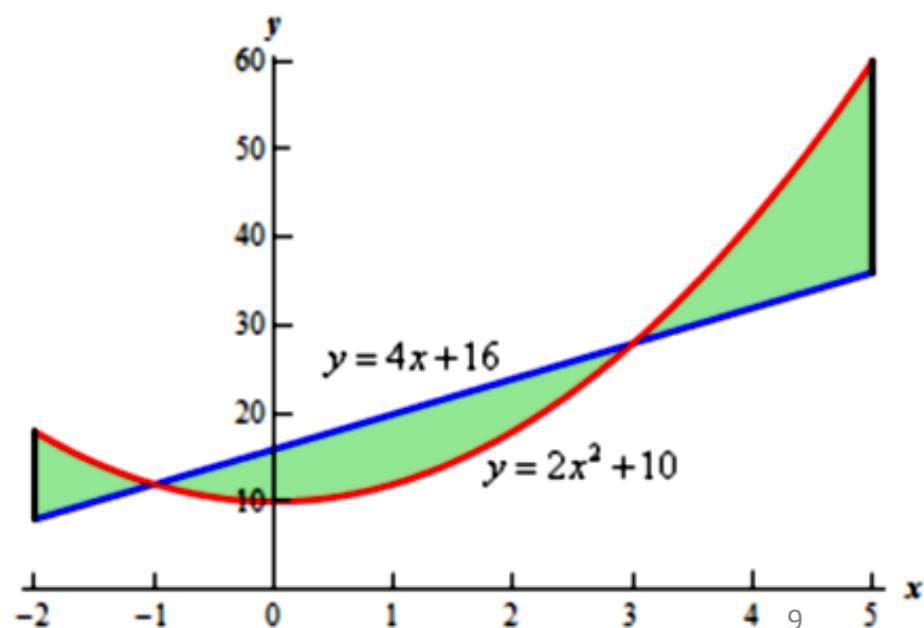
$$= \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \Big|_{-1}^3$$

$$= \frac{64}{3}$$



Example 4 Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

$$\begin{aligned}
 A &= \int_{-2}^{-1} 2x^2 + 10 - (4x + 16) dx + \int_{-1}^3 4x + 16 - (2x^2 + 10) dx + \int_3^5 2x^2 + 10 - (4x + 16) dx \\
 &= \int_{-2}^{-1} 2x^2 - 4x - 6 dx + \int_{-1}^3 -2x^2 + 4x + 6 dx + \int_3^5 2x^2 - 4x - 6 dx \\
 &= \left(\frac{2}{3}x^3 - 2x^2 - 6x \right) \Big|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \Big|_{-1}^3 + \left(\frac{2}{3}x^3 - 2x^2 - 6x \right) \Big|_3^5 \\
 &= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} \\
 &= \frac{142}{3}
 \end{aligned}$$



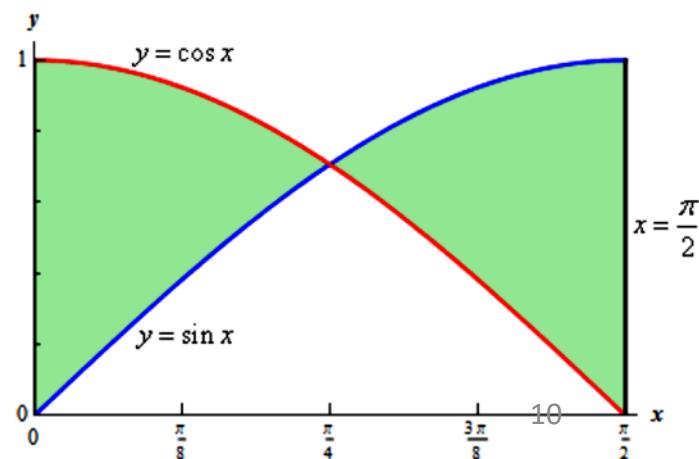
Example 5 Determine the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and the y -axis.

So, we have another situation where we will need to do two integrals to get the area. The intersection point will be where

$$\sin x = \cos x$$

in the interval. We'll leave it to you to verify that this will be $x = \frac{\pi}{4}$. The area is then,

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx \\ &= (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \sqrt{2} - 1 + (\sqrt{2} - 1) \\ &= 2\sqrt{2} - 2 = 0.828427 \end{aligned}$$



Example 6 Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

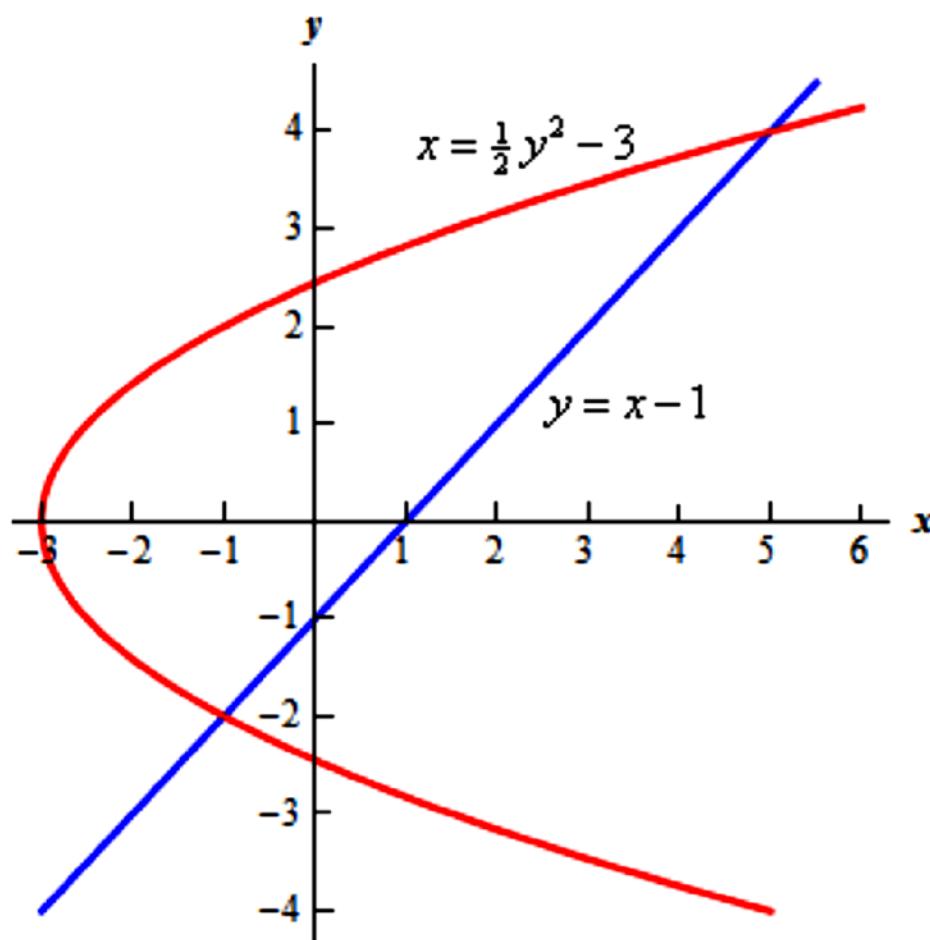
$$y + 1 = \frac{1}{2}y^2 - 3$$

$$2y + 2 = y^2 - 6$$

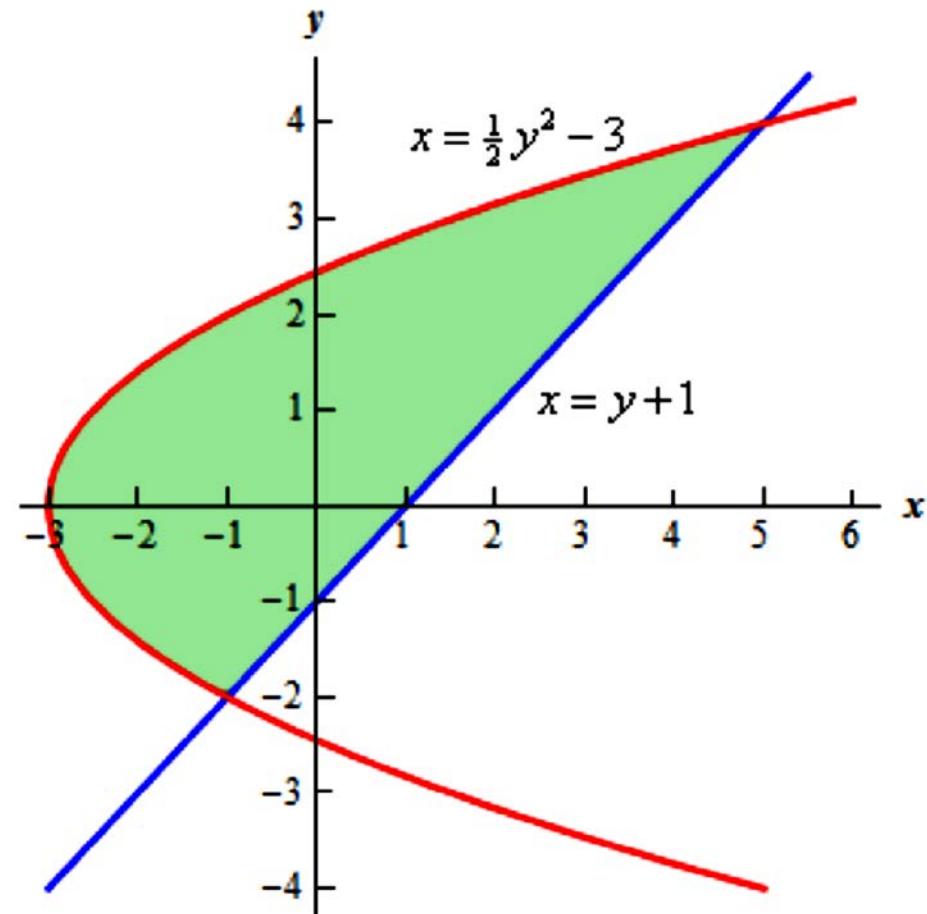
$$0 = y^2 - 2y - 8$$

$$0 = (y - 4)(y + 2)$$

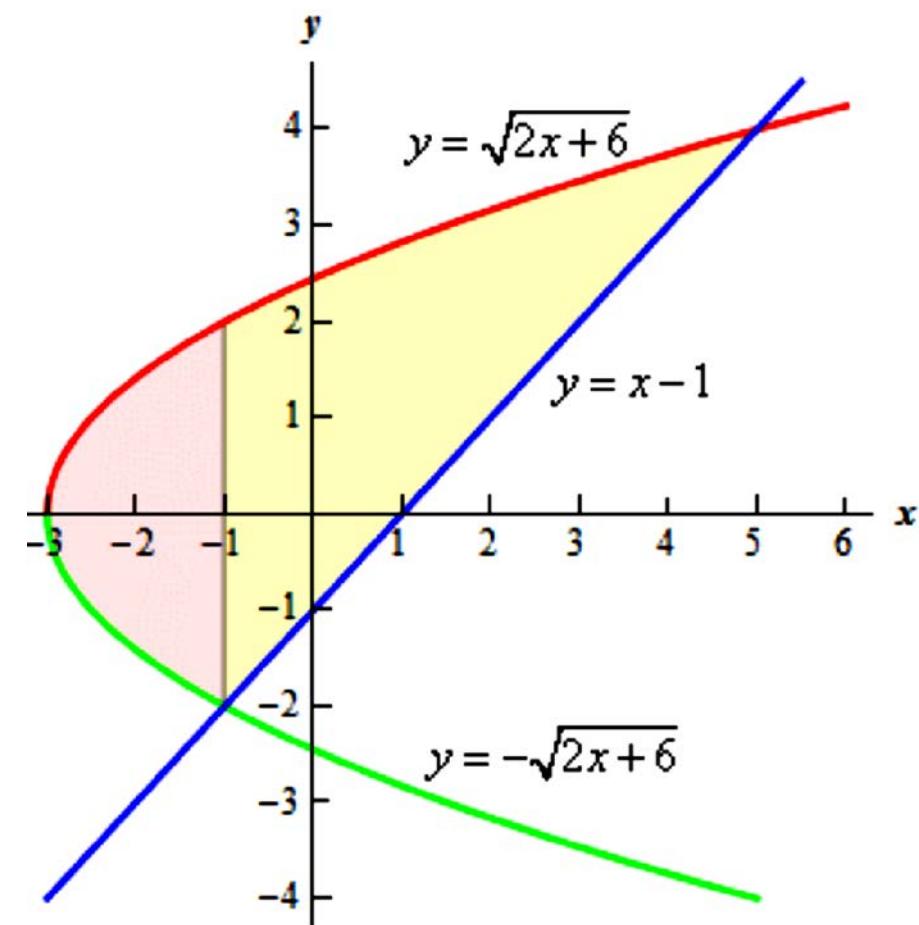
So, it looks like the two curves will intersect at $y = -2$ and $y = 4$ or if we need the full coordinates they will be : $(-1, -2)$ and $(5, 4)$.



$$\begin{aligned}
 A &= \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy \\
 &= \int_{-2}^4 (y+1) - \left(\frac{1}{2}y^2 - 3 \right) dy \\
 &= \int_{-2}^4 -\frac{1}{2}y^2 + y + 4 dy \\
 &= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y \right) \Big|_{-2}^4 \\
 &= 18
 \end{aligned}$$



$$\begin{aligned}
 A &= \int_{-3}^{-1} \sqrt{2x+6} - (-\sqrt{2x+6}) dx + \int_{-1}^5 \sqrt{2x+6} - (x-1) dx \\
 &= \int_{-3}^{-1} 2\sqrt{2x+6} dx + \int_{-1}^5 \sqrt{2x+6} - x + 1 dx \\
 &= \int_{-3}^{-1} 2\sqrt{2x+6} dx + \int_{-1}^5 \sqrt{2x+6} dx + \int_{-1}^5 -x + 1 dx \\
 &= \frac{2}{3} u^{\frac{3}{2}} \Big|_0^4 + \frac{1}{3} u^{\frac{3}{2}} \Big|_4^{16} + \left(-\frac{1}{2} x^2 + x \right) \Big|_{-1}^5 \\
 &= 18
 \end{aligned}$$



Example 7 Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y - 2)^2$.

$$-y^2 + 10 = (y - 2)^2$$

$$-y^2 + 10 = y^2 - 4y + 4$$

$$0 = 2y^2 - 4y - 6$$

$$0 = 2(y+1)(y-3)$$

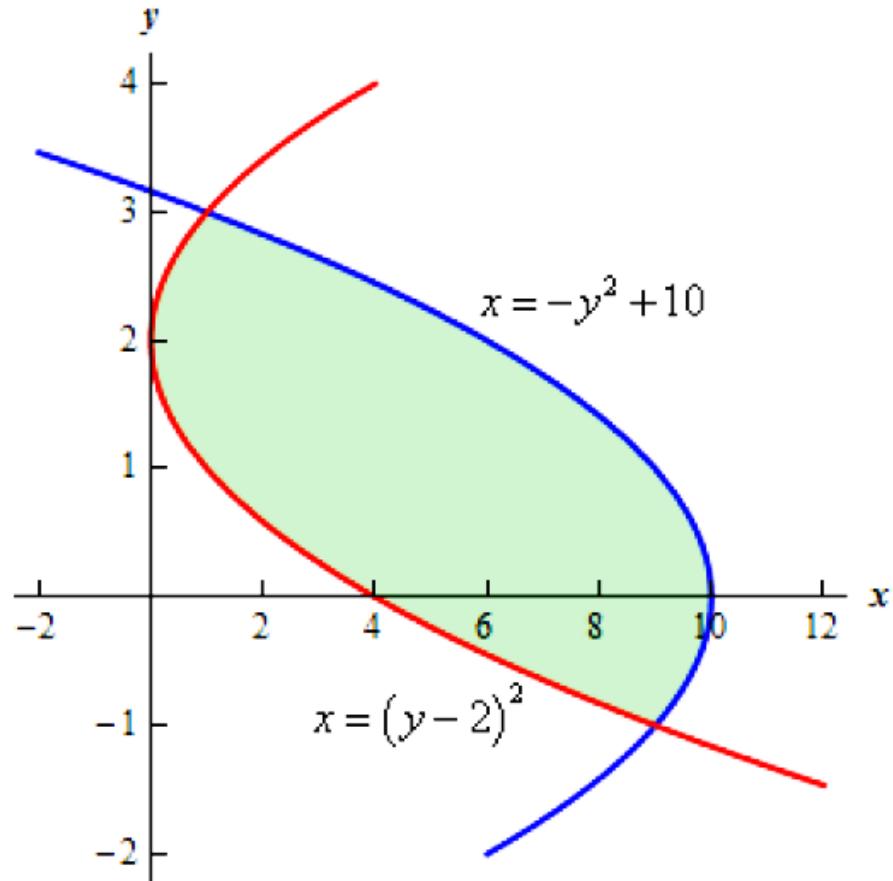
$$A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy$$

$$= \int_{-1}^3 -y^2 + 10 - (y - 2)^2 dy$$

$$= \int_{-1}^3 -2y^2 + 4y + 6 dy$$

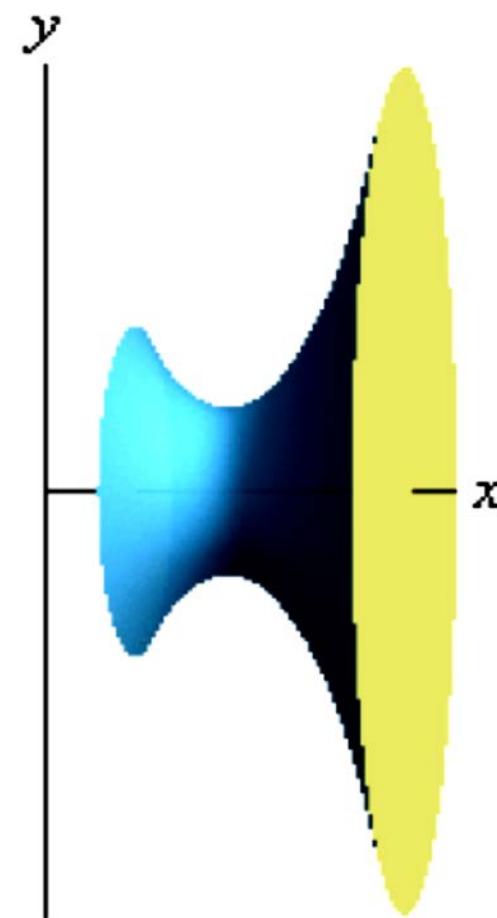
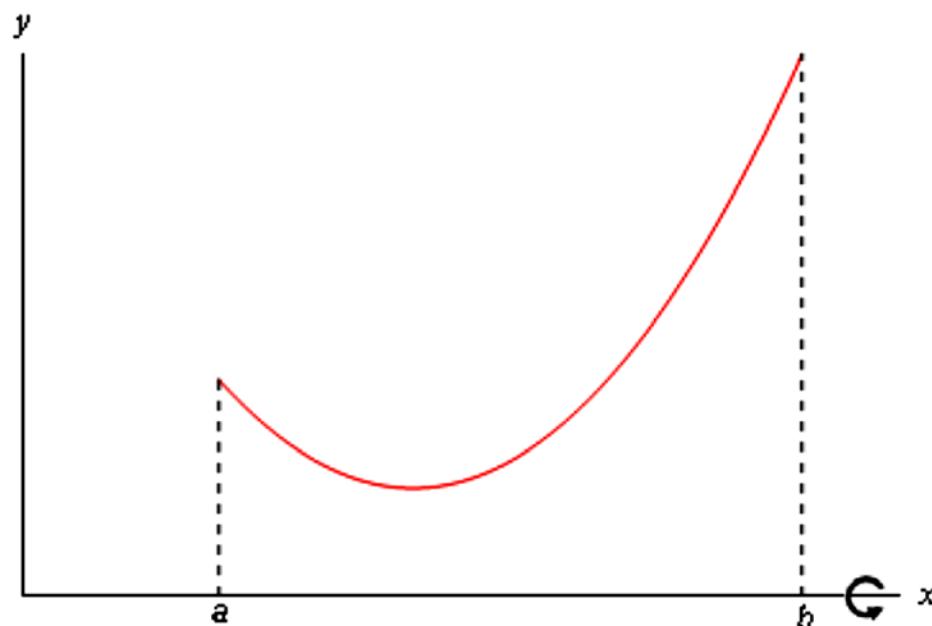
$$= \left[-\frac{2}{3}y^3 + 2y^2 + 6y \right]_{-1}^3 = \frac{64}{3}$$

The intersection points are $y = -1$ and $y = 3$. Here is a sketch of the region.



Volumes of Solids of Revolution / Method of Rings

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval $[a, b]$.



One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).

In the case that we get a solid disk the area is,

$$A = \pi (\text{radius})^2$$

where the radius will depend upon the function and the axis of rotation.

In the case that we get a ring the area is,

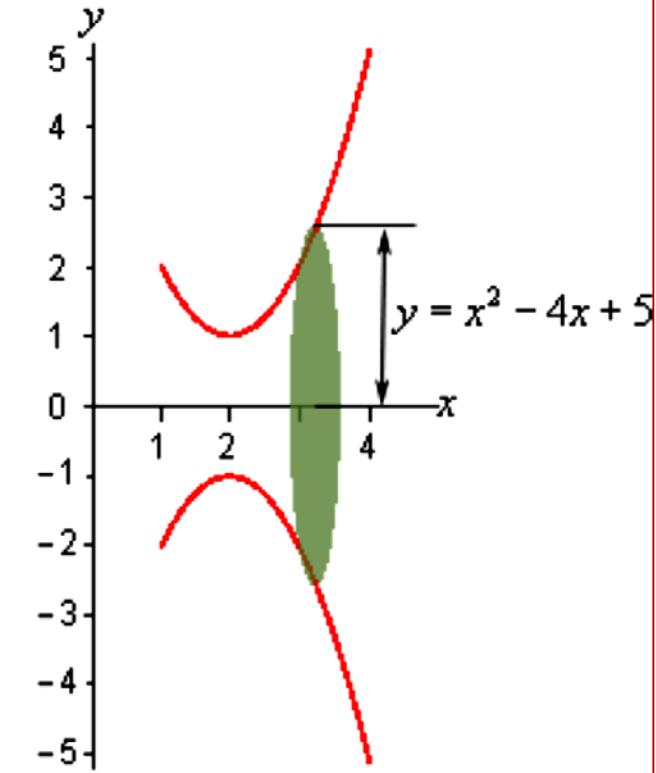
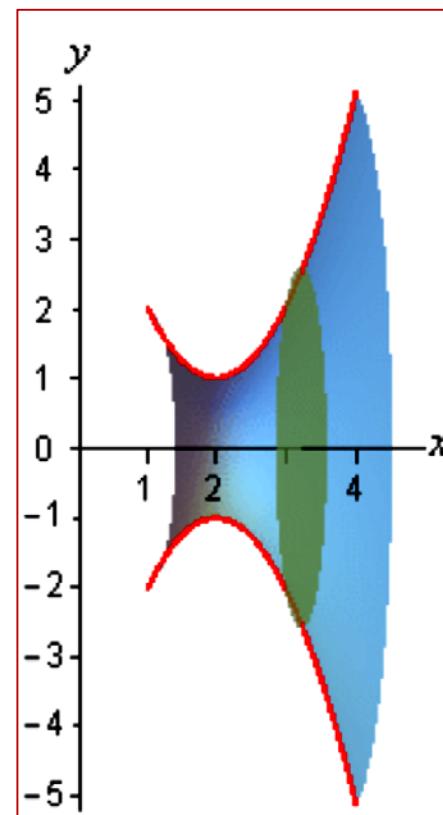
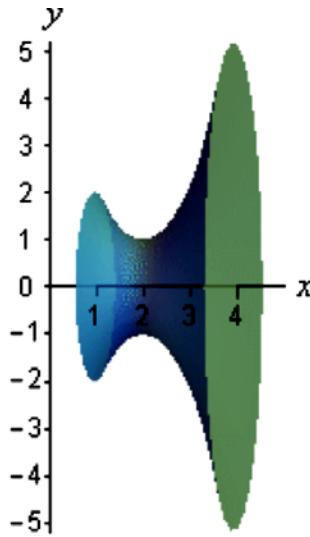
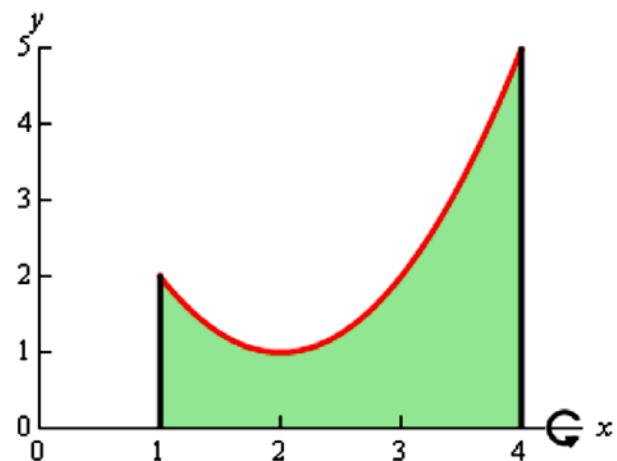
$$A = \pi \left(\left(\frac{\text{outer radius}}{\text{inner radius}} \right)^2 - \left(\frac{\text{inner radius}}{\text{outer radius}} \right)^2 \right)$$

This method is often called the **method of disks** or the **method of rings**.

$$V = \int_a^b A(x) dx$$

$$V = \int_c^d A(y) dy$$

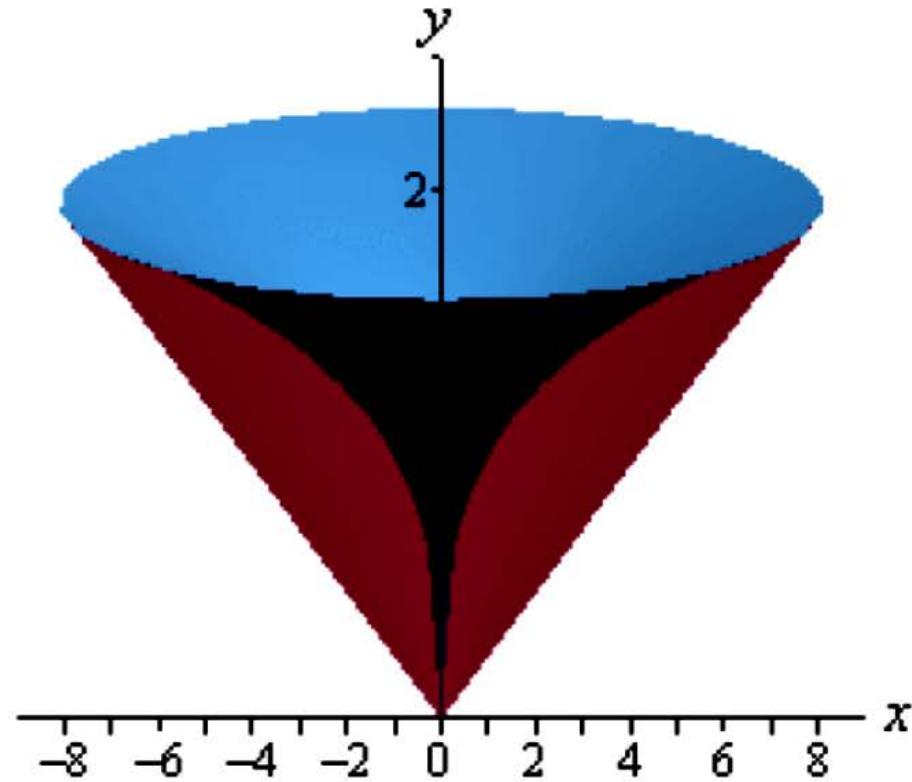
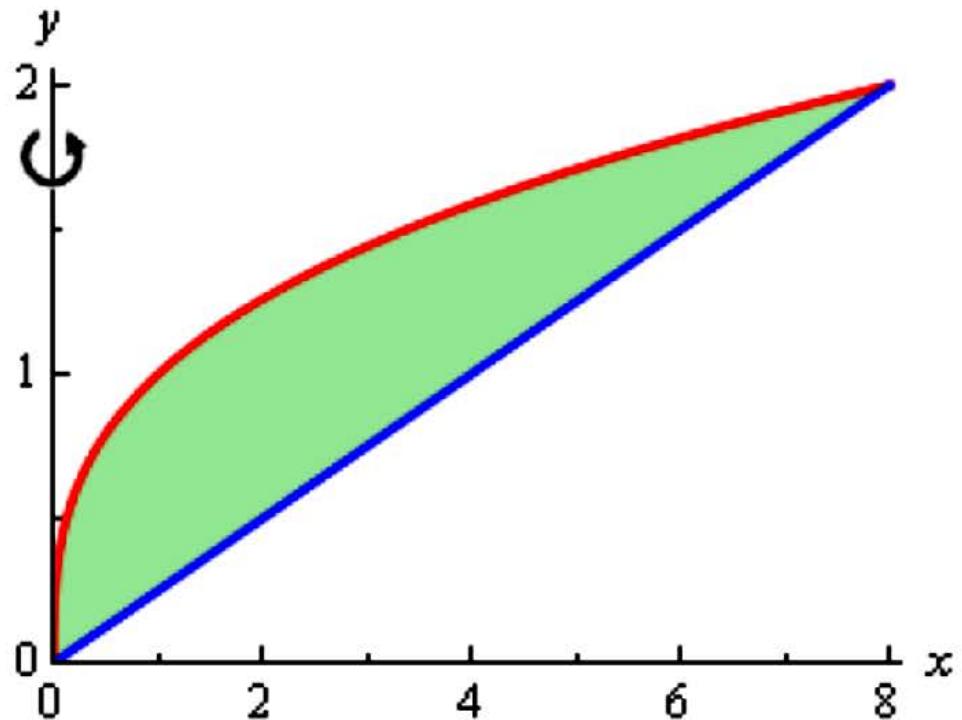
Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x -axis about the x -axis.



$$A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$$

$$\begin{aligned}V &= \int_a^b A(x) dx \\&= \pi \int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25 dx \\&= \pi \left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x \right) \Big|_1^4 \\&= \frac{78\pi}{5}\end{aligned}$$

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the y -axis.



$$y = \sqrt[3]{x}$$

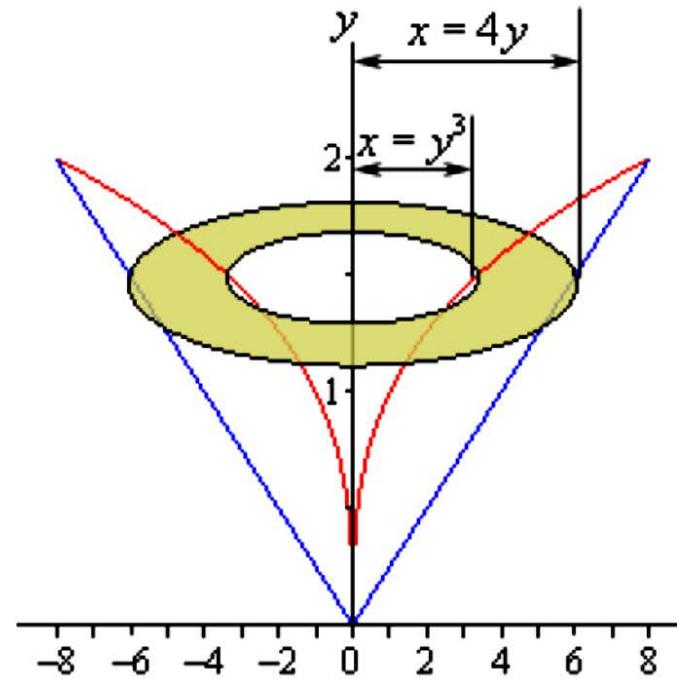
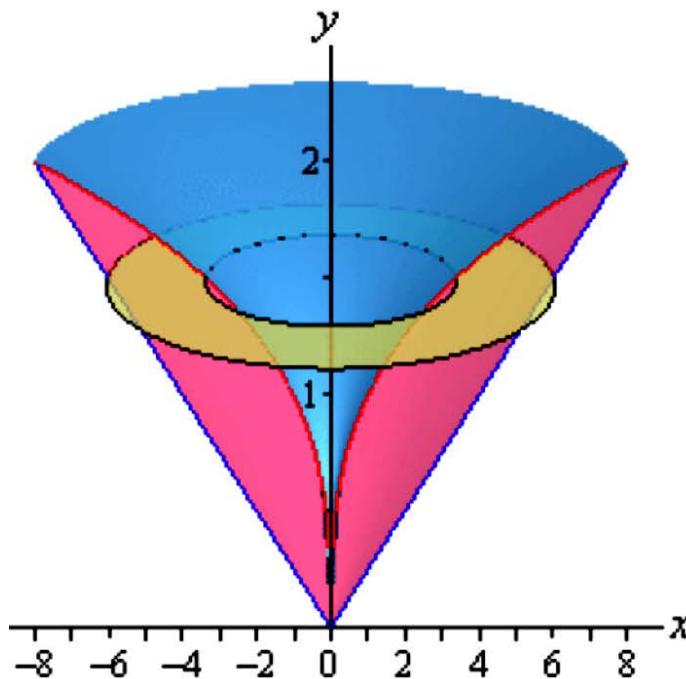
$$\Rightarrow$$

$$x = y^3$$

$$y = \frac{x}{4}$$

$$\Rightarrow$$

$$x = 4y$$



$$A(y) = \pi \left((4y)^2 - (y^3)^2 \right) = \pi (16y^2 - y^6)$$

$$V = \int_c^d A(y) dy$$

$$= \pi \int_0^2 16y^2 - y^6 dy$$

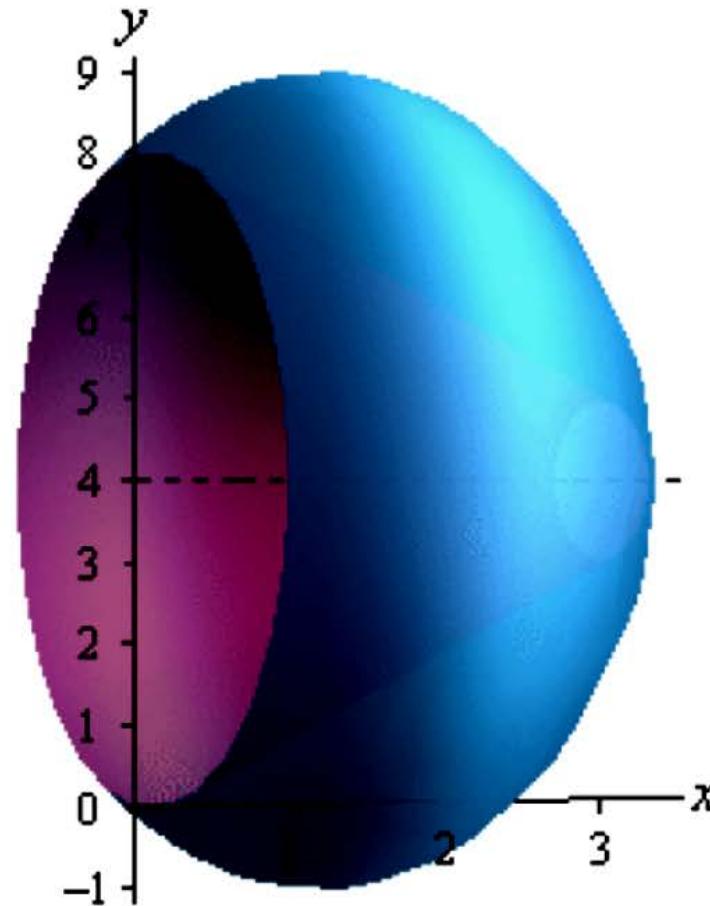
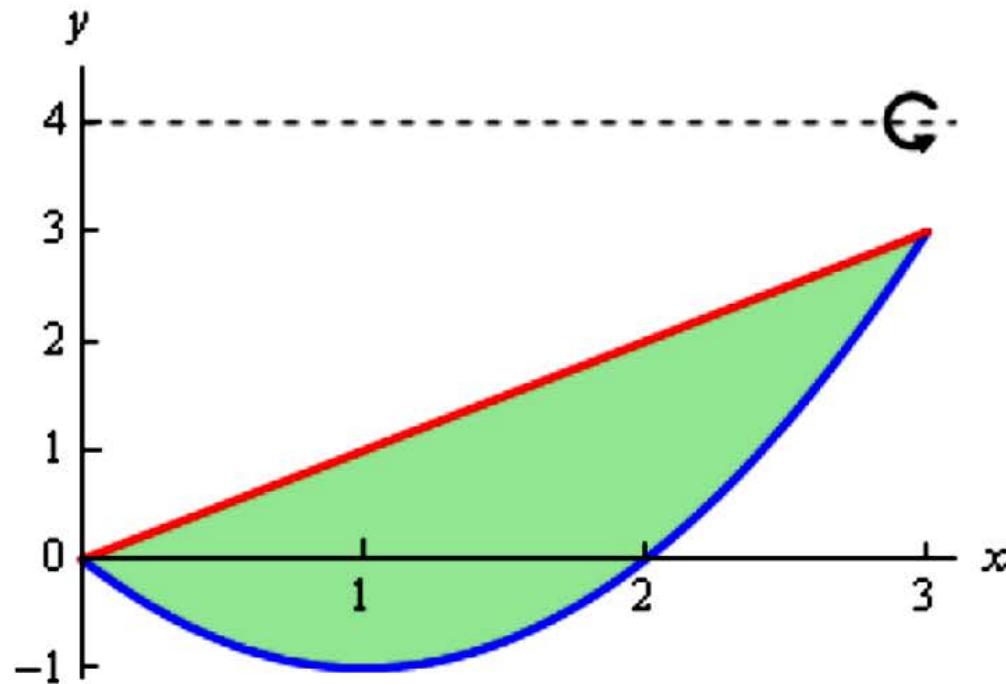
$$= \pi \left(\frac{16}{3} y^3 - \frac{1}{7} y^7 \right) \Big|_0^2$$

$$= \frac{512\pi}{21}$$

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Solution

First let's get the bounding region and the solid graphed.

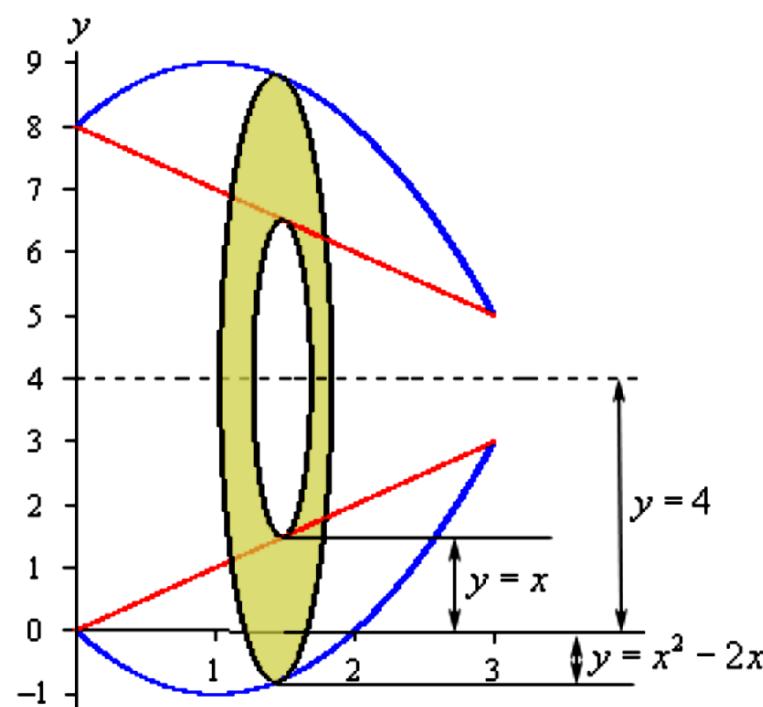
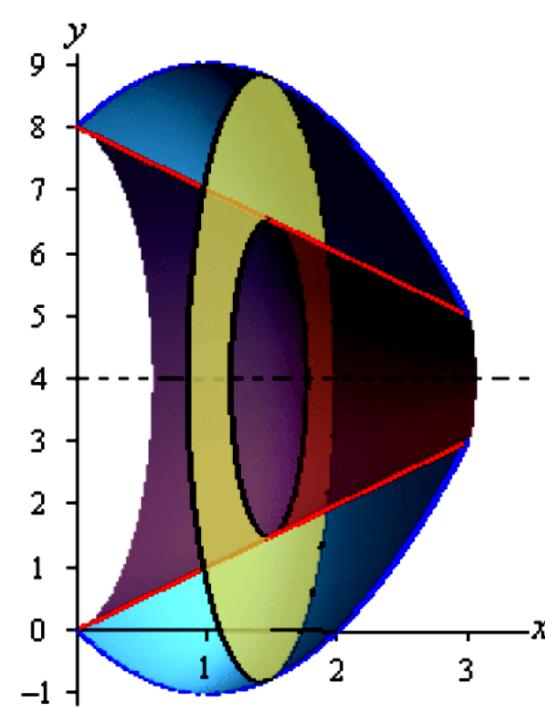


$$\text{inner radius} = 4 - x$$

$$\text{outer radius} = 4 - (x^2 - 2x) = -x^2 + 2x + 4$$

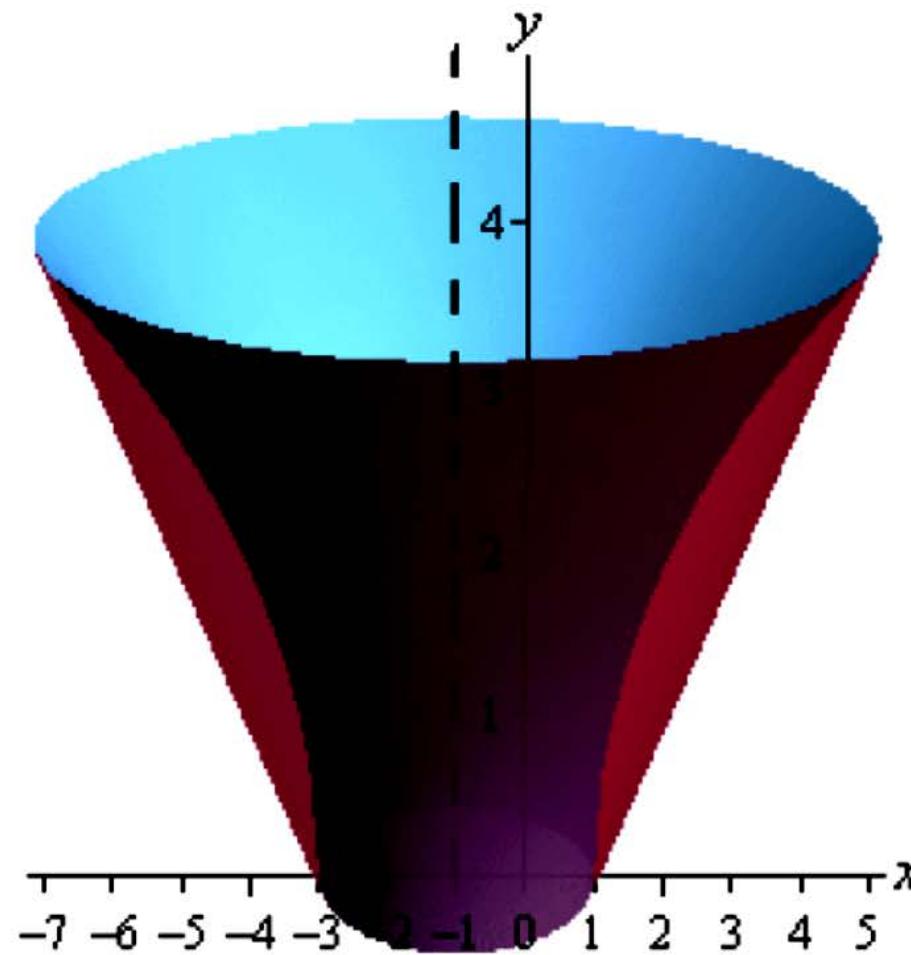
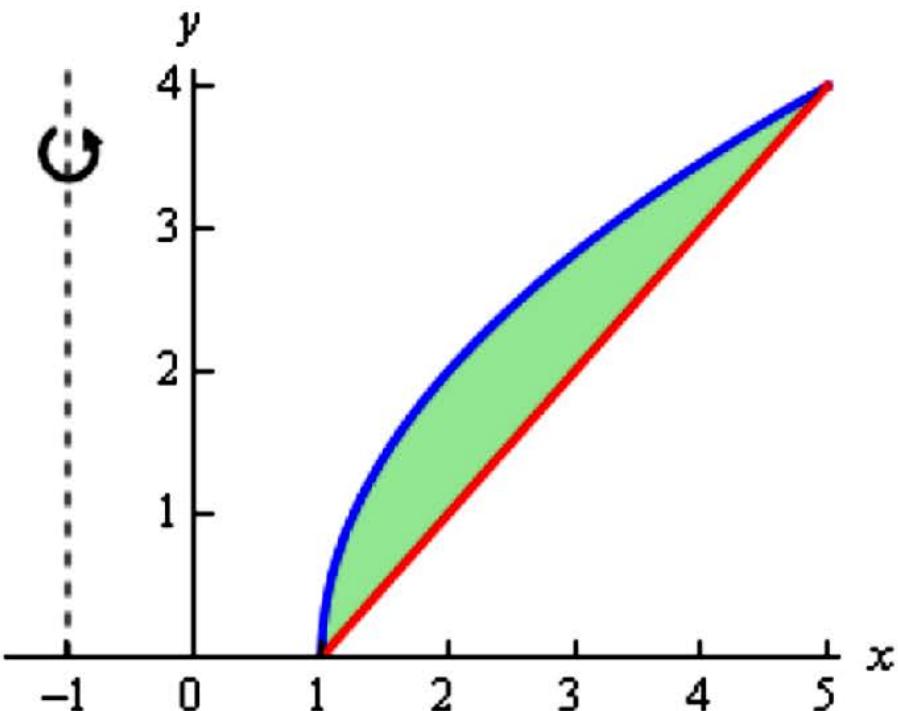
The cross-sectional area for this case is,

$$A(x) = \pi \left((-x^2 + 2x + 4)^2 - (4 - x)^2 \right) = \pi (x^4 - 4x^3 - 5x^2 + 24x)$$



$$\begin{aligned} V &= \int_a^b A(x) dx \\ &= \pi \int_0^3 x^4 - 4x^3 - 5x^2 + 24x dx \\ &= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \Big|_0^3 \\ &= \frac{153\pi}{5} \end{aligned}$$

Example 4 Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.



$$y = 2\sqrt{x-1}$$

$$\Rightarrow x = \frac{y^2}{4} + 1$$

$$y = x - 1$$

$$\Rightarrow x = y + 1$$

$$\text{outer radius} = y + 1 + 1 = y + 2$$

$$\text{inner radius} = \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2$$

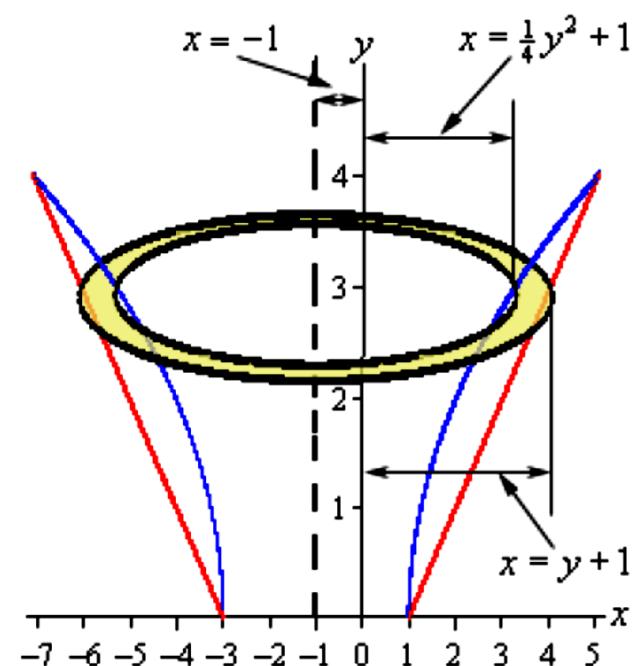
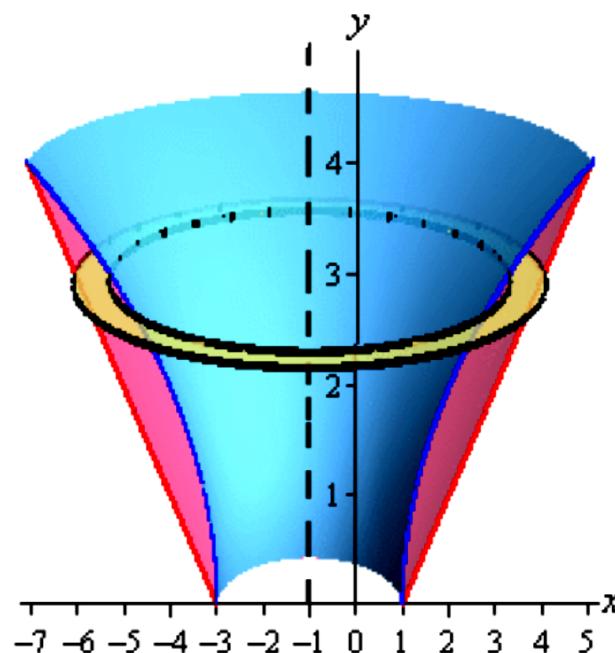
$$A(y) = \pi \left((y+2)^2 - \left(\frac{y^2}{4} + 2 \right)^2 \right) = \pi \left(4y - \frac{y^4}{16} \right)$$

$$V = \int_c^d A(y) dy$$

$$= \pi \int_0^4 4y - \frac{y^4}{16} dy$$

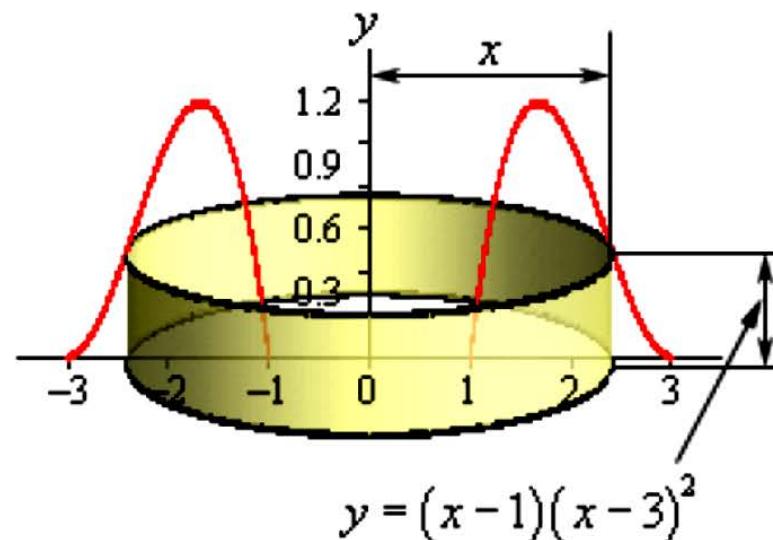
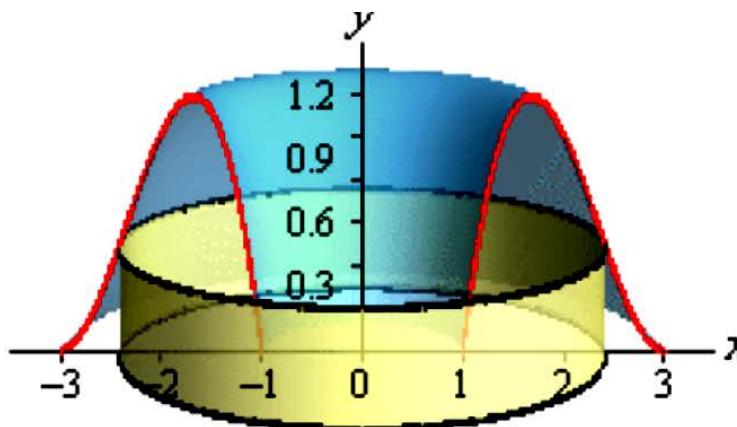
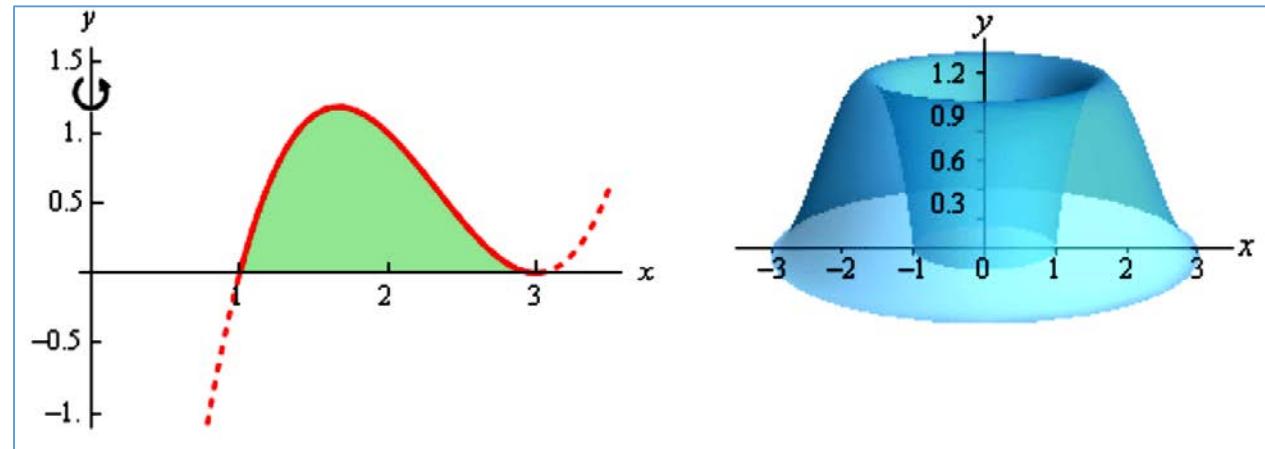
$$= \pi \left(2y^2 - \frac{1}{80} y^5 \right) \Big|_0^4$$

$$= \frac{96\pi}{5}$$



Volumes of Solids of Revolution / Method of Cylinders

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y = (x-1)(x-3)^2$ and the x -axis about the y -axis.



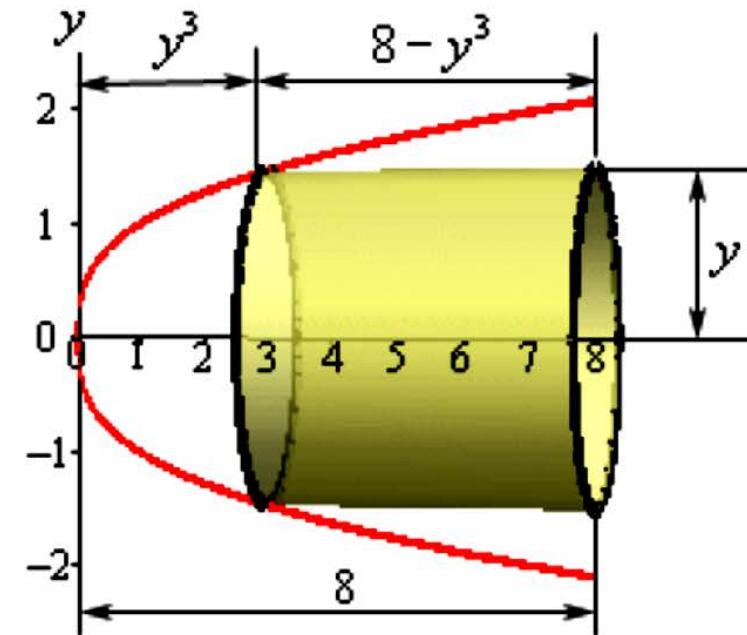
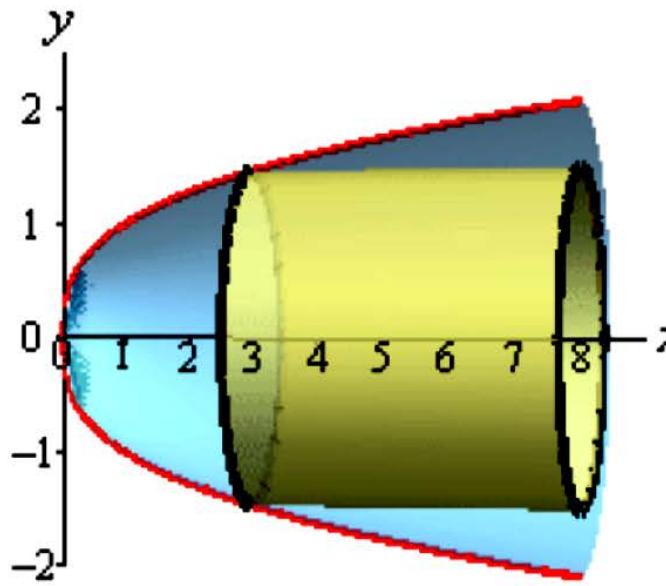
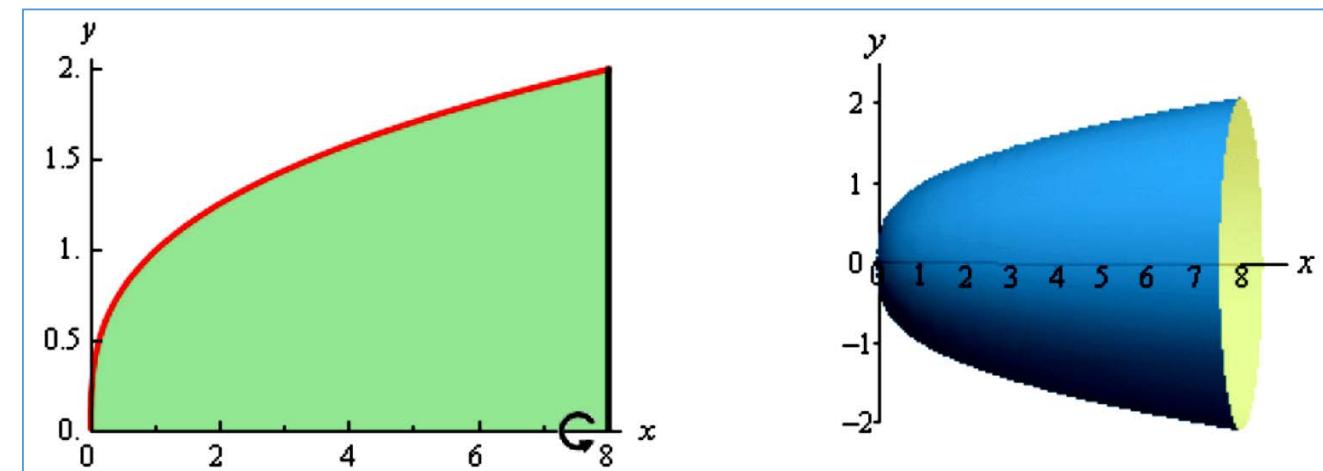
Doing this gives us a cylinder or shell in the object and we can easily find its surface area. The surface area of this cylinder is,

$$\begin{aligned}A(x) &= 2\pi(\text{radius})(\text{height}) \\&= 2\pi(x)((x-1)(x-3)^2) \\&= 2\pi(x^4 - 7x^3 + 15x^2 - 9x)\end{aligned}$$

$$\begin{aligned}V &= \int_a^b A(x) dx \\&= 2\pi \int_1^3 x^4 - 7x^3 + 15x^2 - 9x dx \\&= 2\pi \left(\frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2 \right) \Big|_1^3 \\&= \frac{24\pi}{5}\end{aligned}$$

Example 2 Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the x -axis about the x -axis.

$$y = \sqrt[3]{x} \quad \Rightarrow \quad x = y^3$$



$$A(y) = 2\pi(\text{radius})(\text{width})$$

$$= 2\pi(y)(8 - y^3)$$

$$= 2\pi(8y - y^4)$$

$$V = \int_c^d A(y) dy$$

$$= 2\pi \int_0^2 8y - y^4 dy$$

$$= 2\pi \left(4y^2 - \frac{1}{5}y^5 \right) \Big|_0^2$$

$$= \frac{96\pi}{5}$$

Applications of Derivatives

[Critical Points](#) – In this section we give the definition of critical points. Critical points will show up in most of the sections in this chapter, so it will be important to understand them and how to find them. We will work a number of examples illustrating how to find them for a wide variety of functions.

[Linear Approximations](#) – In this section we discuss using the derivative to compute a linear approximation to a function. We can use the linear approximation to a function to approximate values of the function at certain points. While it might not seem like a useful thing to do with when we have the function there really are reasons that one might want to do this. We give two ways this can be useful in the examples.

[Differentials](#) – In this section we will compute the differential for a function. We will give an application of differentials in this section. However, one of the more important uses of differentials will come in the next chapter and unfortunately we will not be able to discuss it until then.

Critical Points

We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ exists and if either of the following are true.

$$f'(c) = 0$$

OR

$$f'(c) \text{ doesn't exist}$$

Example 1 Determine all the critical points for the function.

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$$

Solution

We first need the derivative of the function in order to find the critical points and so let's get that and notice that we'll factor it as much as possible to make our life easier when we go to find the critical points.

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2(5x^2 + 22x - 15) \\ &= 6x^2(5x - 3)(x + 5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore, the only critical points will be those values of x which make the derivative zero. So, we must solve.

$$6x^2(5x - 3)(x + 5) = 0$$

Because this is the factored form of the derivative it's pretty easy to identify the three critical points. They are,

$$x = -5, \quad x = 0, \quad x = \frac{3}{5}$$

Example 2 Determine all the critical points for the function.

$$g(t) = \sqrt[3]{t^2} (2t - 1)$$

Solution

To find the derivative it's probably easiest to do a little simplification before we actually differentiate. Let's multiply the root through the parenthesis and simplify as much as possible. This will allow us to avoid using the product rule when taking the derivative.

$$g(t) = t^{\frac{2}{3}} (2t - 1) = 2t^{\frac{5}{3}} - t^{\frac{2}{3}}$$

Now differentiate.

$$g'(t) = \frac{10}{3}t^{\frac{2}{3}} - \frac{2}{3}t^{-\frac{1}{3}} = \frac{10t^{\frac{5}{3}}}{3} - \frac{2}{3t^{\frac{1}{3}}}$$

So, in this case we can see that the numerator will be zero if $t = \frac{1}{5}$ and so there are two critical points for this function.

$$t = 0 \quad \text{and} \quad t = \frac{1}{5}$$

1. Determine the critical points of $f(x) = 8x^3 + 81x^2 - 42x - 8$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'(x) = 24x^2 + 162x - 42 = 6(x + 7)(4x - 1)$$

Factoring the derivative as much as possible will help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$6(x + 7)(4x - 1) = 0$$

\Rightarrow

$$\boxed{x = -7, \quad x = \frac{1}{4}}$$

2. Determine the critical points of $R(t) = 1 + 80t^3 + 5t^4 - 2t^5$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$R'(t) = 240t^2 + 20t^3 - 10t^4 = -10t^2(t+4)(t-6)$$

Factoring the derivative as much as possible will help with the next step.

Step 2

Recall that critical points are simply where the derivative is zero and/or doesn't exist. In this case the derivative is just a polynomial and we know that exists everywhere and so we don't need to worry about that. So, all we need to do is set the derivative equal to zero and solve for the critical points.

$$-10t^2(t+4)(t-6) = 0$$

\Rightarrow

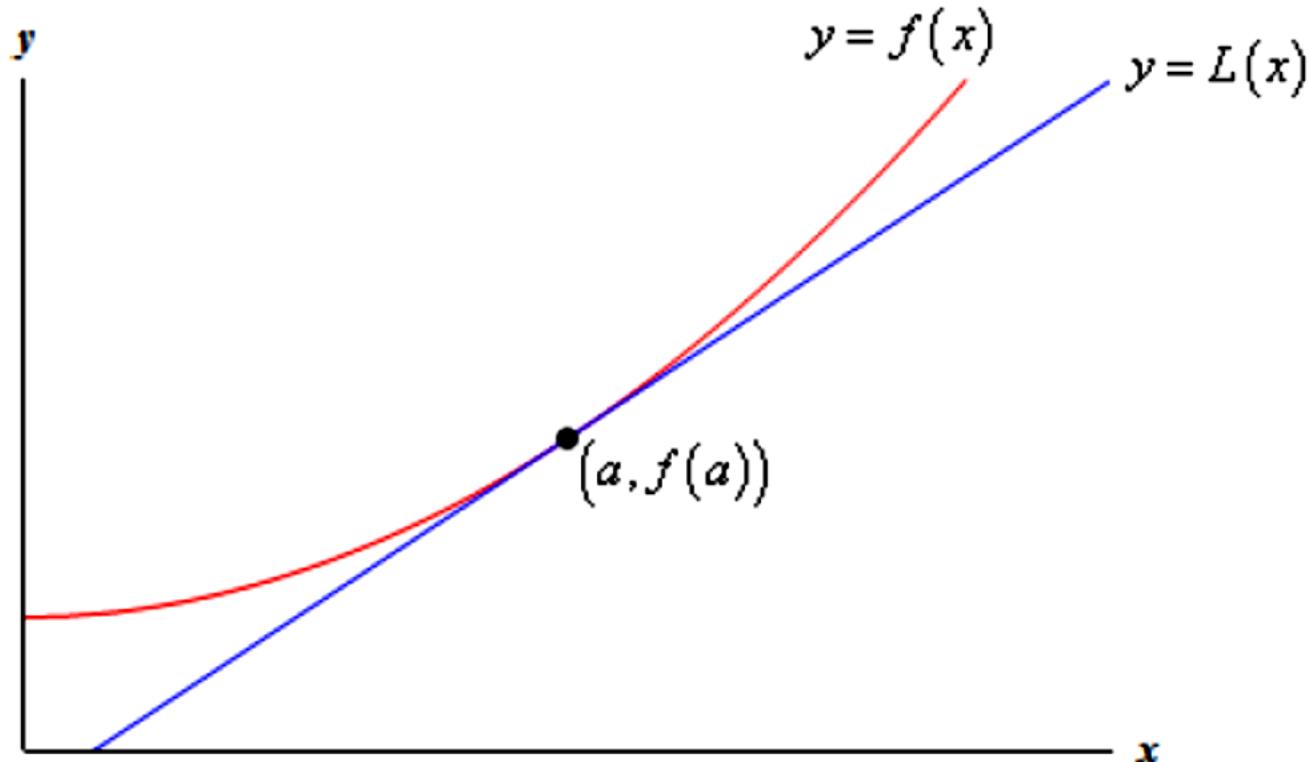
$$t = 0, \quad t = -4, \quad t = 6$$

Linear Approximations

Given a function, $f(x)$, we can find its tangent at $x = a$. The equation of the tangent line, which we'll call $L(x)$ for this discussion, is,

$$L(x) = f(a) + f'(a)(x - a)$$

Take a look at the following graph of a function and its tangent line.



Example 1 Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Solution

Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$
$$f(8) = 2$$
$$f'(8) = \frac{1}{12}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x - 8) = \frac{1}{12}x + \frac{4}{3}$$

Now, the approximations are nothing more than plugging the given values of x into the linear approximation. For comparison purposes we'll also compute the exact values.

$$L(8.05) = 2.00416667$$

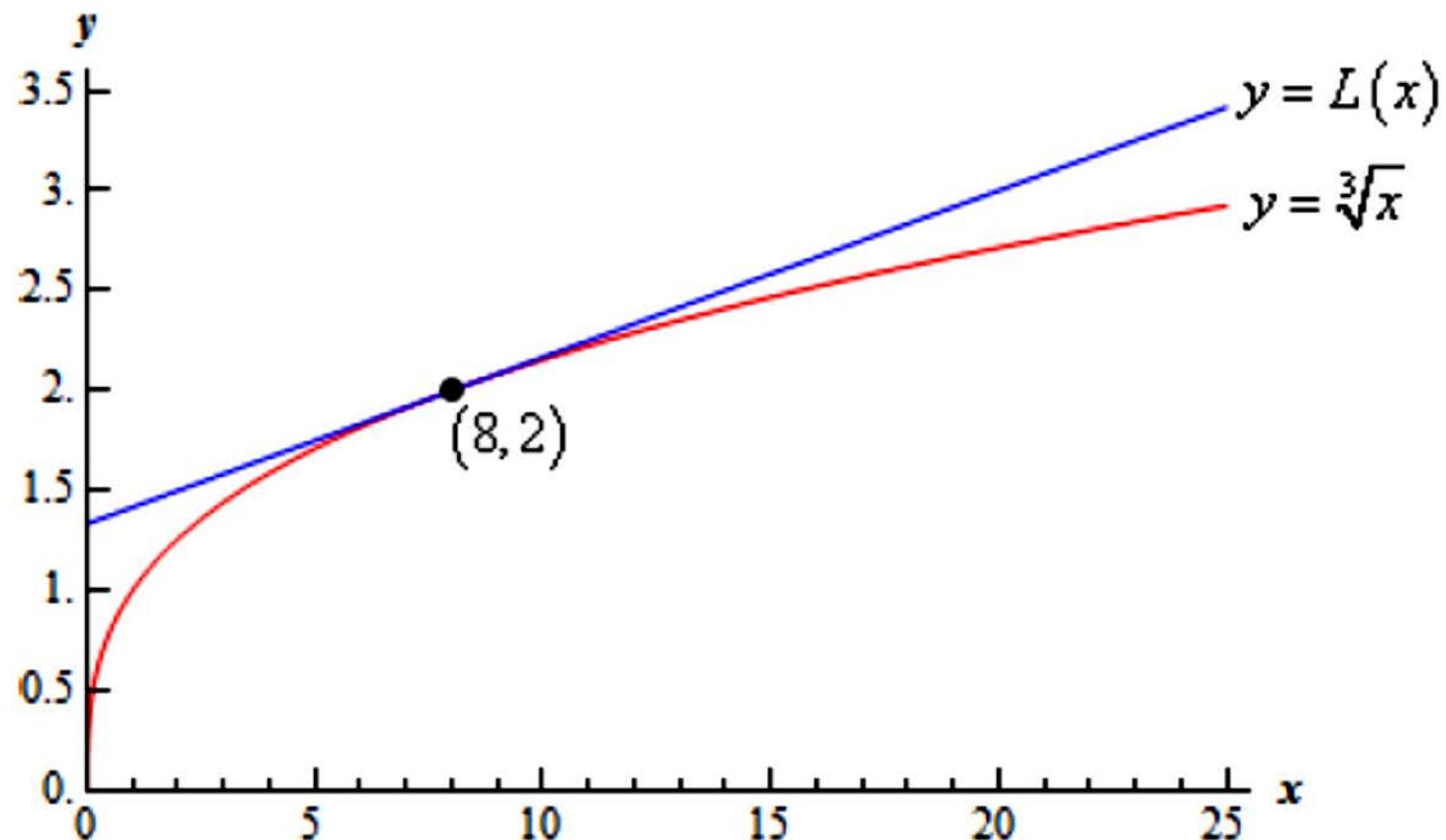
$$\sqrt[3]{8.05} = 2.00415802$$

$$L(25) = 3.41666667$$

$$\sqrt[3]{25} = 2.92401774$$

So, at $x = 8.05$ this linear approximation does a very good job of approximating the actual value. However, at $x = 25$ it doesn't do such a good job.

Here's a quick sketch of the function and its linear approximation at $x = 8$.



Example 2 Determine the linear approximation for $\sin \theta$ at $\theta = 0$.

Solution

Again, there really isn't a whole lot to this example. All that we need to do is compute the tangent line to $\sin \theta$ at $\theta = 0$.

$$f(\theta) = \sin \theta$$

$$f'(\theta) = \cos \theta$$

$$f(0) = 0$$

$$f'(0) = 1$$

The linear approximation is,

$$\begin{aligned} L(\theta) &= f(0) + f'(0)(\theta - a) \\ &= 0 + (1)(\theta - 0) \\ &= \theta \end{aligned}$$

So, as long as θ stays small we can say that $\sin \theta \approx \theta$.

1. Find a linear approximation to $f(x) = 3x e^{2x-10}$ at $x = 5$.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

$$f'(x) = 3e^{2x-10} + 6x e^{2x-10}$$

$$f(5) = 15$$

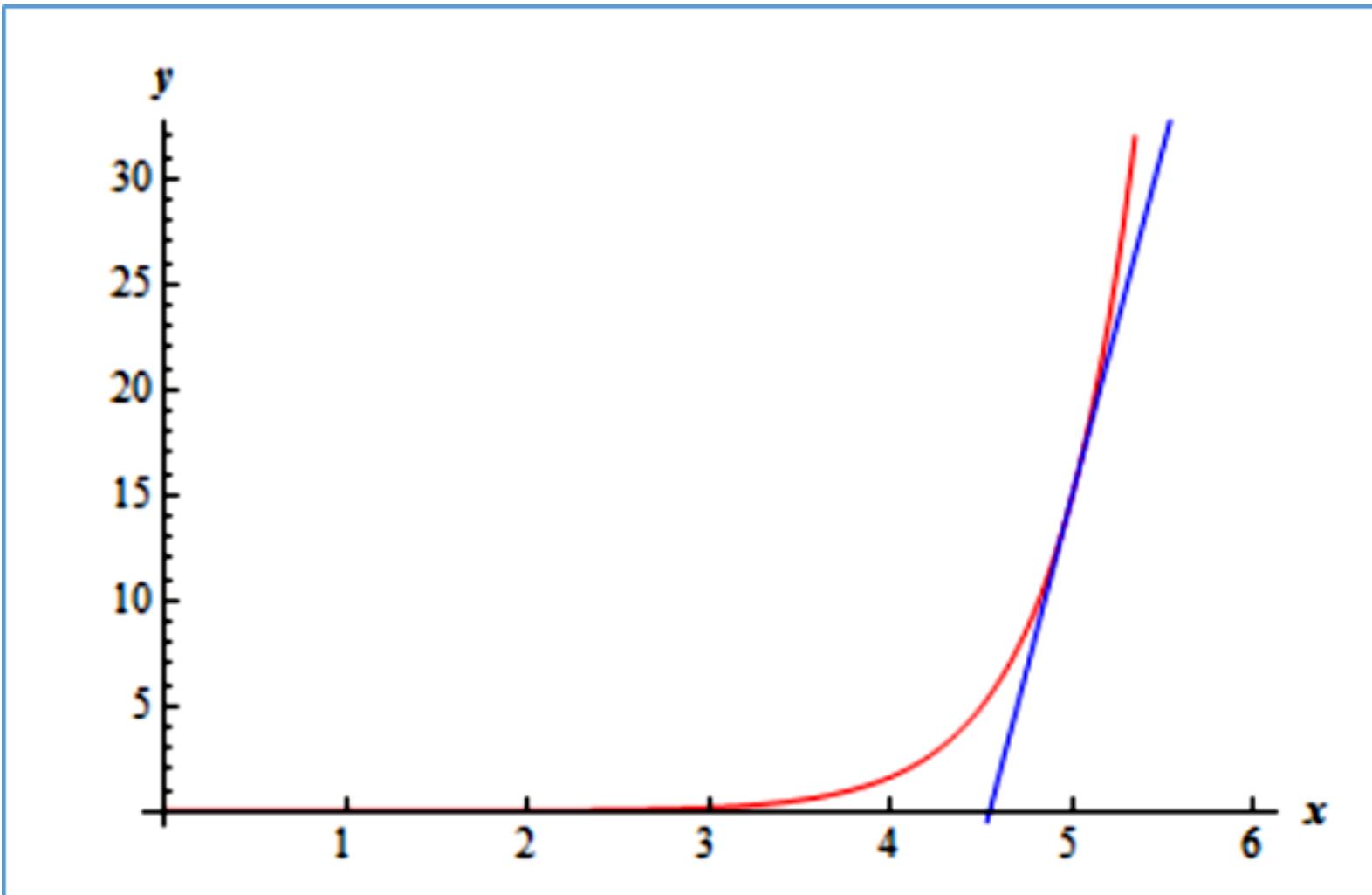
$$f'(5) = 33$$

Step 2

There really isn't much to do at this point other than write down the linear approximation.

$$L(x) = 15 + 33(x - 5) = 33x - 150$$

While it wasn't asked for, here is a quick sketch of the function and the linear approximation.



Differentials

Given a function $y = f(x)$ we call dy and dx differentials and the relationship between them is given by,

$$dy = f'(x)dx$$

Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them in the same manner.

$$df = f'(x)dx$$

We can use the fact that $\Delta y \approx dy$ in the following way.

Example 1 Compute the differential for each of the following.

(a) $y = t^3 - 4t^2 + 7t$

(b) $w = x^2 \sin(2x)$

(c) $f(z) = e^{3-z^4}$

(a) $dy = (3t^2 - 8t + 7)dt$

(b) $dw = (2x \sin(2x) + 2x^2 \cos(2x))dx$

(c) $df = -4z^3 e^{3-z^4} dz$

Example 2 Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$.

Solution

First let's compute actual the change in y , Δy .

$$\Delta y = \cos((2.03)^2 + 1) - 2.03 - (\cos(2^2 + 1) - 2) = 0.083581127$$

Now let's get the formula for dy .

$$dy = (-2x \sin(x^2 + 1) - 1)dx$$

Next, the change in x from $x = 2$ to $x = 2.03$ is $\Delta x = 0.03$ and so we then assume that $dx \approx \Delta x = 0.03$. This gives an approximate change in y of,

$$dy = (-2(2) \sin(2^2 + 1) - 1)(0.03) = 0.085070913$$

We can see that in fact we do have that $\Delta y \approx dy$ provided we keep Δx small.

Example 3 A sphere was measured and its radius was found to be 45 inches with a possible error of no more than 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Solution

First, recall the equation for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

Now, if we start with $r = 45$ and use $dr \approx \Delta r = 0.01$ then $\Delta V \approx dV$ should give us maximum error.

So, first get the formula for the differential.

$$dV = 4\pi r^2 dr$$

Now compute dV .

$$\Delta V \approx dV = 4\pi(45)^2(0.01) = 254.47 \text{ in}^3$$

The maximum error in the volume is then approximately 254.47 in³.

Be careful to not assume this is a large error. On the surface it looks large, however if we compute the actual volume for $r = 45$ we get $V = 381,703.51 \text{ in}^3$. So, in comparison the error in the volume is,

$$\frac{254.47}{381703.51} \times 100 = 0.067\%$$

That's not much possible error at all!

Antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I .

TABLE 4.2 Antiderivative formulas

Function	General antiderivative
1. x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1, n \text{ rational}$
2. $\sin kx$	$-\frac{\cos kx}{k} + C, \quad k \text{ a constant, } k \neq 0$
3. $\cos kx$	$\frac{\sin kx}{k} + C, \quad k \text{ a constant, } k \neq 0$
4. $\sec^2 x$	$\tan x + C$
5. $\csc^2 x$	$-\cot x + C$
6. $\sec x \tan x$	$\sec x + C$
7. $\csc x \cot x$	$-\csc x + C$

TABLE 4.3 Antiderivative linearity rules

	Function	General antiderivative
1.	<i>Constant Multiple Rule:</i> $kf(x)$	$kF(x) + C$, k a constant
2.	<i>Negative Rule:</i> $-f(x)$	$-F(x) + C$,
3.	<i>Sum or Difference Rule:</i> $f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f .

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x , denoted by

$$\int f(x) dx.$$

The symbol \int is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

EXAMPLE 3 Finding Antiderivatives Using Table 4.2

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \frac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \frac{x}{2}$

Solution

(a) $F(x) = \frac{x^6}{6} + C$

Formula 1
with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1
with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$

Formula 2
with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2 \sin \frac{x}{2} + C$

Formula 3
with $k = 1/2$



Find the first and second derivatives.

$$1. \ y = -x^2 + 3$$

$$3. \ s = 5t^3 - 3t^5$$

$$5. \ y = \frac{4x^3}{3} - x$$

$$2. \ y = x^2 + x + 8$$

$$4. \ w = 3z^7 - 7z^3 + 21z^2$$

$$6. \ y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$$

$$1. \ y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$$

$$2. \ y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$$

$$3. \ s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$$

$$4. \ w = 3z^7 - 7z^3 + 21z^2 \Rightarrow \frac{dw}{dz} = 21z^6 - 21z^2 + 42z \Rightarrow \frac{d^2w}{dz^2} = 126z^5 - 42z + 42$$

$$5. \ y = \frac{4}{3}x^3 - x \Rightarrow \frac{dy}{dx} = 4x^2 - 1 \Rightarrow \frac{d^2y}{dx^2} = 8x$$

$$6. \ y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} \Rightarrow \frac{dy}{dx} = x^2 + x + \frac{1}{4} \Rightarrow \frac{d^2y}{dx^2} = 2x + 1 + 0 = 2x + 1$$

Find derivative (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

$$\bullet \quad y = (3 - x^2)(x^3 - x + 1)$$

$$\begin{aligned} \text{(a)} \quad y &= (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2) \\ &= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3 \end{aligned}$$

$$\text{(b)} \quad y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$$

$$\bullet \quad y = (x - 1)(x^2 + x + 1)$$

$$\begin{aligned} \text{(a)} \quad y &= (x - 1)(x^2 + x + 1) \Rightarrow y' = (x - 1)(2x + 1) + (x^2 + x + 1)(1) = 3x^2 \\ \text{(b)} \quad y &= (x - 1)(x^2 + x + 1) = x^3 - 1 \Rightarrow y' = 3x^2 \end{aligned}$$

$$\bullet \quad y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$$

$$\begin{aligned} \text{(a)} \quad y &= (x^2 + 1)\left(x + 5 + \frac{1}{x}\right) \Rightarrow y' = (x^2 + 1) \cdot \frac{d}{dx}\left(x + 5 + \frac{1}{x}\right) + \left(x + 5 + \frac{1}{x}\right) \cdot \frac{d}{dx}(x^2 + 1) \\ &= (x^2 + 1)(1 - x^{-2}) + (x + 5 + x^{-1})(2x) = (x^2 - 1 + 1 - x^{-2}) + (2x^2 + 10x + 2) = 3x^2 + 10x + 2 - \frac{1}{x^2} \\ \text{(b)} \quad y &= x^3 + 5x^2 + 2x + 5 + \frac{1}{x} \Rightarrow y' = 3x^2 + 10x + 2 - \frac{1}{x^2} \end{aligned}$$

$$\bullet \quad y = \frac{2x + 5}{3x - 2}$$

$$\bullet \quad g(x) = \frac{x^2 - 4}{x + 0.5}$$

$$\bullet \quad z = \frac{2x + 1}{x^2 - 1}$$

$$\bullet \quad f(t) = \frac{t^2 - 1}{t^2 + t - 2}$$

- . $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
 $= \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2} = \frac{6x-4 - 6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$
- . $z = \frac{2x+1}{x^2-1} \Rightarrow \frac{dz}{dx} = \frac{(x^2-1)(2) - (2x+1)(2x)}{(x^2-1)^2} = \frac{2x^2-2-4x^2-2x}{(x^2-1)^2} = \frac{-2x^2-2x-2}{(x^2-1)^2} = \frac{-2(x^2+x+1)}{(x^2-1)^2}$
- . $g(x) = \frac{x^2-4}{x+0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$
 $= \frac{(x+0.5)(2x) - (x^2-4)(1)}{(x+0.5)^2} = \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}$
- . $f(t) = \frac{t^2-1}{t^2+t-2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2}, t \neq 1 \Rightarrow f'(t) = \frac{(t+2)(1) - (t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$

Find the first and second derivatives of the functions

$$\cdot \quad y = \frac{x^3 + 7}{x}$$

$$\cdot \quad r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$$

$$\cdot \quad s = \frac{t^2 + 5t - 1}{t^2}$$

$$\cdot \quad u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$$

$$\boxed{\frac{d}{dt} \left(\frac{u}{v} \right) = \frac{v(du/dt) - u(dv/dt)}{v^2}}$$

$$\frac{dy}{dx} = \frac{x(3x^2) - (x^3 + 7)}{x^2} = \frac{(3x^3) - (x^3 + 7)}{x^2} = 2x - 7x^{-2}$$

$$\cdot \quad y = \frac{x^3 + 7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$$

$$\begin{aligned} \cdot \quad s &= \frac{t^2 + 5t - 1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3} \\ &\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4} \end{aligned}$$

$$\cdot \quad r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3} = \frac{\theta^3 - 1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$$

$$\begin{aligned} \cdot \quad u &= \frac{(x^2 + x)(x^2 - x + 1)}{x^4} = \frac{x(x+1)(x^2 - x + 1)}{x^4} = \frac{x(x^3 + 1)}{x^4} = \frac{x^4 + x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3} \\ &\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5} \end{aligned}$$

Using Numerical Values

- Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

- a. $\frac{d}{dx}(uv)$
- b. $\frac{d}{dx}\left(\frac{u}{v}\right)$
- c. $\frac{d}{dx}\left(\frac{v}{u}\right)$
- d. $\frac{d}{dx}(7v - 2u)$

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2$$

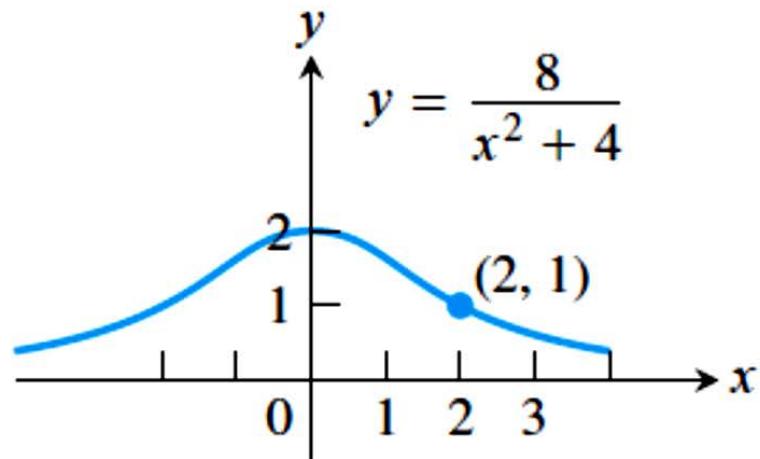
$$(a) \frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$$

$$(b) \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$$

$$(c) \frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(-3)}{(5)^2} = \frac{7}{25}$$

$$(d) \frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)\Big|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$$

Find the tangent to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .

$$y = \frac{8}{x^2 + 4} \Rightarrow y' = \frac{(x^2 + 4)(0) - 8(2x)}{(x^2 + 4)^2} = \frac{-16x}{(x^2 + 4)^2}.$$

When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2 + 4)^2} = -\frac{1}{2}$, so the tangent

line to the curve at $(2, 1)$ has the equation

$$y - 1 = -\frac{1}{2}(x - 2), \text{ or } y = -\frac{x}{2} + 2.$$

Differentiate each of the following functions.

(a) $g(x) = 3 \sec(x) - 10 \cot(x)$

(b) $h(w) = 3w^{-4} - w^2 \tan(w)$

(c) $y = 5 \sin(x) \cos(x) + 4 \csc(x)$

(d) $P(t) = \frac{\sin(t)}{3 - 2 \cos(t)}$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x) \cot(x)$$

$$\begin{aligned} g'(x) &= 3 \sec(x) \tan(x) - 10(-\csc^2(x)) \\ &= 3 \sec(x) \tan(x) + 10 \csc^2(x) \end{aligned}$$

$$\begin{aligned} h'(w) &= -12w^{-5} - (2w \tan(w) + w^2 \sec^2(w)) \\ &= -12w^{-5} - 2w \tan(w) - w^2 \sec^2(w) \end{aligned}$$

$$\begin{aligned} P'(t) &= \frac{\cos(t)(3 - 2 \cos(t)) - \sin(t)(2 \sin(t))}{(3 - 2 \cos(t))^2} \\ &= \frac{3 \cos(t) - 2 \cos^2(t) - 2 \sin^2(t)}{(3 - 2 \cos(t))^2} \end{aligned}$$

$$\begin{aligned} y' &= 5 \cos(x) \cos(x) + 5 \sin(x)(-\sin(x)) - 4 \csc(x) \cot(x) \\ &= 5 \cos^2(x) - 5 \sin^2(x) - 4 \csc(x) \cot(x) \end{aligned}$$

$$1. \ y = -10x + 3 \cos x$$

$$2. \ y = \frac{3}{x} + 5 \sin x$$

$$3. \ y = \csc x - 4\sqrt{x} + 7$$

$$4. \ y = x^2 \cot x - \frac{1}{x^2}$$

$$5. \ y = (\sec x + \tan x)(\sec x - \tan x)$$

$$6. \ y = (\sin x + \cos x) \sec x$$

$$1. \ y = -10x + 3 \cos x \Rightarrow \frac{dy}{dx} = -10 + 3 \frac{d}{dx}(\cos x) = -10 - 3 \sin x$$

$$2. \ y = \frac{3}{x} + 5 \sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5 \frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5 \cos x$$

$$3. \ y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$$

$$4. \ y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$$
$$= -x^2 \csc^2 x + 2x \cot x + \frac{2}{x^3}$$

$$\begin{aligned}5. \quad y &= (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x) \frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x) \frac{d}{dx}(\sec x + \tan x) \\&= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x) \\&= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.\end{aligned}$$

$$\begin{aligned}6. \quad y &= (\sin x + \cos x) \sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x) \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x) \\&= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x) \sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x} \\&= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

Differentiate each of the following functions.

$$\begin{aligned}r &= 4 - \theta^2 \sin \theta \\r &= \sec \theta \csc \theta\end{aligned}$$

$$\begin{aligned}. \quad r &= \theta \sin \theta + \cos \theta \\. \quad r &= (1 + \sec \theta) \sin \theta\end{aligned}$$

$$\begin{aligned}1 + \tan^2 \theta &= \sec^2 \theta. \\1 + \cot^2 \theta &= \csc^2 \theta.\end{aligned}$$

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}\end{aligned}$$

- . $r = 4 - \theta^2 \sin \theta \Rightarrow \frac{dr}{d\theta} = -(\theta^2 \frac{d}{d\theta}(\sin \theta) + (\sin \theta)(2\theta)) = -(\theta^2 \cos \theta + 2\theta \sin \theta) = -\theta(\theta \cos \theta + 2 \sin \theta)$
- . $r = \theta \sin \theta + \cos \theta \Rightarrow \frac{dr}{d\theta} = (\theta \cos \theta + (\sin \theta)(1)) - \sin \theta = \theta \cos \theta$
- . $r = \sec \theta \csc \theta \Rightarrow \frac{dr}{d\theta} = (\sec \theta)(-\csc \theta \cot \theta) + (\csc \theta)(\sec \theta \tan \theta)$
 $= \left(\frac{-1}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right) + \left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right) = \frac{-1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta - \csc^2 \theta$
- . $r = (1 + \sec \theta) \sin \theta \Rightarrow \frac{dr}{d\theta} = (1 + \sec \theta) \cos \theta + (\sin \theta)(\sec \theta \tan \theta) = (\cos \theta + 1) + \tan^2 \theta = \cos \theta + \sec^2 \theta$

Tangent Lines

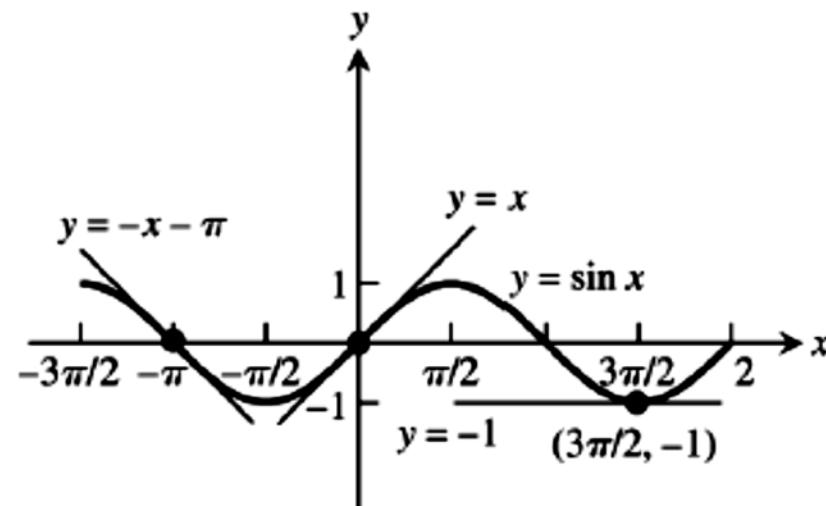
- $y = \sin x, -3\pi/2 \leq x \leq 2\pi$
- $x = -\pi, 0, 3\pi/2$

$y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos \frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0, 0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .



given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9$, $u = (1/2)x^4$ 2. $y = 2u^3$, $u = 8x - 1$
3. $y = \sin u$, $u = 3x + 1$ 4. $y = \cos u$, $u = -x/3$

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x - 1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = 6(8x - 1)^2 \cdot 8 = 48(8x - 1)^2$

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x + 1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = (\cos(3x + 1))(3) = 3 \cos(3x + 1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin\left(\frac{-x}{3}\right)$; $g(x) = \frac{-x}{3} \Rightarrow g'(x) = -\frac{1}{3}$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -\sin\left(\frac{-x}{3}\right) \cdot \left(-\frac{1}{3}\right) = \frac{1}{3} \sin\left(\frac{-x}{3}\right)$

$$\bullet \quad y = \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^4$$

$$\bullet \quad y = \sec(\tan x)$$

$$\bullet \quad y = \left(\frac{x}{5} + \frac{1}{5x} \right)^5$$

$$\bullet \quad y = \cot\left(\pi - \frac{1}{x}\right)$$

$$u = \left(\frac{x^2}{8} + x - \frac{1}{x} \right), y = u^4: \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 4u^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2} \right) = 4 \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2} \right)$$

$$u = \left(\frac{x}{5} + \frac{1}{5x} \right), y = u^5: \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 5u^4 \cdot \left(\frac{1}{5} - \frac{1}{5x^2} \right) = \left(\frac{x}{5} + \frac{1}{5x} \right)^4 \left(1 - \frac{1}{x^2} \right)$$

$$u = \tan x, y = \sec u: \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x) \tan(\tan x)) \sec^2 x$$

$$u = \pi - \frac{1}{x}, y = \cot u: \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (-\csc^2 u) \left(\frac{1}{x^2} \right) = -\frac{1}{x^2} \csc^2 \left(\pi - \frac{1}{x} \right)$$

Differentiate each of the following functions.

$$\frac{d}{dx}(\mathrm{e}^x) = \mathrm{e}^x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$1. \ f(x) = 2\mathrm{e}^x - 8^x$$

$$f'(x) = 2\mathrm{e}^x - 8^x \ln(8)$$

$$2. \ g(t) = 4 \log_3(t) - \ln(t)$$

$$g'(t) = \frac{4}{t \ln(3)} - \frac{1}{t}$$

$$3. \ R(w) = 3^w \log(w)$$

$$R'(w) = 3^w \ln(3) \log(w) + \frac{3^w}{w \ln(10)}$$

$$4. \ y = z^5 - \mathrm{e}^z \ln(z)$$

$$y' = 5z^4 - \mathrm{e}^z \ln(z) - \frac{\mathrm{e}^z}{z}$$

. Find the tangent line to $f(x) = 7^x + 4e^x$ at $x = 0$.

. Find the tangent line to $f(x) = \ln(x)\log_2(x)$ at $x = 2$.

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .

$$f'(x) = 7^x \ln(7) + 4e^x$$

$$f(0) = 5 \quad f'(0) = \ln(7) + 4 = 5.9459$$

$$y = f(0) + f'(0)(x - 0) = \boxed{5 + (\ln(7) + 4)x = 5 + 5.9459x}$$

$$f'(x) = \frac{\log_2(x)}{x} + \frac{\ln(x)}{x \ln(2)}$$

$$f(2) = \ln(2)\log_2(2) = \ln(2) \quad f'(2) = \frac{\log_2(2)}{2} + \frac{\ln(2)}{2 \ln(2)} = 1$$

$$y = f(2) + f'(2)(x - 2) = \ln(2) + (1)(x - 2) = \boxed{x - 2 + \ln(2)}$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$

$$T'(z) = -2\sin(z) - \frac{6}{\sqrt{1-z^2}}$$

2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$

$$g'(t) = -\frac{1}{|t|\sqrt{t^2-1}} + \frac{4}{t^2+1}$$

3. $y = 5x^6 - \sec^{-1}(x)$

$$\frac{dy}{dx} = 30x^5 - \frac{1}{x\sqrt{x^2-1}}$$

4. $f(w) = \sin(w) + w^2 \tan^{-1}(w)$

$$f'(w) = \cos(w) + 2w\tan^{-1}(w) + \frac{w^2}{1+w^2}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$1. f(x) = \sinh(x) + 2 \cosh(x) - \operatorname{sech}(x) \quad f'(x) = \cosh(x) + 2 \sinh(x) + \operatorname{sech}(x) \tanh(x)$$

$$2. R(t) = \tan(t) + t^2 \operatorname{csch}(t) \quad R'(t) = \sec^2(t) + 2t \operatorname{csch}(t) - t^2 \operatorname{csch}(t) \coth(t)$$

$$3. g(z) = \frac{z+1}{\tanh(z)}$$

$$g'(z) = \frac{\tanh(z) - (z+1) \operatorname{sech}^2(z)}{\tanh^2(z)}$$

Implicit Differentiation

$$\therefore 2y^3 + 4x^2 - y = x^6$$

$$6y^2 y' + 8x - y' = 6x^5$$

$$(6y^2 - 1)y' = 6x^5 - 8x \quad \Rightarrow$$

$$y' = \frac{6x^5 - 8x}{6y^2 - 1}$$

$$\therefore 7y^2 + \sin(3x) = 12 - y^4$$

$$14y y' + 3\cos(3x) = -4y^3 y'$$

$$(14y + 4y^3)y' = -3\cos(3x) \quad \Rightarrow$$

$$y' = \frac{-3\cos(3x)}{14y + 4y^3}$$

$$\cos(x^2 + 2y) + x e^{y^2} = 1.$$

$$-(2x + 2y') \sin(x^2 + 2y) + e^{y^2} + 2y y' x e^{y^2} = 0$$

$$-2x \sin(x^2 + 2y) - 2y' \sin(x^2 + 2y) + e^{y^2} + 2y y' x e^{y^2} = 0$$

$$(2yx e^{y^2} - 2 \sin(x^2 + 2y)) y' = 0 + 2x \sin(x^2 + 2y) - e^{y^2}$$

$$y' = \boxed{\frac{2x \sin(x^2 + 2y) - e^{y^2}}{2yx e^{y^2} - 2 \sin(x^2 + 2y)}}$$

Indefinite Integrals

$$\int c f(x) dx = c \int f(x) dx$$

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$$

$$\int 4x^6 - 2x^3 + 7x - 4 \, dx = \frac{4}{7}x^7 - \frac{2}{4}x^4 + \frac{7}{2}x^2 - 4x + c = \boxed{\frac{4}{7}x^7 - \frac{1}{2}x^4 + \frac{7}{2}x^2 - 4x + c}$$

$$\int z^7 - 48z^{11} - 5z^{16} \, dz = \frac{1}{8}z^8 - \frac{48}{12}z^{12} - \frac{5}{17}z^{17} + c = \boxed{\frac{1}{8}z^8 - 4z^{12} - \frac{5}{17}z^{17} + c}$$

$$\int 10t^{-3} + 12t^{-9} + 4t^3 \, dt = \frac{10}{-2}t^{-2} + \frac{12}{-8}t^{-8} + \frac{4}{4}t^4 + c = \boxed{-5t^{-2} - \frac{3}{2}t^{-8} + t^4 + c}$$

$$\int w^{-2} + 10w^{-5} - 8 \, dw = \frac{1}{-1}w^{-1} + \frac{10}{-4}w^{-4} - 8w + c = \boxed{-w^{-1} - \frac{5}{2}w^{-4} - 8w + c}$$

$$\int (t^2 - 1)(4 + 3t) dt = \int 3t^3 + 4t^2 - 3t - 4 dt = \boxed{\frac{3}{4}t^4 + \frac{4}{3}t^3 - \frac{3}{2}t^2 - 4t + c}$$

$$\int \sqrt{z} \left(z^2 - \frac{1}{4z} \right) dz = \int z^{\frac{5}{2}} - \frac{1}{4} z^{-\frac{1}{2}} dz = \boxed{\frac{2}{7}z^{\frac{7}{2}} - \frac{1}{2}z^{\frac{1}{2}} + c}$$

$$\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4} dz = \int z^4 - 6z + \frac{4}{z} - 2z^{-4} dz = \boxed{\frac{1}{5}z^5 - 3z^2 + 4 \ln|z| + \frac{2}{3}z^{-3} + c}$$

$$\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}} dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{-\frac{1}{6}} dx = \boxed{\frac{1}{27}x^{\frac{9}{2}} - \frac{1}{5}x^{\frac{5}{6}} + c}$$

$$\int (8x-12)(4x^2-12x)^4 dx .$$

$$u = 4x^2 - 12x$$

$$du = (8x-12)dx$$

$$\int (8x-12)(4x^2-12x)^4 dx = \int u^4 du = \frac{1}{5}u^5 + c = \boxed{\frac{1}{5}(4x^2-12x)^5 + c}$$

$$\int 3t^{-4}(2+4t^{-3})^{-7} dt .$$

$$u = 2 + 4t^{-3}$$

$$du = -12t^{-4} dt$$

$$3t^{-4} dt = -\frac{1}{4}du$$

$$\int 3t^{-4}(2+4t^{-3})^{-7} dt = -\frac{1}{4} \int u^{-7} du = \frac{1}{24}u^{-6} + c = \boxed{\frac{1}{24}(2+4t^{-3})^{-6} + c}$$

$$\int (7y - 2y^3) e^{y^4 - 7y^2} dy.$$

$$u = y^4 - 7y^2$$

$$du = (4y^3 - 14y)dy = -2(7y - 2y^3)dy \quad \Rightarrow \quad (7y - 2y^3)dy = -\frac{1}{2}du$$

$$\int (7y - 2y^3) e^{y^4 - 7y^2} dy = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = \boxed{-\frac{1}{2} e^{y^4 - 7y^2} + C}$$

$$\int 10 \sin(2x) \cos(2x) \sqrt{\cos^2(2x) - 5} dx.$$

$$u = \cos^2(2x) - 5$$

$$du = -4 \cos(2x) \sin(2x) dx = -2(2)(\tfrac{5}{5}) \cos(2x) \sin(2x) dx$$

$$\Rightarrow 10 \cos(2x) \sin(2x) dx = -\frac{5}{2} du$$

$$\int 10 \sin(2x) \cos(2x) \sqrt{\cos^2(2x) - 5} dx = -\frac{5}{2} \int u^{\frac{1}{2}} du = -\frac{5}{3} u^{\frac{3}{2}} + C = \boxed{-\frac{5}{3} (\cos^2(2x) - 5)^{\frac{3}{2}} + C}$$

“Outside-Inside” Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

$$\frac{d}{dx} [\cos^2(2x)]$$

$$= 2 \cos(2x) \cdot \frac{d}{dx} [\cos(2x)]$$

$$= 2 \cos(2x) (-\sin(2x)) \cdot \frac{d}{dx} [2x]$$

$$= -2 \cos(2x) \cdot 2 \cdot \frac{d}{dx} [x] \cdot \sin(2x)$$

$$= -4 \cos(2x) \cdot 1 \sin(2x)$$

$$= -4 \cos(2x) \sin(2x)$$

By the Chain Rule,

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

$$\int \frac{6e^{7w}}{(1-8e^{7w})^3} + \frac{14e^{7w}}{1-8e^{7w}} dw. \quad u = 1-8e^{7w}$$

$$du = -56e^{7w}dw \quad \rightarrow \quad e^{7w}dw = -\frac{1}{56}du$$

$$\int \frac{6e^{7w}}{(1-8e^{7w})^3} + \frac{14e^{7w}}{1-8e^{7w}} dw = \int \left[\frac{6}{(1-8e^{7w})^3} + \frac{14}{1-8e^{7w}} \right] e^{7w} dw$$

$$\begin{aligned} \int \frac{6e^{7w}}{(1-8e^{7w})^3} + \frac{14e^{7w}}{1-8e^{7w}} dw &= -\frac{1}{56} \int 6u^{-3} + \frac{14}{u} du = -\frac{1}{56} \left(-3u^{-2} + 14 \ln|u| \right) + c \\ &= \boxed{-\frac{1}{56} \left(-3(1-8e^{7w})^{-2} + 14 \ln|1-8e^{7w}| \right) + c} \end{aligned}$$

Evaluate each of the following integrals.

$$\int \cos(x) - \frac{3}{x^5} dx = \int \cos(x) - 3x^{-5} dx = \sin(x) + \frac{3}{4}x^{-4} + c = \boxed{\sin(x) + \frac{3}{4x^4} + c}$$

$$\begin{aligned}\int_1^4 \cos(x) - \frac{3}{x^5} dx &= \int_1^4 \cos(x) - 3x^{-5} dx = \left(\sin(x) + \frac{3}{4x^4} \right) \Big|_1^4 \\&= \sin(4) + \frac{3}{4(4^4)} - \left(\sin(1) - \frac{3}{4(1^4)} \right) \\&= \sin(4) + \frac{3}{1024} - \left(\sin(1) - \frac{3}{4} \right) = \boxed{\sin(4) - \sin(1) - \frac{765}{1024}}\end{aligned}$$

$$\int_1^6 12x^3 - 9x^2 + 2 \, dx = \left(3x^4 - 3x^3 + 2x \right) \Big|_1^6 \quad \int_1^6 12x^3 - 9x^2 + 2 \, dx = 3252 - 2 = \boxed{3250}$$

$$\int_{-2}^1 5z^2 - 7z + 3 \, dz = \left(\frac{5}{3}z^3 - \frac{7}{2}z^2 + 3z \right) \Big|_{-2}^1 \quad \int_{-2}^1 5z^2 - 7z + 3 \, dz = \frac{7}{6} - \left(-\frac{100}{3} \right) = \boxed{\frac{69}{2}}$$

$$\int_1^4 \frac{8}{\sqrt{t}} - 12\sqrt{t^3} \, dt = \int_1^4 8t^{-\frac{1}{2}} - 12t^{\frac{3}{2}} \, dt = \left(16t^{\frac{1}{2}} - \frac{24}{5}t^{\frac{5}{2}} \right) \Big|_1^4 \quad \int_1^4 \frac{8}{\sqrt{t}} - 12\sqrt{t^3} \, dt = -\frac{608}{5} - \frac{56}{5} = \boxed{-\frac{664}{5}}$$

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3} \, dz = \int_1^2 \frac{1}{7} \frac{1}{z} + \frac{1}{4} z^{\frac{2}{3}} - \frac{1}{2} z^{-3} \, dz = \left(\frac{1}{7} \ln |z| + \frac{3}{20} z^{\frac{5}{3}} + \frac{1}{4} z^{-2} \right) \Big|_1^2$$

$$\int_1^2 \frac{1}{7z} + \frac{\sqrt[3]{z^2}}{4} - \frac{1}{2z^3} \, dz = \left(\frac{1}{7} \ln(2) + \frac{3}{20} \left(2^{\frac{5}{3}} \right) + \frac{1}{16} \right) - \left(\frac{1}{7} \ln(1) + \frac{2}{5} \right) = \boxed{\frac{1}{7} \ln(2) + \frac{3}{20} \left(2^{\frac{5}{3}} \right) - \frac{27}{80}}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2 \sec^2(w) - 8 \csc(w) \cot(w) dw = (2 \tan(w) + 8 \csc(w)) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\frac{16}{\sqrt{3}} + 2\sqrt{3} \right) - \left(16 + \frac{2}{\sqrt{3}} \right) = \boxed{\frac{14}{\sqrt{3}} + 2\sqrt{3} - 16}$$

$$\int_0^2 e^x + \frac{1}{x^2 + 1} dx = (e^x + \tan^{-1}(x)) \Big|_0^2 = (e^2 + \tan^{-1}(2)) - (e^0 + \tan^{-1}(0)) = \boxed{e^2 + \tan^{-1}(2) - 1}$$

$$\int_{-5}^{-2} 7e^y + \frac{2}{y} dy = (7e^y + 2 \ln|y|) \Big|_{-5}^{-2}$$

$$= (7e^{-2} + 2 \ln|-2|) - (7e^{-5} + 2 \ln|-5|) = \boxed{7(e^{-2} - e^{-5}) + 2(\ln(2) - \ln(5))}$$

Substitution Rule for Definite Integrals

$$\int_0^1 3(4x + x^4)(10x^2 + x^5 - 2)^6 dx$$

$$u = 10x^2 + x^5 - 2$$

$$du = (20x + 5x^4)dx = 5(4x + x^4)dx \quad \rightarrow \quad (4x + x^4)dx = \frac{1}{5}du$$
$$x = 0 : u = -2 \qquad \qquad \qquad x = 1 : u = 9$$

$$\int_0^1 3(4x + x^4)(10x^2 + x^5 - 2)^6 dx = \frac{3}{5} \int_{-2}^9 u^6 du$$

$$= \frac{3}{35} u^7 \Big|_{-2}^9 = \frac{3}{35} (4,782,969 - (-128)) = \boxed{\frac{14,349,291}{35}}$$

$$\int_0^{\frac{\pi}{4}} \frac{8 \cos(2t)}{\sqrt{9 - 5 \sin(2t)}} dt$$

$$u = 9 - 5 \sin(2t)$$

$$du = -10 \cos(2t) dt \quad \rightarrow \quad \cos(2t) dt = -\frac{1}{10} du$$

$$t = 0 : u = 9 \quad \quad \quad t = \frac{\pi}{4} : u = 4$$

$$\int_0^{\frac{\pi}{4}} \frac{8 \cos(2t)}{\sqrt{9 - 5 \sin(2t)}} dt = -\frac{8}{10} \int_9^4 u^{-\frac{1}{2}} du$$

$$= -\frac{8}{5} u^{\frac{1}{2}} \Big|_9^4 = -\frac{16}{5} - \left(-\frac{24}{5}\right) = \boxed{\frac{8}{5}}$$

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y} dy$$

$$u = 5 - 2y$$

$$\begin{aligned} du &= -2dy & \rightarrow & \quad dy = -\frac{1}{2}du \\ y = -4 : u &= 13 & y = -1 : u &= 7 \end{aligned}$$

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y} dy = -\frac{1}{2} \int_{13}^7 u^{\frac{1}{3}} + \frac{7}{u} du$$

$$\begin{aligned} &= \left(-\frac{1}{2} \left[\frac{3}{4}u^{\frac{4}{3}} + 7 \ln|u| \right] \right) \Big|_{13}^7 \\ &= -\frac{3}{8}7^{\frac{4}{3}} - \frac{7}{2} \ln|7| - \left(-\frac{3}{8}13^{\frac{4}{3}} - \frac{7}{2} \ln|13| \right) \\ &= \boxed{\frac{3}{8}(13^{\frac{4}{3}} - 7^{\frac{4}{3}}) + \frac{7}{2}(\ln(13) - \ln(7))} \end{aligned}$$

$$\int_{-2}^0 t\sqrt{3+t^2} + \frac{3}{(6t-1)^2} dt = \int_{-2}^0 t\sqrt{3+t^2} dt + \int_{-2}^0 \frac{3}{(6t-1)^2} dt$$

$$u = 3 + t^2$$

$$v = 6t - 1$$

$$\begin{aligned} du &= 2t \, dt & \rightarrow & \quad t \, dt = \frac{1}{2} du \\ t = -2 : u &= 7 & t = 0 : u &= 3 \end{aligned}$$

$$\begin{aligned} dv &= 6 \, dt & \rightarrow & \quad dt = \frac{1}{6} dv \\ t = -2 : v &= -13 & t = 0 : v &= -1 \end{aligned}$$

$$\int_{-2}^0 t\sqrt{3+t^2} + \frac{3}{(6t-1)^2} dt = \frac{1}{2} \int_7^3 u^{\frac{1}{2}} \, du + \frac{3}{6} \int_{-13}^{-1} v^{-2} \, dv$$

$$= \frac{1}{3} u^{\frac{3}{2}} \Big|_7^3 - \frac{1}{2} v^{-1} \Big|_{-13}^{-1} = \frac{1}{3} \left(3^{\frac{3}{2}} - 7^{\frac{3}{2}} \right) - \frac{1}{2} \left(-1 - \left(-\frac{1}{13} \right) \right) = \boxed{\frac{1}{3} \left(3^{\frac{3}{2}} - 7^{\frac{3}{2}} \right) + \frac{6}{13}}$$

$$\int_{-2}^1 (2-z)^3 + \sin(\pi z) [3+2\cos(\pi z)]^3 dz = \int_{-2}^1 (2-z)^3 dz + \int_{-2}^1 \sin(\pi z) [3+2\cos(\pi z)]^3 dz$$

$$u = 2 - z$$

$$v = 3 + 2\cos(\pi z)$$

$$du = -dz$$

$$\rightarrow$$

$$dz = -du$$

$$z = -2 : u = 4$$

$$z = 1 : u = 1$$

$$dv = -2\pi \sin(\pi z) dz$$

$$z = -2 : v = 5$$

$$\sin(\pi z) dz = -\frac{1}{2\pi} dv$$

$$z = 1 : v = 1$$

$$\int_{-2}^1 (2-z)^3 + \sin(\pi z) [3+2\cos(\pi z)]^3 dz = -\int_4^1 u^3 du - \frac{1}{2\pi} \int_5^1 v^3 dv$$

$$= -\frac{1}{4} u^4 \Big|_4^1 - \frac{1}{8\pi} v^4 \Big|_5^1$$

$$= -\frac{1}{4}(1-256) - \frac{1}{8\pi}(1-625) = \boxed{\frac{255}{4} + \frac{78}{\pi}}$$

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k du = ku + C \quad (\text{any number } k)$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
5. $\int \frac{du}{u} = \ln |u| + C$
6. $\int \sin u du = -\cos u + C$
7. $\int \cos u du = \sin u + C$
8. $\int \sec^2 u du = \tan u + C$
9. $\int \csc^2 u du = -\cot u + C$
10. $\int \sec u \tan u du = \sec u + C$
11. $\int \csc u \cot u du = -\csc u + C$
12. $\int \tan u du = -\ln |\cos u| + C$
 $= \ln |\sec u| + C$

13. $\int \cot u du = \ln |\sin u| + C$
 $= -\ln |\csc u| + C$
14. $\int e^u du = e^u + C$
15. $\int a^u du = \frac{a^u}{\ln a} + C \quad (a > 0, a \neq 1)$
16. $\int \sinh u du = \cosh u + C$
17. $\int \cosh u du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$
20. $\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C \quad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left(\frac{u}{a} \right) + C \quad (u > a > 0)$

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u du = \ln |\sec u + \tan u| + C$
2. $\int \csc u du = -\ln |\csc u + \cot u| + C$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$$

Basic Substitutions

- . $\int \frac{16x \, dx}{\sqrt{8x^2 + 1}}$; $\begin{bmatrix} u = 8x^2 + 1 \\ du = 16x \, dx \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2 + 1} + C$
- . $\int \frac{3 \cos x \, dx}{\sqrt{1 + 3 \sin x}}$; $\begin{bmatrix} u = 1 + 3 \sin x \\ du = 3 \cos x \, dx \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1 + 3 \sin x} + C$
- . $\int 3\sqrt{\sin v} \cos v \, dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v \, dv \end{bmatrix} \rightarrow \int 3\sqrt{u} \, du = 3 \cdot \frac{2}{3} u^{3/2} + C = 2(\sin v)^{3/2} + C$
- . $\int \cot^3 y \csc^2 y \, dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y \, dy \end{bmatrix} \rightarrow \int u^3(-du) = -\frac{u^4}{4} + C = -\frac{\cot^4 y}{4} + C$

Completing the Square

$$\begin{aligned} \cdot \int_1^2 \frac{8 \, dx}{x^2 - 2x + 2} &= 8 \int_1^2 \frac{dx}{1 + (x-1)^2}; \left[\begin{array}{l} u = x - 1 \\ du = dx \\ x = 1 \Rightarrow u = 0, x = 2 \Rightarrow u = 1 \end{array} \right] \rightarrow 8 \int_0^1 \frac{du}{1+u^2} = 8 [\tan^{-1} u]_0^1 \\ &= 8 (\tan^{-1} 1 - \tan^{-1} 0) = 8 \left(\frac{\pi}{4} - 0 \right) = 2\pi \end{aligned}$$

$$\begin{aligned} \cdot \int_2^4 \frac{2 \, dx}{x^2 - 6x + 10} &= 2 \int_2^4 \frac{dx}{(x-3)^2 + 1}; \left[\begin{array}{l} u = x - 3 \\ du = dx \\ x = 2 \Rightarrow u = -1, x = 4 \Rightarrow u = 1 \end{array} \right] \rightarrow 2 \int_{-1}^1 \frac{du}{u^2 + 1} = 2 [\tan^{-1} u]_{-1}^1 \\ &= 2 [\tan^{-1} 1 - \tan^{-1} (-1)] = 2 \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \pi \end{aligned}$$

$$\cdot \int \frac{dt}{\sqrt{-t^2 + 4t - 3}} = \int \frac{dt}{\sqrt{1 - (t-2)^2}}; \left[\begin{array}{l} u = t - 2 \\ du = dt \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} (t-2) + C$$

$$\cdot \int \frac{d\theta}{\sqrt{2\theta - \theta^2}} = \int \frac{d\theta}{\sqrt{1 - (\theta-1)^2}}; \left[\begin{array}{l} u = \theta - 1 \\ du = d\theta \end{array} \right] \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} (\theta-1) + C$$

Trigonometric Identities

- .
$$\begin{aligned}\int (\sec x + \cot x)^2 dx &= \int (\sec^2 x + 2 \sec x \cot x + \cot^2 x) dx = \int \sec^2 x dx + \int 2 \csc x dx + \int (\csc^2 x - 1) dx \\ &= \tan x - 2 \ln |\csc x + \cot x| - \cot x - x + C\end{aligned}$$
- .
$$\begin{aligned}\int (\csc x - \tan x)^2 dx &= \int (\csc^2 x - 2 \csc x \tan x + \tan^2 x) dx = \int \csc^2 x dx - \int 2 \sec x dx + \int (\sec^2 x - 1) dx \\ &= -\cot x - 2 \ln |\sec x + \tan x| + \tan x - x + C\end{aligned}$$
- .
$$\begin{aligned}\int \csc x \sin 3x dx &= \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x) dx = \int (\csc x)(2 \sin x \cos^2 x + \sin x \cos 2x) dx \\ &= \int (2 \cos^2 x + \cos 2x) dx = \int [(1 + \cos 2x) + \cos 2x] dx = \int (1 + 2 \cos 2x) dx = x + \sin 2x + C\end{aligned}$$
- .
$$\int (\sin 3x \cos 2x - \cos 3x \sin 2x) dx = \int \sin(3x - 2x) dx = \int \sin x dx = -\cos x + C$$

$$\cos^2 \theta + \sin^2 \theta = 1.$$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B\end{aligned}$$

$$\tan^2 x + 1 = \sec^2 x, \quad \tan^2 x = \sec^2 x - 1.$$

$$\begin{aligned}1 + \tan^2 \theta &= \sec^2 \theta. \\ 1 + \cot^2 \theta &= \csc^2 \theta.\end{aligned}$$

$$\begin{array}{ll}\tan \theta = \frac{\sin \theta}{\cos \theta} & \cot \theta = \frac{1}{\tan \theta} \\ \sec \theta = \frac{1}{\cos \theta} & \csc \theta = \frac{1}{\sin \theta}\end{array}$$

Double-Angle Formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

Improper Fractions

$$\cdot \int \frac{x}{x+1} dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$$

$$\cdot \int \frac{x^2}{x^2+1} dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$$

$$\cdot \int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt = \int \left[(4t - 1) + \frac{4}{t^2 + 4} \right] dt = 2t^2 - t + 2 \tan^{-1} \left(\frac{t}{2} \right) + C$$

$$\cdot \int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta - 5} \right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2} \ln|2\theta - 5| + C$$

Separating Fractions

$$\cdot \int \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$$

$$\cdot \int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}} dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$$

$$\cdot \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx = [\tan x + \sec x]_0^{\pi/4} = (1 + \sqrt{2}) - (0 + 1) = \sqrt{2}$$

$$\begin{aligned}\cdot \int_0^{1/2} \frac{2-8x}{1+4x^2} dx &= \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2} \right) dx = [\tan^{-1}(2x) - \ln|1+4x^2|]_0^{1/2} \\ &= (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2\end{aligned}$$

Eliminating Square Roots

- . $\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t} dt = \int_{\pi/2}^{\pi} \sqrt{2} |\cos t| dt;$
 $= -\sqrt{2} (\sin \pi - \sin \frac{\pi}{2}) = \sqrt{2}$

 $\rightarrow \int_{\pi/2}^{\pi} -\sqrt{2} \cos t dt = [-\sqrt{2} \sin t]_{\pi/2}^{\pi}$
- . $\int_{-\pi}^0 \sqrt{1 + \cos t} dt = \int_{-\pi}^0 \sqrt{2} |\cos \frac{t}{2}| dt;$
 $= 2\sqrt{2} [\sin 0 - \sin(-\frac{\pi}{2})] = 2\sqrt{2}$

 $\rightarrow \int_{-\pi}^0 \sqrt{2} \cos \frac{t}{2} dt = [2\sqrt{2} \sin \frac{t}{2}]_{-\pi}^0$
- . $\int_{-\pi}^0 \sqrt{1 - \cos^2 \theta} d\theta = \int_{-\pi}^0 |\sin \theta| d\theta;$
 $= 1 - (-1) = 2$

 $\rightarrow \int_{-\pi}^0 -\sin \theta d\theta = [\cos \theta]_{-\pi}^0 = \cos 0 - \cos(-\pi)$
- . $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta} d\theta = \int_{\pi/2}^{\pi} |\cos \theta| d\theta;$

 $\rightarrow \int_{\pi/2}^{\pi} -\cos \theta d\theta = [-\sin \theta]_{\pi/2}^{\pi} = -\sin \pi + \sin \frac{\pi}{2} = 1$

Integration by Parts

Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du$$

1. $u = \ln x, du = \frac{dx}{x}; dv = x \, dx, v = \frac{x^2}{2};$

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4} \right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

2. $u = \ln x, du = \frac{dx}{x}; dv = x^3 \, dx, v = \frac{x^4}{4};$

$$\int_1^e x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x \right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e = \frac{3e^4 + 1}{16}$$

3. $u = \tan^{-1} y, du = \frac{dy}{1+y^2}; dv = dy, v = y;$

$$\int \tan^{-1} y \, dy = y \tan^{-1} y - \int \frac{y \, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$$

4. $u = \sin^{-1} y, du = \frac{dy}{\sqrt{1-y^2}}; dv = dy, v = y;$

$$\int \sin^{-1} y \, dy = y \sin^{-1} y - \int \frac{y \, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$$

1. Determine f_{avg} for $f(x) = 8x - 3 + 5e^{2-x}$ on $[0, 2]$.

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \frac{1}{2-0} \int_0^2 8x - 3 + 5e^{2-x} dx = \frac{1}{2} \left(4x^2 - 3x - 5e^{2-x} \right) \Big|_0^2 = \boxed{\frac{1}{2} (5 + 5e^2)}$$

2. Determine f_{avg} for $f(x) = \cos(2x) - \sin(\frac{x}{2})$ on $[-\frac{\pi}{2}, \pi]$.

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \frac{1}{\pi - (-\frac{\pi}{2})} \int_{-\frac{\pi}{2}}^{\pi} \cos(2x) - \sin(\frac{x}{2}) dx = \frac{2}{3\pi} \left(\frac{1}{2} \sin(2x) + 2 \cos(\frac{x}{2}) \right) \Big|_{-\frac{\pi}{2}}^{\pi} = \boxed{-\frac{2\sqrt{2}}{3\pi}}$$

3. Find f_{avg} for $f(x) = 4x^2 - x + 5$ on $[-2, 3]$ and determine the value(s) of c in $[-2, 3]$ for which $f(c) = f_{\text{avg}}$.

If $f(x)$ is a continuous function on $[a, b]$ then there is a number c in $[a, b]$ such that,

$$\int_a^b f(x) dx = f(c)(b-a)$$

Step 1

First, we need to use the formula for the notes in this section to find f_{avg} .

$$f_{\text{avg}} = \frac{1}{3 - (-2)} \int_{-2}^3 4x^2 - x + 5 dx = \frac{1}{5} \left(\frac{4}{3}x^3 - \frac{1}{2}x^2 + 5x \right) \Big|_{-2}^3 = \boxed{\frac{83}{6}}$$

Step 2

Note that for the second part of this problem we are really just asking to find the value of c that satisfies the Mean Value Theorem for Integrals.

There really isn't much to do here other than solve $f(c) = f_{\text{avg}}$.

$$4c^2 - c + 5 = \frac{83}{6}$$

$$4c^2 - c - \frac{53}{6} = 0 \quad \Rightarrow \quad c = \frac{1 \pm \sqrt{1-4(4)\left(-\frac{53}{6}\right)}}{2(4)} = \frac{1 \pm \sqrt{\frac{427}{3}}}{2(4)} = \boxed{-1.3663, 1.6163}$$

In the first case we will use,

$$A = \int_a^b \left(\begin{array}{c} \text{upper} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{lower} \\ \text{function} \end{array} \right) dx, \quad a \leq x \leq b \quad (5)$$

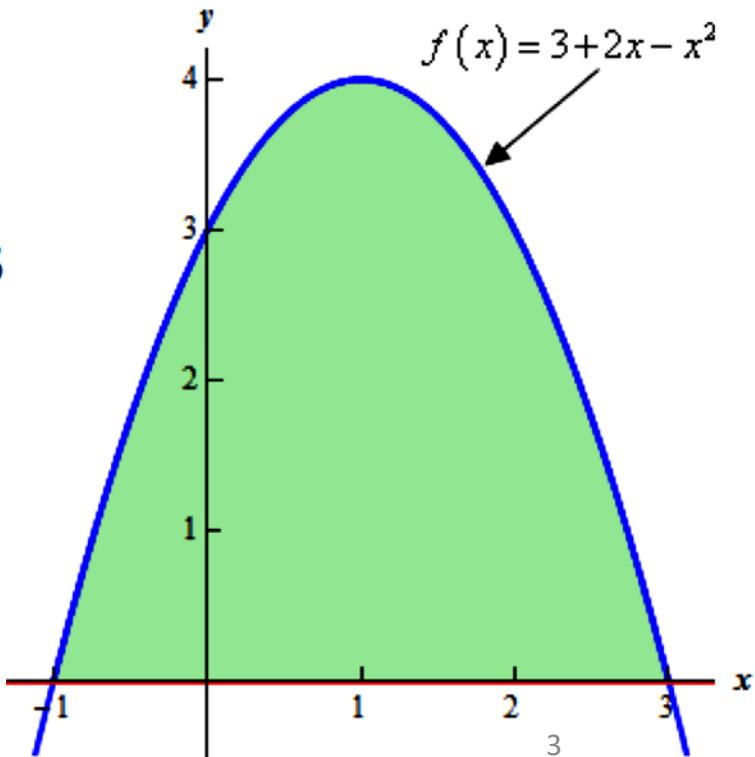
In the second case we will use,

$$A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy, \quad c \leq y \leq d \quad (6)$$

.. Determine the area below $f(x) = 3 + 2x - x^2$ and above the x-axis.

$$3 + 2x - x^2 = 0 \rightarrow -(x+1)(x-3) = 0 \rightarrow x = -1, \quad x = 3$$

$$A = \int_{-1}^3 3 + 2x - x^2 dx = \left(3x + x^2 - \frac{1}{3}x^3 \right) \Big|_{-1}^3 = \boxed{\frac{32}{3}}$$

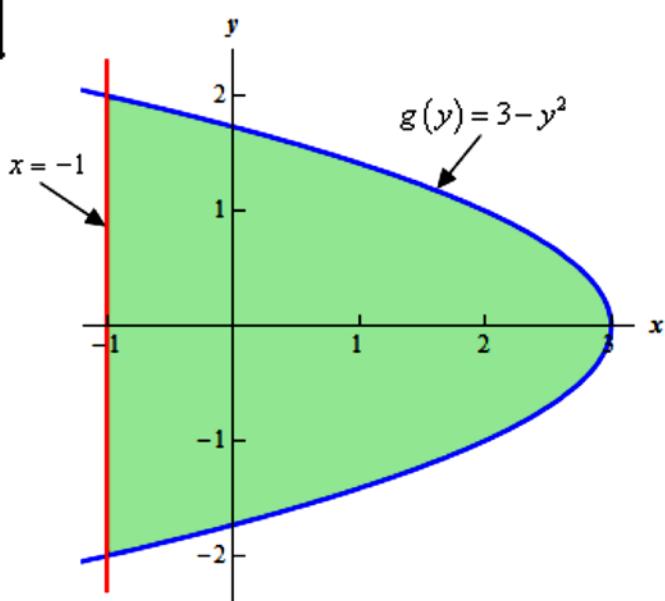


. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

$$3 - y^2 = -1 \quad \rightarrow \quad y^2 = 4 \quad \rightarrow \quad y = -2, \quad y = 2$$

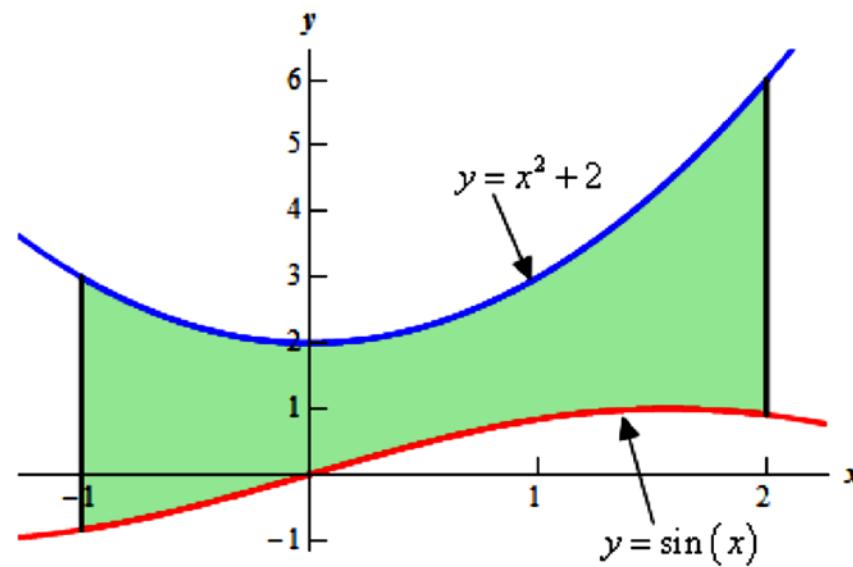
So, the limits on y are : $-2 \leq y \leq 2$.

$$A = \int_{-2}^2 [3 - y^2 - (-1)] dy = \int_{-2}^2 [4 - y^2] dy = \left(4y - \frac{1}{3}y^3 \right) \Big|_{-2}^2 = \boxed{\frac{32}{3}}$$



i. Determine the area of the region bounded by $y = x^2 + 2$, $y = \sin(x)$, $x = -1$ and $x = 2$.

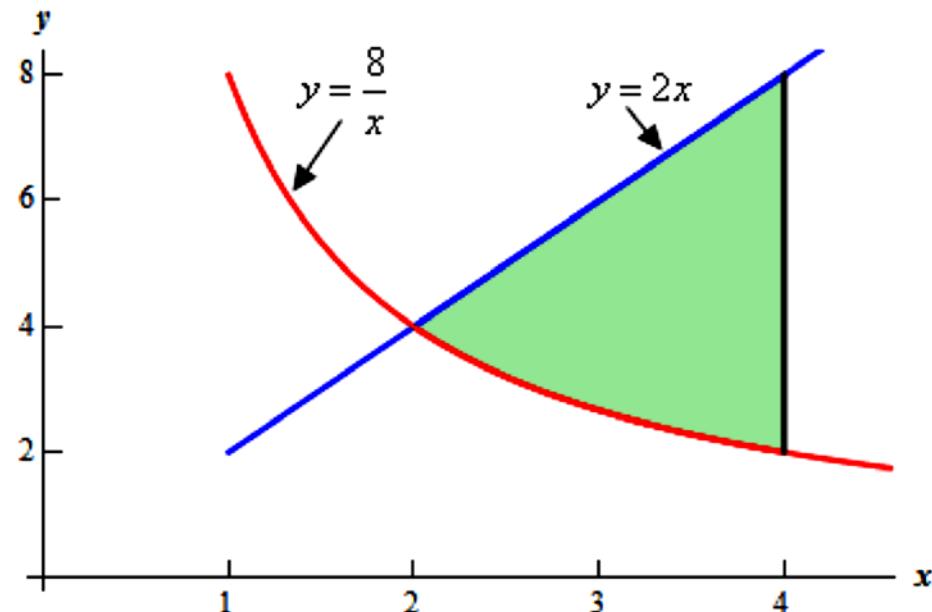
$$-1 \leq x \leq 2.$$



$$A = \int_{-1}^2 x^2 + 2 - \sin(x) dx = \left(\frac{1}{3}x^3 + 2x + \cos(x) \right) \Big|_{-1}^2 = [9 + \cos(2) - \cos(1)] = 8.04355$$

Determine the area of the region bounded by $y = \frac{8}{x}$, $y = 2x$ and $x = 4$.

$$\frac{8}{x} = 2x \rightarrow x^2 = 4 \rightarrow x = -2, x = 2$$

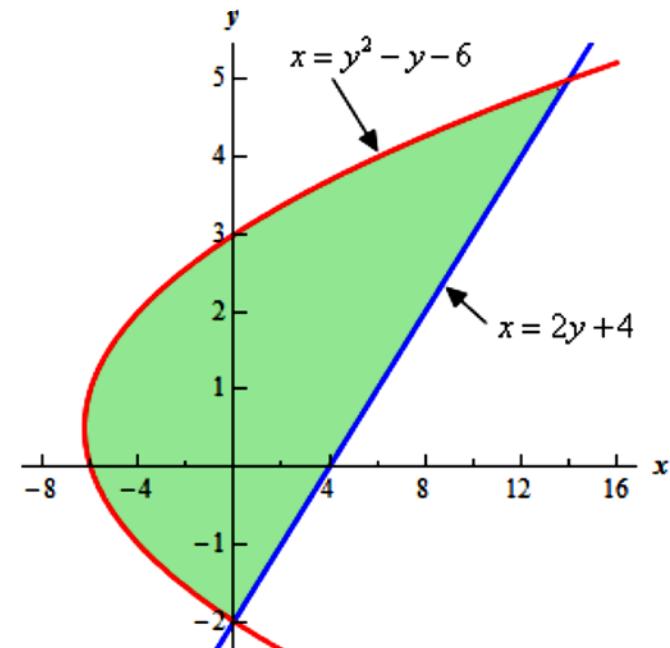


$$A = \int_2^4 2x - \frac{8}{x} dx = \left(x^2 - 8 \ln|x| \right) \Big|_2^4 = \boxed{12 - 8 \ln(4) + 8 \ln(2) = 6.4548}$$

. Determine the area of the region bounded by $x = y^2 - y - 6$ and $x = 2y + 4$.

$$y^2 - y - 6 = 2y + 4 \quad \rightarrow \quad y^2 - 3y - 10 = (y - 5)(y + 2) = 0 \quad \rightarrow \quad y = -2, \quad y = 5$$

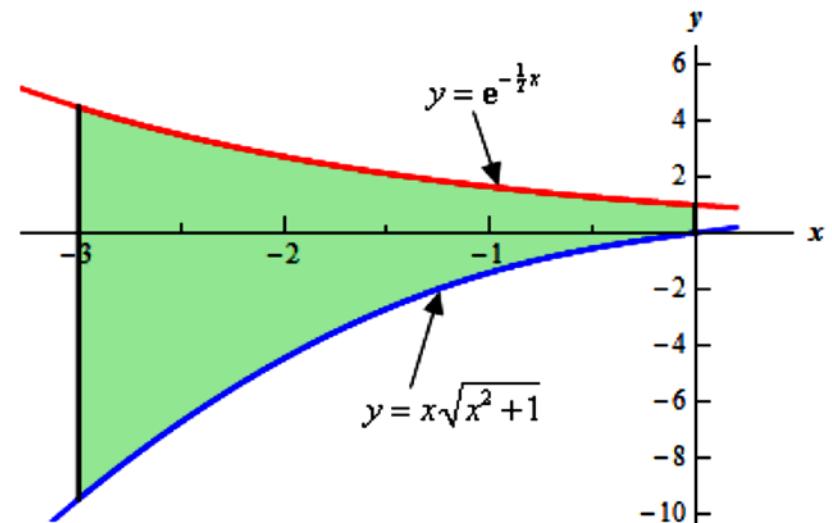
Therefore the limits on y are : $-2 \leq y \leq 5$.



$$A = \int_{-2}^5 2y + 4 - (y^2 - y - 6) dy = \int_{-2}^5 10 + 3y - y^2 dy = \left(10y + \frac{3}{2}y^2 - \frac{1}{3}y^3 \right) \Big|_{-2}^5 = \boxed{\frac{343}{6}}$$

Determine the area of the region bounded by $y = x\sqrt{x^2 + 1}$, $y = e^{-\frac{1}{2}x}$, $x = -3$ and the y -axis.

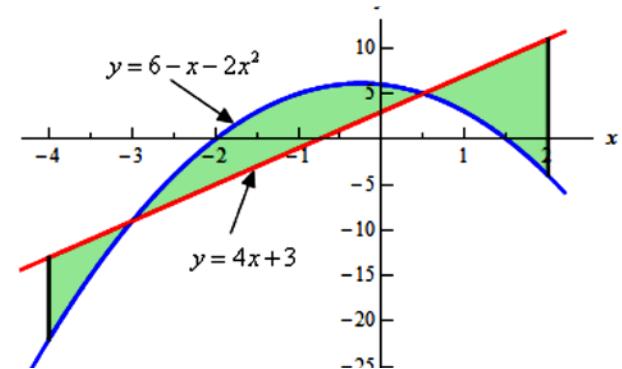
$$-3 \leq x \leq 0.$$



$$A = \int_{-3}^0 e^{-\frac{1}{2}x} - x\sqrt{x^2 + 1} dx = \left(-2e^{-\frac{1}{2}x} - \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} \right) \Big|_{-3}^0 = \boxed{-\frac{7}{3} + 2e^{\frac{3}{2}} + \frac{1}{3}10^{\frac{3}{2}} = 17.17097}$$

i. Determine the area of the region bounded by $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$ and $x = 2$.

$$6 - x - 2x^2 = 4x + 3 \quad \rightarrow \quad 2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0 \quad \rightarrow \quad x = -3, \quad x = \frac{1}{2}$$

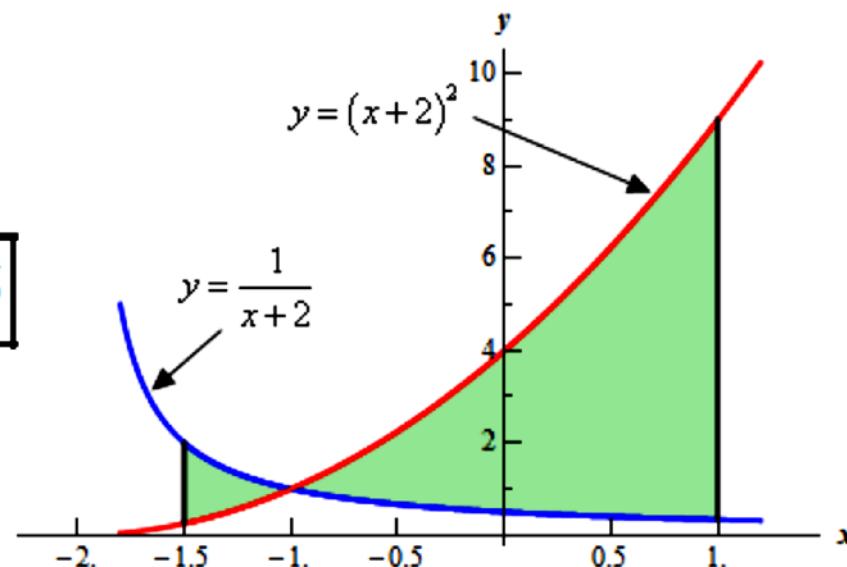


$$\begin{aligned}
 A &= \int_{-4}^{-3} 4x + 3 - (6 - x - 2x^2) dx + \int_{-3}^{\frac{1}{2}} 6 - x - 2x^2 - (4x + 3) dx + \int_{\frac{1}{2}}^2 4x + 3 - (6 - x - 2x^2) dx \\
 &= \int_{-4}^{-3} 2x^2 + 5x - 3 dx + \int_{-3}^{\frac{1}{2}} 3 - 5x - 2x^2 dx + \int_{\frac{1}{2}}^2 2x^2 + 5x - 3 dx \\
 &= \left(\frac{2}{3}x^3 + \frac{5}{2}x^2 - 3x \right) \Big|_{-4}^{-3} + \left(3x - \frac{5}{2}x^2 - \frac{2}{3}x^3 \right) \Big|_{-3}^{\frac{1}{2}} + \left(\frac{2}{3}x^3 + \frac{5}{2}x^2 - 3x \right) \Big|_{\frac{1}{2}}^2 \\
 &= \frac{25}{6} + \frac{343}{24} + \frac{81}{8} = \boxed{\frac{343}{12}}
 \end{aligned}$$

. Determine the area of the region bounded by $y = \frac{1}{x+2}$, $y = (x+2)^2$, $x = -\frac{3}{2}$, $x = 1$.

$$\frac{1}{x+2} = (x+2)^2 \rightarrow (x+2)^3 = 1 \rightarrow x+2 = \sqrt[3]{1} = 1 \rightarrow x = -1$$

$$\begin{aligned} A &= \int_{-\frac{3}{2}}^{-1} \frac{1}{x+2} - (x+2)^2 dx + \int_{-1}^1 (x+2)^2 - \frac{1}{x+2} dx \\ &= \left(\ln|x+2| - \frac{1}{3}(x+2)^3 \right) \Big|_{-\frac{3}{2}}^{-1} + \left(\frac{1}{3}(x+2)^3 - \ln|x+2| \right) \Big|_{-1}^1 \\ &= \left[-\frac{7}{24} - \ln\left(\frac{1}{2}\right) \right] + \left[\frac{26}{3} - \ln(3) \right] = \boxed{\frac{67}{8} - \ln\left(\frac{1}{2}\right) - \ln(3) = 7.9695} \end{aligned}$$



Critical Points

We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ exists and if either of the following are true.

$$f'(c) = 0$$

OR

$$f'(c) \text{ doesn't exist}$$

. Determine the critical points of $g(w) = 2w^3 - 7w^2 - 3w - 2$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$g'(w) = 6w^2 - 14w - 3$$

Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$6w^2 - 14w - 3 = 0 \quad \Rightarrow \quad w = \frac{14 \pm \sqrt{268}}{12} = \frac{7 \pm \sqrt{67}}{6}$$

. Determine the critical points of $Q(x) = (2 - 8x)^4 (x^2 - 9)^3$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$\begin{aligned} Q'(x) &= 4(2 - 8x)^3(-8)(x^2 - 9)^3 + (2 - 8x)^4(3)(x^2 - 9)^2(2x) \\ &= 2(2 - 8x)^3(x^2 - 9)^2[-16(x^2 - 9) + 3x(2 - 8x)] \\ &= 2(2 - 8x)^3(x^2 - 9)^2[-40x^2 + 6x + 144] = -4(2 - 8x)^3(x^2 - 9)^2[20x^2 - 3x - 72] \end{aligned}$$

$$-4(2 - 8x)^3(x^2 - 9)^2[20x^2 - 3x - 72] = 0$$

$$\begin{array}{lll} (2 - 8x)^3 = 0 & 2 - 8x = 0 & \Rightarrow x = \frac{1}{4} \\ (x^2 - 9)^2 = 0 & x^2 - 9 = 0 & \Rightarrow x = \pm 3 \\ 20x^2 - 3x - 72 = 0 & 20x^2 - 3x - 72 = 0 & \Rightarrow x = \frac{3 \pm \sqrt{3^2 - 4(20)(-72)}}{2(20)} = \frac{3 \pm \sqrt{5769}}{40} \end{array}$$

Determine the critical points of $f(z) = \frac{z+4}{2z^2+z+8}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$f'(z) = \frac{(1)(2z^2 + z + 8) - (z + 4)(4z + 1)}{(2z^2 + z + 8)^2} = \frac{-2z^2 - 16z + 4}{(2z^2 + z + 8)^2} = \frac{-2(z^2 + 8z - 2)}{(2z^2 + z + 8)^2}$$

$$z^2 + 8z - 2 = 0 \quad \Rightarrow \quad z = \frac{-8 \pm \sqrt{72}}{2} = -4 \pm 3\sqrt{2}$$

$$2z^2 + z + 8 = 0 \quad \Rightarrow \quad z = \frac{-1 \pm \sqrt{-63}}{4} = \frac{-1 \pm \sqrt{63}i}{4}$$

$$x = -4 \pm 3\sqrt{2}$$

. Determine the critical points of $r(y) = \sqrt[5]{y^2 - 6y}$.

Step 1

We'll need the first derivative to get the answer to this problem so let's get that.

$$r'(y) = \frac{1}{5}(2y-6)(y^2-6y)^{-\frac{4}{5}} = \frac{2y-6}{5(y^2-6y)^{\frac{4}{5}}}$$

$$2y-6=0 \quad \Rightarrow \quad y=3$$

$$y^2-6y=y(y-6)=0 \quad \Rightarrow \quad y=0,6$$

Linear Approximations

Find the linear approximation to $g(z) = \sqrt[4]{z}$ at $z = 2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

$$L(x) = f(a) + f'(a)(x - a)$$

$$g'(z) = \frac{1}{4}z^{-\frac{3}{4}}$$

$$g(2) = 2^{\frac{1}{4}}$$

$$g'(2) = \frac{1}{4}(2^{-\frac{3}{4}})$$

Step 2

Here is the linear approximation.

$$L(z) = 2^{\frac{1}{4}} + \frac{1}{4}(2^{-\frac{3}{4}})(z - 2)$$

Step 3

Finally, here are the approximations of the values along with the exact values.

$$L(3) = 1.33786$$

$$g(3) = 1.31607$$

$$L(10) = 2.37841$$

$$g(10) = 1.77828$$

Find the linear approximation to $f(t) = \cos(2t)$ at $t = \frac{1}{2}$. Use the linear approximation to approximate the value of $\cos(2)$ and $\cos(18)$. Compare the approximated values to the exact values.

Step 1

We'll need the derivative first as well as a couple of function evaluations.

$$f'(t) = -2 \sin(2t)$$

$$f\left(\frac{1}{2}\right) = \cos(1)$$

$$f'\left(\frac{1}{2}\right) = -2 \sin(1)$$

Step 2

Here is the linear approximation.

$$L(t) = \cos(1) - 2 \sin(1)\left(t - \frac{1}{2}\right) = 0.5403 - 1.6829\left(t - \frac{1}{2}\right)$$

$$L(1) = -0.301169$$

$$f(1) = -0.416147$$

$$L(9) = -13.7647$$

$$f(9) = 0.660317$$

Differentials

1. Compute dy and Δy for $y = e^{x^2}$ as x changes from 3 to 3.01.

Step 1

First let's get the actual change, Δy .

$$\Delta y = e^{3.01^2} - e^{3^2} = 501.927$$

Step 2

Next, we'll need the differential.

$$dy = 2x e^{x^2} dx$$

Step 3

As x changes from 3 to 3.01 we have $\Delta x = 3.01 - 3 = 0.01$ and we'll assume that $dx \approx \Delta x = 0.01$. The approximate change, dy , is then,

$$dy = 2(3)e^{3^2}(0.01) = 486.185$$

Don't forget to use the "starting" value of x (i.e. $x = 3$) for all the x 's in the differential.

. Compute dy and Δy for $y = x^5 - 2x^3 + 7x$ as x changes from 6 to 5.9.

Step 1

First let's get the actual change, Δy .

$$\Delta y = (5.9^5 - 2(5.9^3) + 7(5.9)) - (6^5 - 2(6^3) + 7(6)) = -606.215$$

Step 2

Next, we'll need the differential.

$$dy = (5x^4 - 6x^2 + 7)dx$$

Step 3

As x changes from 6 to 5.9 we have $\Delta x = 5.9 - 6 = -0.1$ and we'll assume that $dx \approx \Delta x = -0.1$. The approximate change, dy , is then,

$$dy = (5(6^4) - 6(6^2) + 7)(-0.1) = -627.1$$

Don't forget to use the "starting" value of x (i.e. $x = 6$) for all the x 's in the differential.