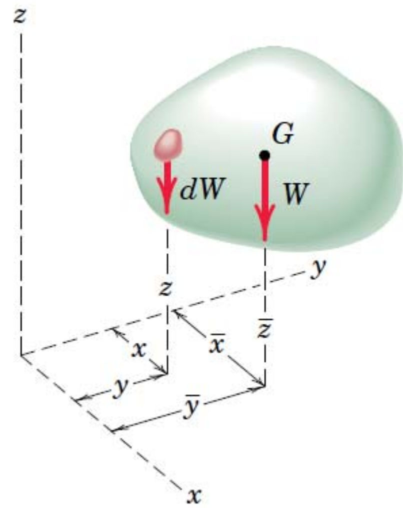
Center of Gravity and Centroid

CENTER OF MASS

Center of Gravity:

To determine mathematically the location of the center of gravity of any body, we apply the principle of moments (recall the moment chapter) to the parallel system of gravitational forces. The moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body. The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum W=∫ dW. If we apply the moment principle about the y-axis, for example, the moment about this axis of the elemental weight is x dW, and the sum of these moments for all elements of the body is ∫ x dW. This sum of moments must equal W $\bar{x}$, the moment of the sum. Thus, $\bar{x}$W = ∫ x dW.



With similar expressions for the other two components, we may express the coordinates of the center of gravity G as

$$\bar{x} = \frac{\int x\,dW}{W} \qquad \bar{y} = \frac{\int y\,dW}{W} \qquad \bar{z} = \frac{\int z\,dW}{W}$$

With the substitution of W= mg and dW= g dm, the expressions for the coordinates of the center of gravity become

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$

CENTROIDS OF LINES, AREAS, AND VOLUMES

1. Lines

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L}$$

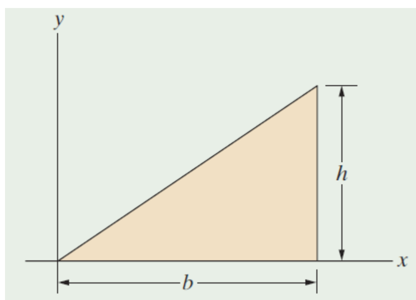
2. Areas

$$\bar{x} = \frac{\int x\,dA}{A} \qquad \bar{y} = \frac{\int y\,dA}{A} \qquad \bar{z} = \frac{\int z\,dA}{A}$$

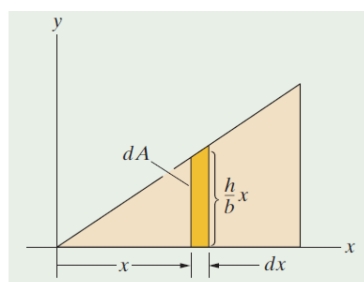
3. Volumes

$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V}$$

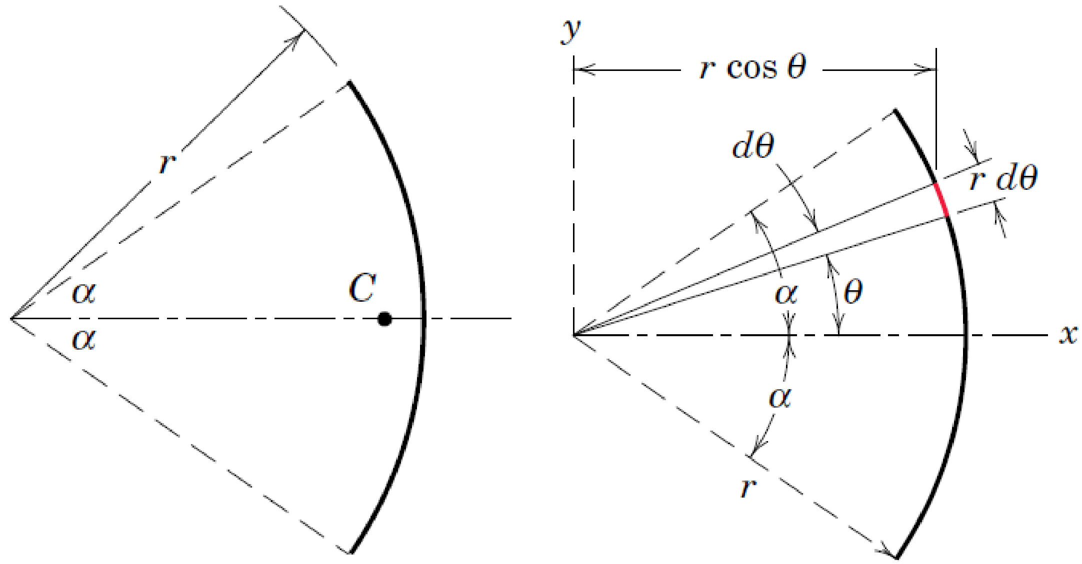
Example: Determine the x coordinate of the centroid of the triangular area.



$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^b x\left(\frac{h}{b}x\,dx\right)}{\int_0^b \frac{h}{b}x\,dx} = \frac{\frac{h}{b}\left[\frac{x^3}{3}\right]_0^b}{\frac{h}{b}\left[\frac{x^2}{2}\right]_0^b} = \frac{2}{3}b.$$



Example: Locate the centroid of a circular arc as shown in the figure.



Solution:

Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length dL= r dθ expressed in polar coordinates, and the x-coordinate of the element is r cosθ.

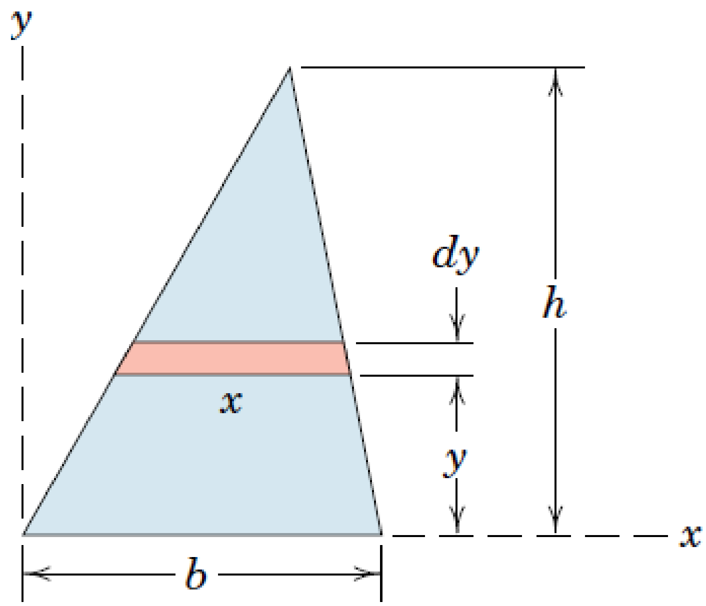
L= 2αr:

$$[L\bar{x} = \int x\, dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos\theta)\, r\, d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r \sin\alpha}{\alpha}$$

Example: Centroid of a triangular area. Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.



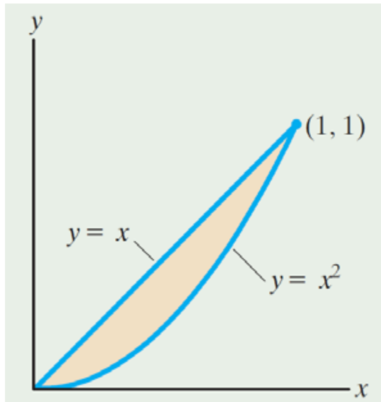
Solution:

The x-axis is taken to coincide with the base. A differential strip of area dA= x dy is chosen. By similar triangles x/ (h - y) = b/h.

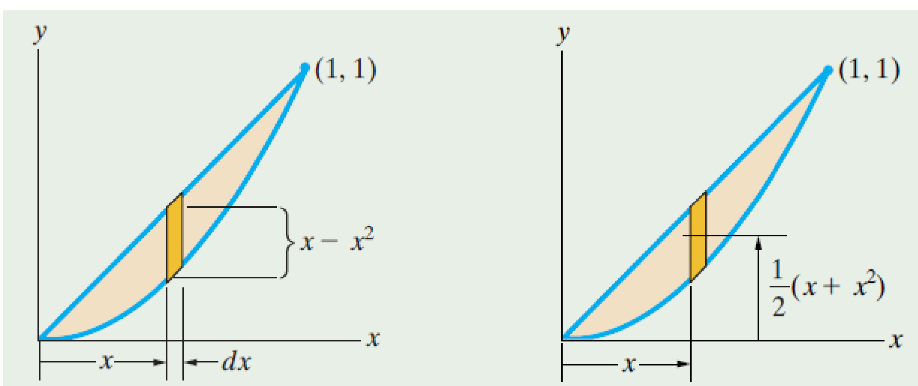
$$[A\bar{y} = \int y_c \, dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{h} dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3}$$

Example: Determine the centroid of the area.
Let dA be the vertical strip. The height of the strip is x-x^2
So, dA=(x-x^2) dx and the x coordinate of the centroid is:



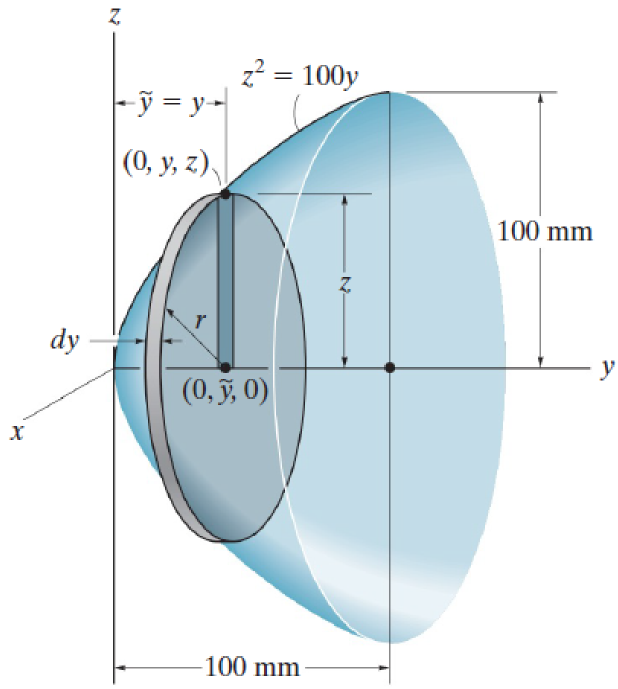
$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^1 x(x - x^2)\,dx}{\int_0^1 (x - x^2)\,dx} = \frac{\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1}{\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1} = \frac{1}{2}.$$



The y coordinate of the mid point of the strip is:
x^2+1/2(x-x2) = 1/2(x+x^2)

$$\bar{y} = \frac{\int_A y\,dA}{\int_A dA} = \frac{\int_0^1 \left[\frac{1}{2}(x + x^2)\right](x - x^2)\,dx}{\int_0^1 (x - x^2)\,dx} = \frac{\frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1}{\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1} = \frac{2}{5}$$

EXAMPLE: Locate the y centroid for the paraboloid of revolution,



Differential Element. An element having the shape of a thin disk is chosen. This element has a thickness dy, it intersects the generating curve at the arbitrary point (0, y, z), and so its radius is r = z.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)dy$, and its centroid is located at y.

$$\bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\, dy}{\int_0^{100\text{ mm}} (\pi z^2)\, dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\, dy}{100\pi \int_0^{100\text{ mm}} y\, dy} = 66.7\text{ mm}$$

Composite Bodies

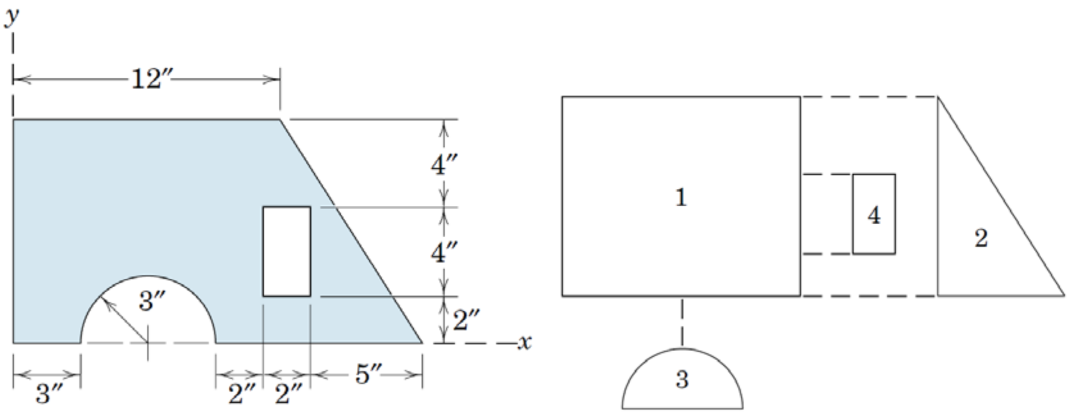
A composite body consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the weight and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. The formula for calculating the coordinates of the center of gravity for composite bodies:

$$\overline{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \overline{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \overline{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$

Centroid of Area

$$\bar{x} = \frac{\Sigma A x_c}{\Sigma A} \qquad \bar{y} = \frac{\Sigma A y_c}{\Sigma A}$$

Example: Locate the centroid of the shaded area.



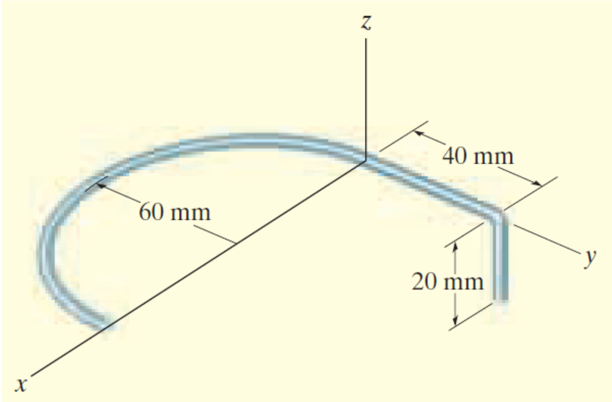
Solution. The composite area is divided into the four elementary shapes shown in the figure above. The centroid locations of all these shapes may be obtained from Table 1 at the end of this lecture. Note that the areas of the "holes" (parts 3 and 4) are taken as negative in the following table:

PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	−14.14	6	1.273	−84.8	−18
4	−8	12	4	−96	−32
TOTALS	127.9			959	650

$$\left[\overline{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \overline{X} = \frac{959}{127.9} = 7.50 \text{ in}$$

$$\left[\overline{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \overline{Y} = \frac{650}{127.9} = 5.08 \text{ in}$$

Example: Locate the centroid of the wire shown in the figure below.



Solution:

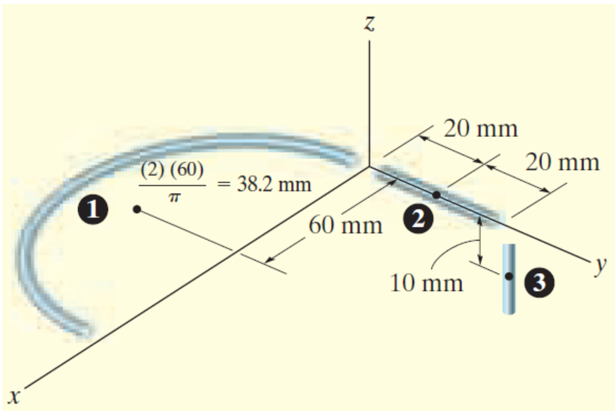
1. The wire is divided into three segments as shown in the figure below.
2. Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table 1.

Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	−38.2	0	11 310	−7200	0
2	40	0	20	0	0	800	0
3	20	0	40	−10	0	800	−200
	$\Sigma L = 248.5$				$\Sigma\tilde{x}L = 11\,310$	$\Sigma\tilde{y}L = -5600$	$\Sigma\tilde{z}L = -200$

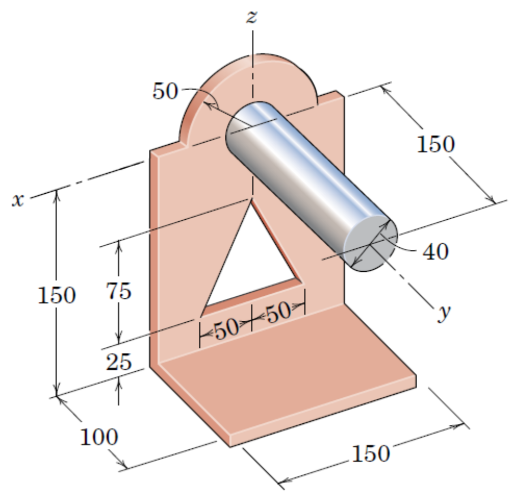
$$\bar{x} = \frac{\Sigma\tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm}$$

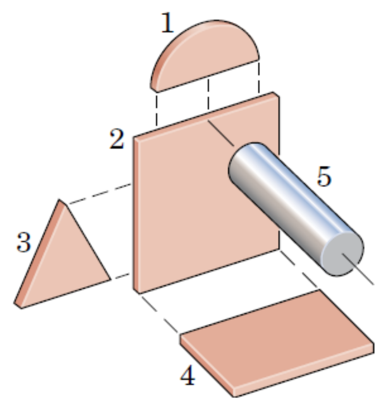
$$\bar{y} = \frac{\Sigma\tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm}$$

$$\bar{z} = \frac{\Sigma\tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm}$$



Example: Locate the center of mass of the bracket-and-shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m2. The material of the horizontal base has a mass of 40 kg/m2, and the steel shaft has a density of 7830 Kg/m3. (All dimensions in the figure are in millimeters)





Solution. The composite body may be considered to be composed of the five elements shown in the figure below. The triangular part will be taken as a negative mass. For the reference axes indicated, it is clear by symmetry that the x-coordinate of the center of mass is zero.

For Part 1 we have from table 1

$$\bar{z} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2 \text{ mm}$$

For Part 3 we know that the centroid of the triangular mass is one-third of its altitude above its base. Measurement from the coordinate axes becomes:

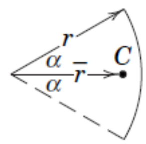
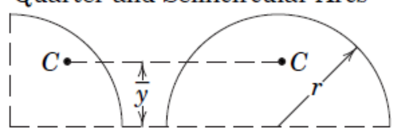
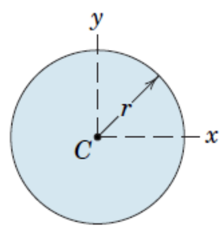
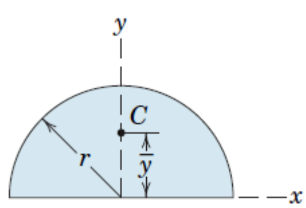
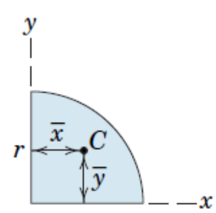
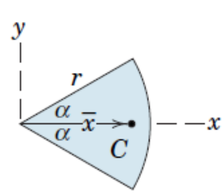
$$\bar{z} = -[150 - 25 - \tfrac{1}{3}(75)] = -100 \text{ mm}$$

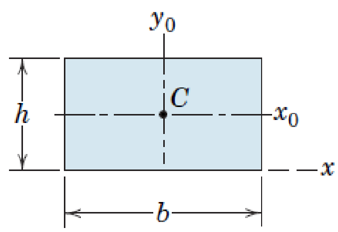
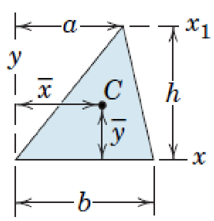
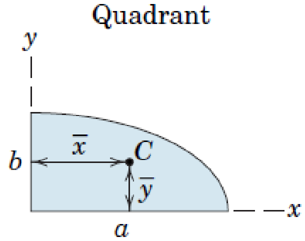
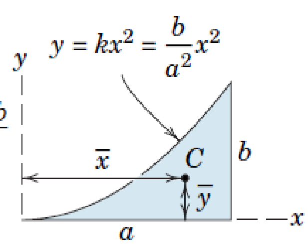
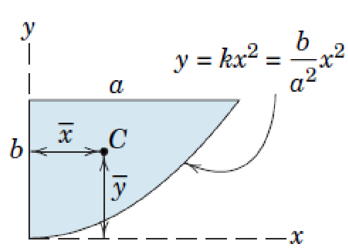
PART	m kg	$\bar{y}$ mm	$\bar{z}$ mm	$m\bar{y}$ kg·m	$m\bar{z}$ kg·mm
1	0.098	0	21.2	0	2.08
2	0.562	0	−75.0	0	−42.19
3	−0.094	0	−100.0	0	9.38
4	0.600	50.0	−150.0	30.0	−90.00
5	1.476	75.0	0	110.7	0
TOTALS	2.642			140.7	−120.73

$$\left[\bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m}\right] \qquad \bar{Y} = \frac{140.7}{2.642} = 53.3 \text{ mm}$$

$$\left[\bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}\right] \qquad \bar{Z} = \frac{-120.73}{2.642} = -45.7 \text{ mm}$$

TABLE 1 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3}\dfrac{r \sin \alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \dfrac{1}{2}\sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \dfrac{1}{2}\sin 2\alpha)$ $I_z = \dfrac{1}{2} r^4 \alpha$

Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^3}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

Moments of Inertia

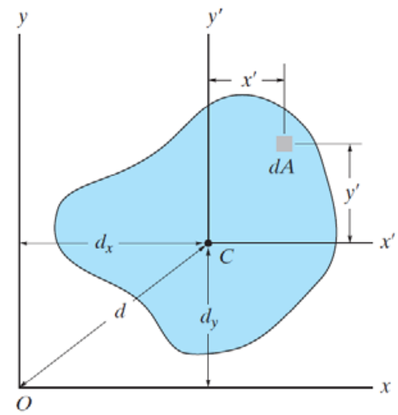
Moments of Inertia of an area

The area moment of inertia represents the second moment of the area about an axis. It is frequently used in formulas related to the strength and stability of structural members or mechanical elements.

For the entire area A the moments of inertia are determined by integration:

$$I_x = \int_A y^2\, dA$$

$$I_y = \int_A x^2\, dA$$



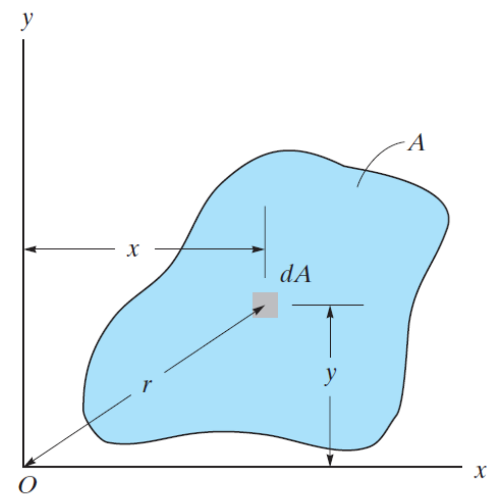
From the above formulations it is seen that Ix and Iy, will always be positive since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4, mm^4, or ft^4, in^4.

Parallel-Axis Theorem for an Area

The parallel-axis theorem can be used to find the moment of inertia of an area about any axis that is parallel to an axis passing through the centroid and about which the moment of inertia is known.

$$I_x = \bar{I}_{x'} + A d_y^2$$

$$I_y = \bar{I}_{y'} + A d_x^2$$

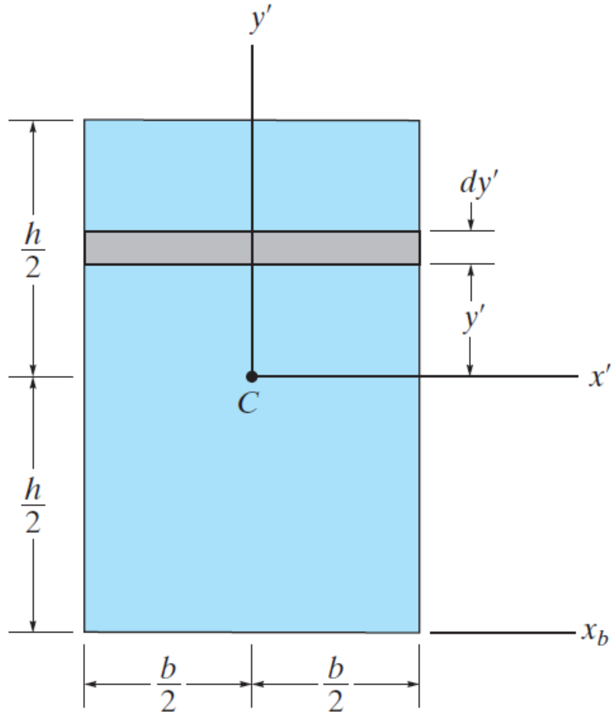


$\bar{I}_{x'}$ and $\bar{I}_{y'}$: The first integral represents the moment of inertia of the area about the centroidal axis.

A: The total area.

d_y and d_x: The distance between the parallel x’ and x and y’ and y respectively.

EXAMPLE: Determine the moment of inertia for the rectangular area shown in the figure below with respect to (a) the centroidal x′ axis, and (b) the axis x_b.



a. The differential element shown in Figure is chosen for integration. Because of its location and orientation, the entire element is at a distance y from the x axis. Here it is necessary to integrate from
$y = -h/2$ to $y = h/2$. Since $dA = b\,dy$, then:

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

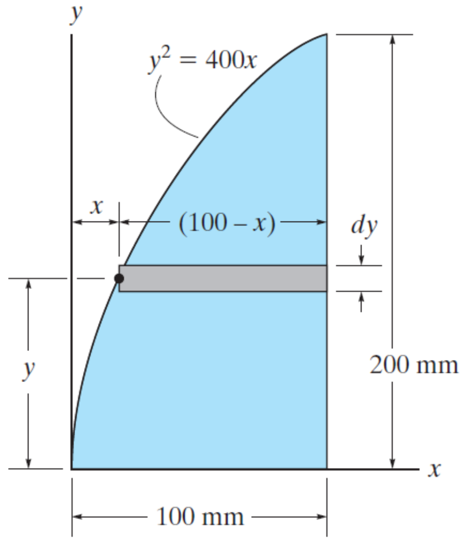
$$\bar{I}_{x'} = \frac{1}{12}bh^3$$

b. The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem.

$$I_{x_b} = \bar{I}_{x'} + A\,d_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

EXAMPLE: Determine the moment of inertia for the shaded area shown in the figure below about the x-axis.



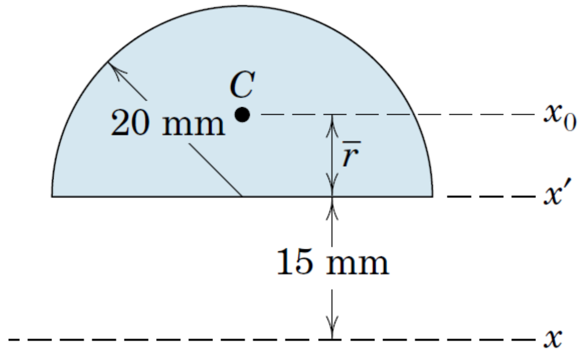
A differential element of area that is parallel to the x axis, as shown in Figure, is chosen for integration. Since this element has a thickness d_y and intersects the curve at the arbitrary point (x, y), its area is dA = (100 - x) d_y. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from y = 0 to y = 200 mm, yields

$$I_x = \int_A y^2\, dA = \int_0^{200\text{ mm}} y^2(100 - x)\, dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right) dy = \int_0^{200\text{ mm}} \left(100y^2 - \frac{y^4}{400}\right) dy$$

$$= 107(10^6)\text{ mm}^4$$

Example: Find the moment of inertia about the x-axis of the semicircular area.



Solution:

The moment of inertia of the semicircular area about the x′-axis is shown in table 1

$I_{x'}=\pi r^4/8$

$=\pi(20)^2/8=2\pi(10)^4$ mm^4

We obtain the moment of inertia I about the parallel centroidal axis x_0 next. Transfer is made through the distance $r= 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)$ mm by the parallel-axis theorem. Hence,

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

Finally, we transfer from the centroidal x_0-axis to the x-axis. Thus,

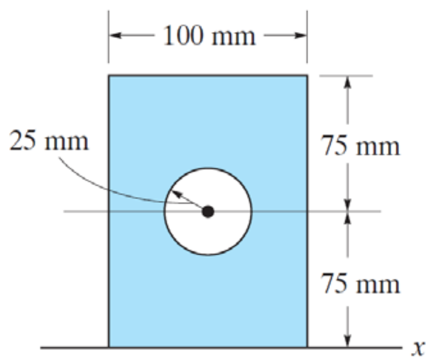
$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2$$

$= 1.755(10^4) + 34.7(10^4) = 36.4(10^4)$ mm^4

Moments of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the algebraic sum of the moments of inertia of all its parts.

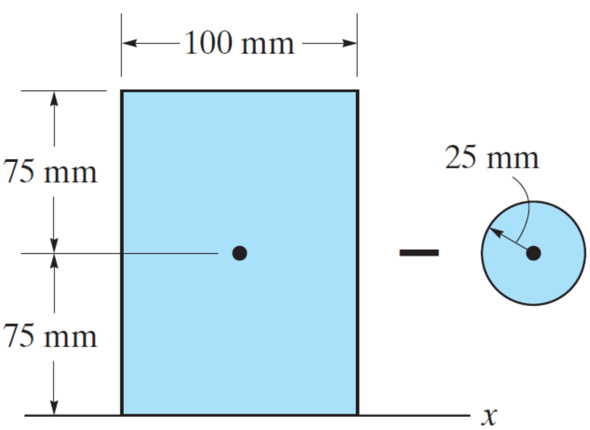
EXAMPLE: Determine the moment of inertia of the area shown in figure below about the x-axis.



Solution:

<u>Composite Parts</u>. The area can be obtained by subtracting the circle from the rectangle shown in figure below. The centroid of each area is located in the figure.

<u>Parallel-Axis Theorem</u>. The moments of inertia about the x-axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \pi r^4/4$; $I_x = bh^3/12$, found in table 1.



Circle

$$I_x = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ \text{mm}^4$$

Rectangle

$$I_x = \bar{I}_{x'} + A d_y^2$$

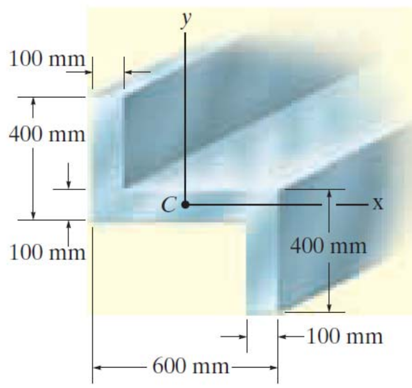
$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ \text{mm}^4$$

Summation. The moment of inertia for the area is therefore

$Ix = -11.4(10^6) + 112.5(10^6)$

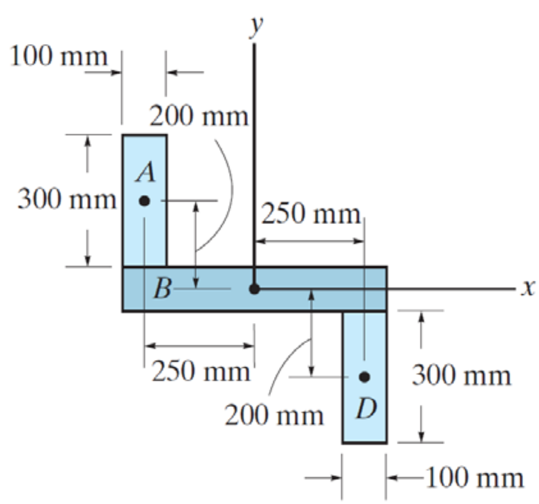
$= 101(10^6)\ \text{mm}^4$

EXAMPLE: Determine the moments of inertia for the cross-sectional area of the member shown in the figure below about the x and y centroidal axes.



Solution:

Composite Parts. The cross section can be subdivided into the three rectangular areas A, B, and D shown in the figure below. For the calculation, the centroid of each of these rectangles is located in the figure.



Parallel-Axis Theorem. From table 1, the moment of inertia of a rectangle about its centroidal axis is $I = 1/12\ bh^3$. Hence, using the parallel-axis theorem for rectangles A and D, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + A d_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\text{ mm}^4$$

$$I_y = \bar{I}_{y'} + A d_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\text{ mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\text{ mm}^4$$

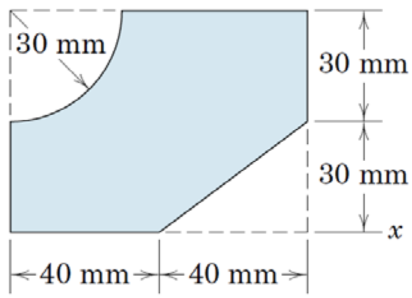
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\text{ mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\text{ mm}^4$$

$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\text{ mm}^4$$

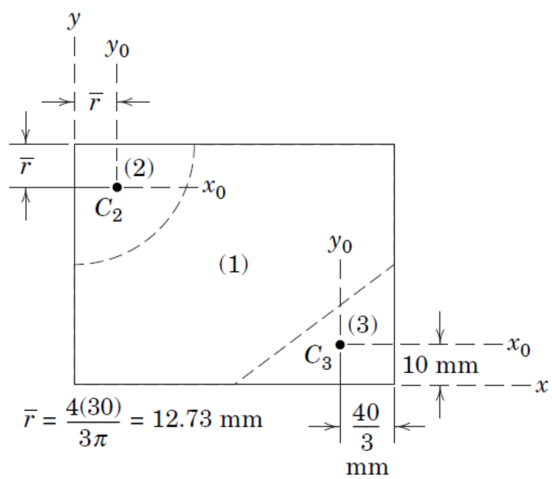
Example: Determine the moments of inertia about the x- and y-axes for the shaded area. Make direct use of the expressions given in Table 1 for the centroidal moments of inertia of the constituent parts.



Solution:

The given area is subdivided into the three subareas shown—a rectangular (1), a quarter-circular (2), and a triangular (3) area. Two of the subareas are "holes" with negative areas. Centroidal x_0 - y_0 axes are shown for areas (2) and (3), and the locations of centroids C_2 and C_3 are from Table 1.

The following table will facilitate the calculations.

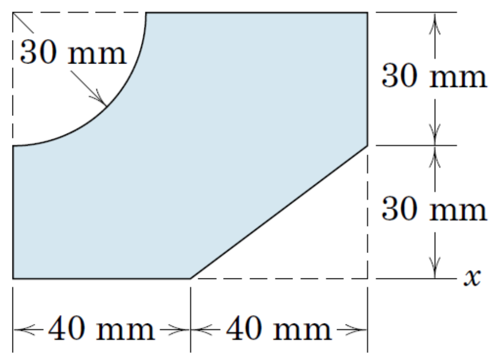


Part	A (mm)	dy (mm)	dx (mm)	A dy^2 (mm^3)	A dx^2 (mm^3)	Ix (mm^4)	Iy (mm^4)
1	80(60)	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	(60 – 12.73)	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$-\frac{1}{36}40(30)^3$	$-\frac{1}{36}(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

$$[I_x = \Sigma \bar{I}_x + \Sigma A d_y^2] \qquad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6)\ \text{mm}^4$$

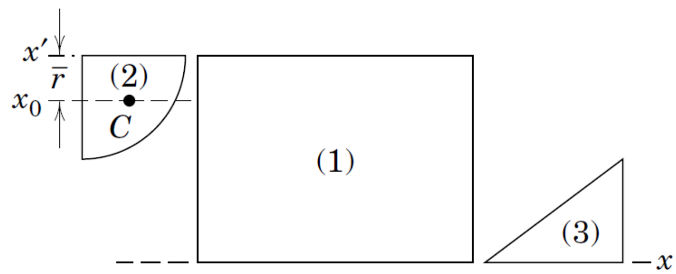
$$[I_y = \Sigma \bar{I}_y + \Sigma A d_x^2] \qquad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6)\ \text{mm}^4$$

Example: Calculate the moment of inertia about the x-axis for the shaded area shown. Wherever possible, make expedient use of tabulated moments of inertia.



Solution:

The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from table 1.



$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\ \text{mm}^4$$

The moment of inertia of the negative quarter-circular area about its base axis x′ is:

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\ \text{mm}^4$$

We now transfer this result through the distance

$$\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73\ \text{mm}$$

by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table 1 directly).

1.

$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$$
$$= -0.0445(10^6)\ \text{mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

2.

$$[I = \bar{I} + Ad^2] \qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$$
$$= -1.624(10^6)\ \text{mm}^4$$

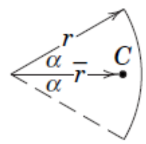
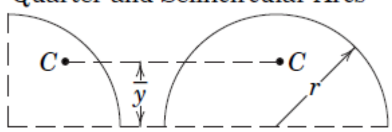
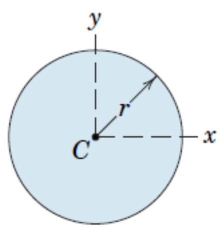
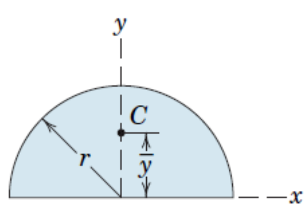
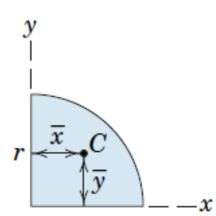
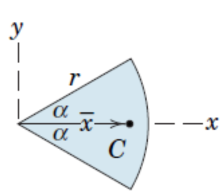
Finally, the moment of inertia of the negative triangular area (3) about its base, is

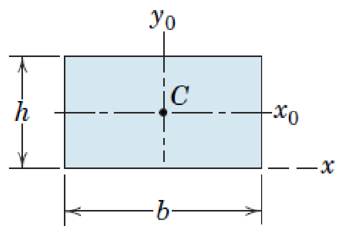
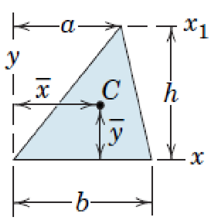
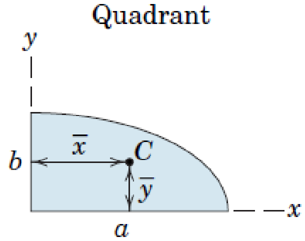
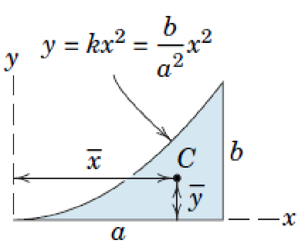
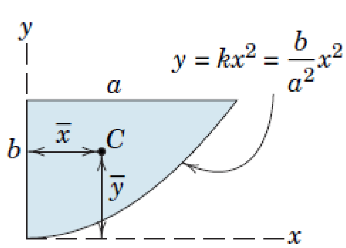
$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.90(10^6)\ \text{mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4$$

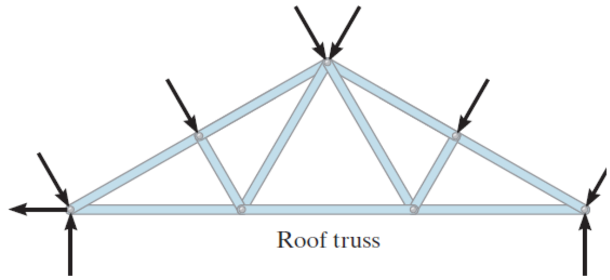
TABLE 1 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \frac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \frac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2} \sin 2\alpha)$ $I_y = \frac{r^4}{4}(\alpha + \frac{1}{2} \sin 2\alpha)$ $I_z = \frac{1}{2} r^4 \alpha$

Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}$, $\bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}$, $\bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

Structural Analysis (Trusses)

A truss is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts or metal bars.



Simple trusses are composed of triangular elements. The members are assumed to be pin connected at their ends and loads applied at the joints.

Assumptions for Design

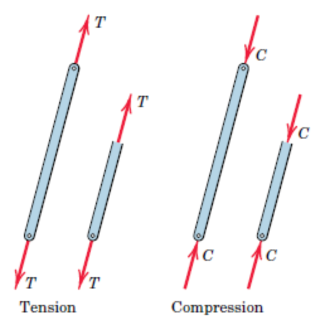
- ✓ All loadings are applied at the joints.
- ✓ The members are joined together by smooth pins.

Analysis of trusses:

In order to analyze or design a truss, it is necessary to determine the force in each of its members. There are two main methods:

1. Method of joints.
2. Method of sections.

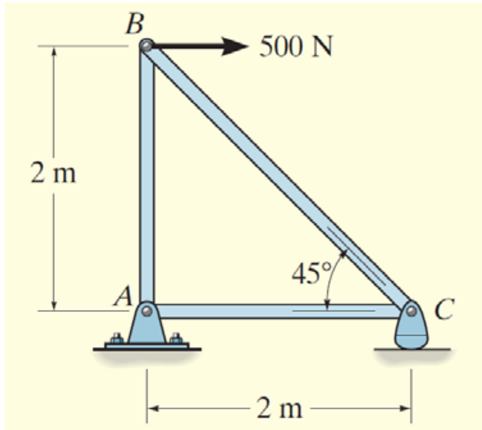
Each member of a truss is normally a straight link joining the two points of application of force. The two forces are applied at the ends of the member and are necessarily equal, opposite, and collinear for equilibrium. The member may be in tension or compression, as shown in the figure.



The Method of Joints

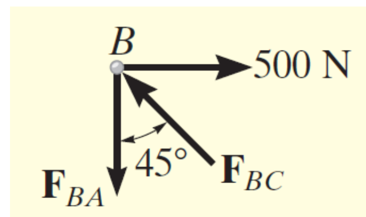
This method is based on the fact that if the entire truss is in equilibrium, then each of its joints is also in equilibrium. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations can then be used to obtain the member forces acting on each joint.

Example 1: Determine the force in each member of the truss shown in the figure below and indicate whether the members are in tension or compression.



Solution:

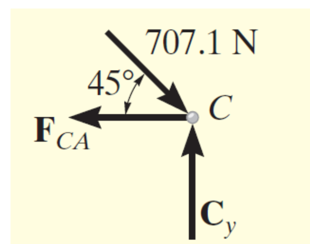
Since we should have no more than two unknown forces at the joint and at least one known force acting there, we will begin our analysis at joint B.



Joint B.

$$\xrightarrow{+} \Sigma F_x = 0; \quad 500\text{ N} - F_{BC}\sin 45^\circ = 0 \quad F_{BC} = 707.1\text{ N (C)}$$

$$+\uparrow \Sigma F_y = 0; \quad F_{BC}\cos 45^\circ - F_{BA} = 0 \quad F_{BA} = 500\text{ N (T)}$$

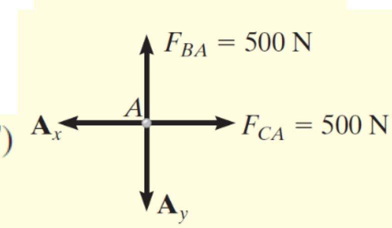


Since the force in member BC has been calculated, we can proceed to analyze joint C to determine the force in member CA and the support reaction at the rocker.

Joint C.

$$\xrightarrow{+} \Sigma F_x = 0; \quad -F_{CA} + 707.1\cos 45^\circ\text{ N} = 0 \quad F_{CA} = 500\text{ N (T)}$$

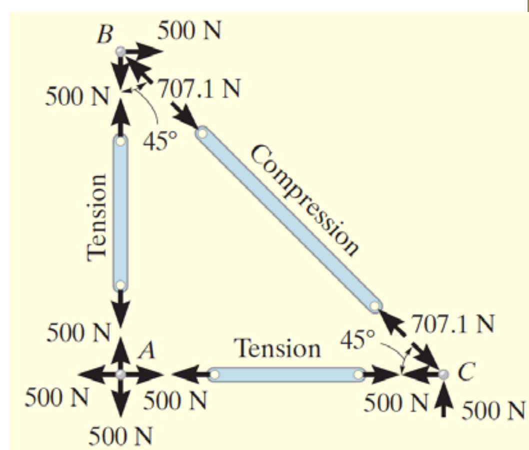
$$+\uparrow \Sigma F_y = 0; \quad C_y - 707.1\sin 45^\circ\text{ N} = 0 \quad C_y = 500\text{ N}$$



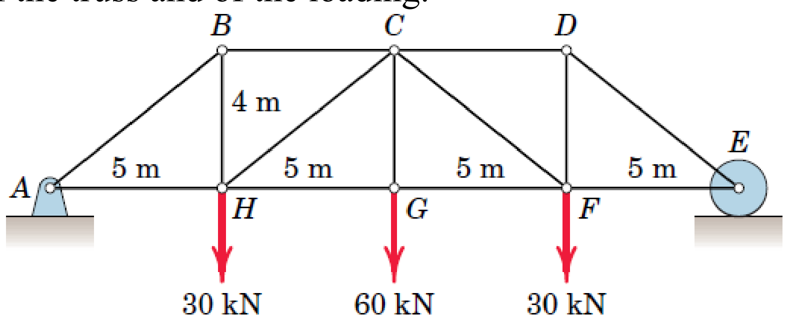
Joint A.

$$\xrightarrow{+} \Sigma F_x = 0; \quad 500\text{ N} - A_x = 0 \quad A_x = 500\text{ N}$$

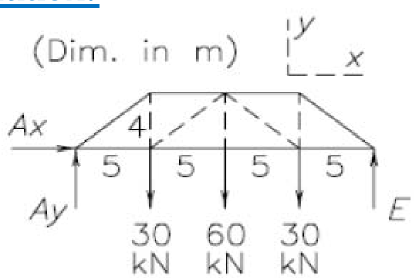
$$+\uparrow \Sigma F_y = 0; \quad 500\text{ N} - A_y = 0 \quad A_y = 500\text{ N}$$



Example 2: Determine the force in each member of the loaded truss. Make use of the symmetry of the truss and of the loading.



Solution:

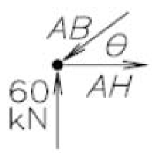


As a whole: $\Sigma F_x = 0 \Rightarrow A_x = 0$

$A_y = E = 60$ kN by

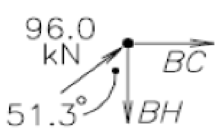
$\Sigma F_y = 0$ and symmetry.

Joint A: $\left(\theta = \tan^{-1}(4/5) = 38.7°\right)$



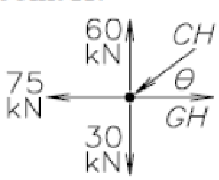
$$\begin{cases} \Sigma F_y = 0: 60 - AB\sin\theta = 0,\ \underline{AB = 96.0 \text{ kN } C} \\ \Sigma F_x = 0: AH - 96.0\cos\theta,\ \underline{AH = 75 \text{ kN } T} \end{cases}$$

Joint B:



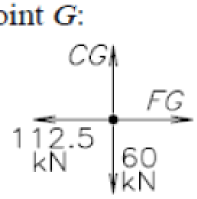
$$\begin{cases} \Sigma F_x = 0: BC + 96.0\sin 51.3° = 0,\ \underline{BC = -75 \text{ kN } (C)} \\ \Sigma F_y = 0: -BH + 96.0\cos 51.3° = 0,\ \underline{BH = 60 \text{ kN } T} \end{cases}$$

Joint H:



$$\begin{cases} \Sigma F_y = 0: -CH\sin\theta + 30 = 0,\ \underline{CH = 48.0 \text{ kN } C} \\ \Sigma F_x = 0: 48.0\cos\theta + GH - 75 = 0,\ \underline{GH = 112.5 \text{ kN } T} \end{cases}$$

Joint G:



$\Sigma F_y = 0 \Rightarrow CG = 60$ kN T

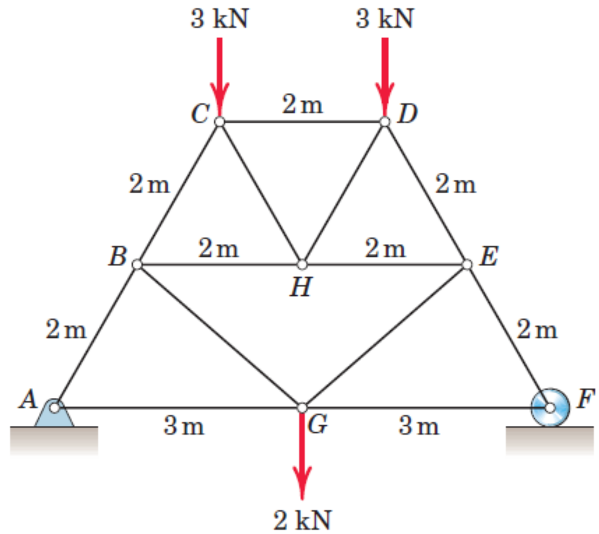
By symmetry:

$FG = 112.5$ kN T, $CF = 48.0$ kN C

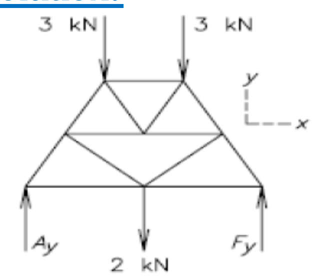
$CD = 75$ kN C, $DF = 60$ kN T

$\underline{EF = 75 \text{ kN } T,\ DE = 96.0 \text{ kN } C}$

Example 3: Determine the forces in members BC and BG of the loaded truss.

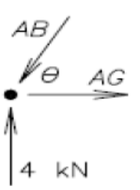


Solution:



By symmetry, $A_y = F_y = 4$ kN

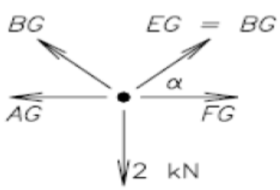
Joint A:



$$\theta = \cos^{-1}\frac{1}{2} = 60°$$

$$\begin{cases} \Sigma F_y = 0: & 4\text{ kN} - AB\sin 60° = 0,\ AB = 4.62\text{ kN } C \\ \Sigma F_x = 0: & AG - 4.62\cos 60° = 0,\ AG = 2.31\text{ kN } T \end{cases}$$

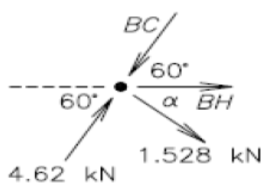
Joint G:



$$\alpha = \tan^{-1}\frac{2\sin 60°}{2} = 40.9°$$

$$\Sigma F_y = 0: \ 2BG\sin 40.9° - 2 = 0,\ \underline{BG = 1.528\text{ kN } T}$$

Joint B:



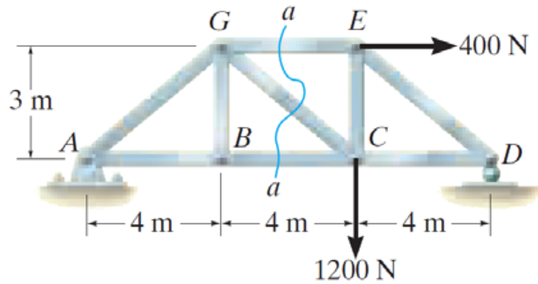
$$\Sigma F_y = 0: \ 4.62\sin 60° - BC\sin 60° - 1.528\sin 40.9° = 0$$

$$\underline{BC = 3.46\text{ kN } C}$$

The Method of Sections

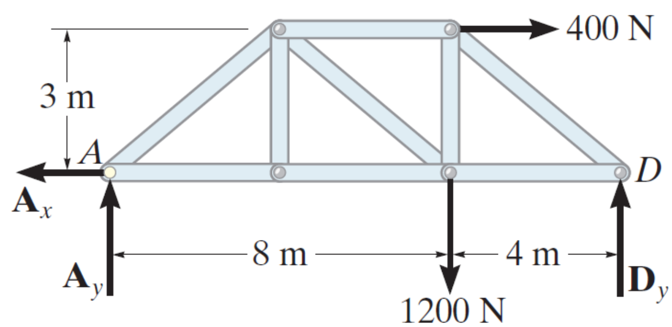
When we need to find the force in only a few members of a truss, we can analyze the truss using the method of sections. It is based on the principle that if the truss is in equilibrium then any segment of the truss is also in equilibrium.

Example: Determine the force in members GE, GC, and BC of the truss shown in the figure below. Indicate whether the members are in tension or compression.



Solution:

Section a-a in the figure has been chosen since it cuts through the three members whose forces are to be determined. In order to use the method of sections, however, it is first necessary to determine the external reactions at A or D. Why? A free-body diagram of the entire truss is shown in the figure. applying the equations of equilibrium, we have:



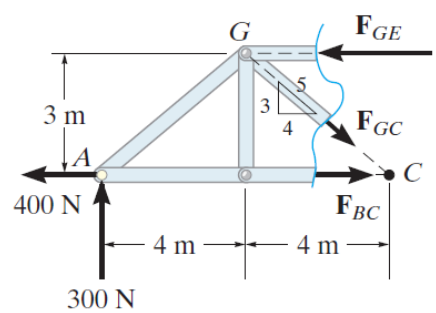
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 400\text{ N} - A_x = 0 \qquad A_x = 400\text{ N}$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad -1200\text{ N}(8\text{ m}) - 400\text{ N}(3\text{ m}) + D_y(12\text{ m}) = 0$$

$$D_y = 900\text{ N}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 1200\text{ N} + 900\text{ N} = 0 \qquad A_y = 300\text{ N}$$

Free-Body Diagram. For the analysis, the free-body diagram of the left portion of the sectioned truss will be used, since it involves the least number of forces.



Equations of Equilibrium. Summing moments about point G eliminates FGE and FGC and yields a direct solution for FBC.

$$\circlearrowleft +\Sigma M_G = 0; \quad -300 \text{ N}(4 \text{ m}) - 400 \text{ N}(3 \text{ m}) + F_{BC}(3 \text{ m}) = 0$$

$$F_{BC} = 800 \text{ N} \quad (\text{T})$$

In the same manner, by summing moments about point C we obtain a direct solution for FGE.

$$\circlearrowleft +\Sigma M_C = 0; \quad -300 \text{ N}(8 \text{ m}) + F_{GE}(3 \text{ m}) = 0$$

$$F_{GE} = 800 \text{ N} \quad (\text{C})$$

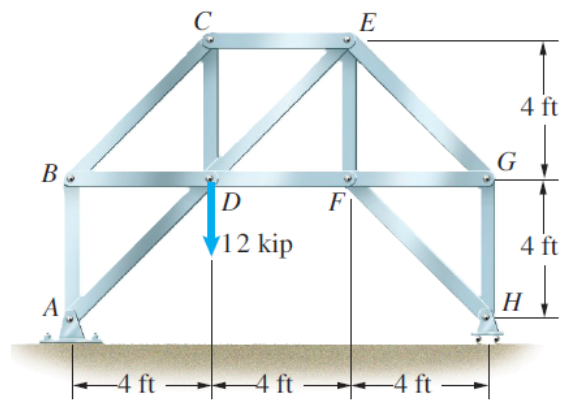
Since FBC and FGE have no vertical components, summing forces in the y direction directly yields FGC, i.e.,

$$+\uparrow \Sigma F_y = 0; \qquad 300 \text{ N} - \tfrac{3}{5} F_{GC} = 0$$

$$F_{GC} = 500 \text{ N} \quad (\text{T})$$

NOTE: Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example, MC = 0 requires F_{GE} to be compressive because it must balance the moment of the 300-N force about C.

Example: Use the method of sections to determine the axial forces in member CE, DE, and DF.



Solution:

The free-body diagrams for the entire structure and the section to the right of the cut are shown.

From the entire structure:

ΣM_A = -12 kip (4 ft) + H (12 ft) = 0

H = 4 kips

Using the section to the right of the cut, we have

ΣM_E: H(4 ft) – T_{DF} (4 ft) = 0

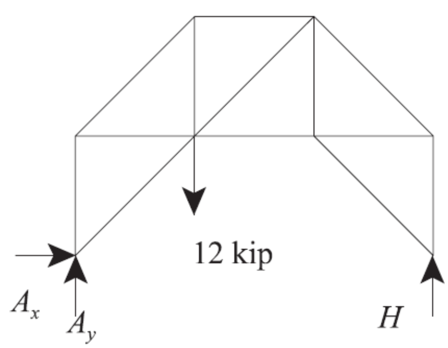
T_{DF} = 4 kips

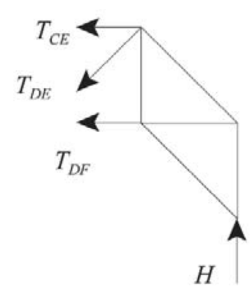
ΣM_D : H (8 ft) + T_{CE} (4 ft) = 0

T_{CE} = -8 kips

ΣFy : H - T_{DE} sin 45° = 0

T_{DE} = 5.66 kips





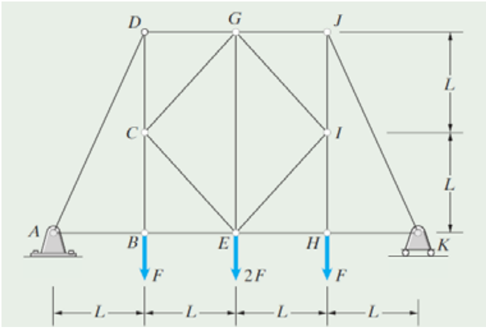
Therefore, we have:

DF = 4 kips (T)

CE = 8 kips (C)

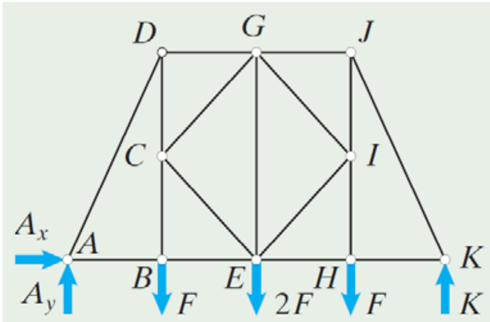
DE = 5.66 kips (T)

Example: Determine the axial forces in members DG and BE of the truss.



Solution:

We cannot obtain a section that involves cutting members DG and BE without cutting more than three members. However, cutting members DG, BE, CD, and BC results in a section with which we can determine the axial forces in members DG and BE.



From the free-body diagram of the entire truss as shown in figure:

$A_x = 0$

From symmetry:

$A_y = K = 2F$

OR:

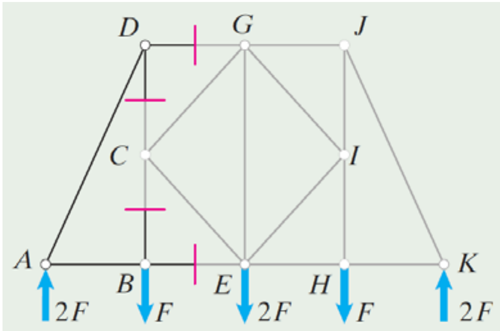
$\Sigma M_A = -LF - (2L)(2F) - (3L)F + (4L)K = 0$

$K = 2F$

$\Sigma F_y = A_y + K - F - 2F - F = 0$

$= A_y + 2F - F - 2F - F = 0$

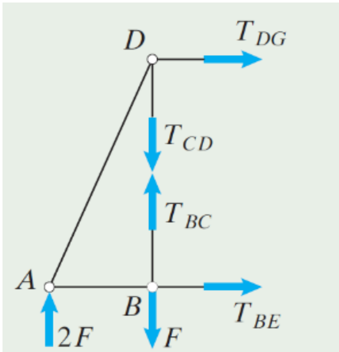
$A_y = 2F$



In the figure, we obtain a section by cutting members DG, CD, BC, and BE. Because the lines of action of T_{BE}, T_{BC}, and T_{CD} pass through point B, we can determine T_{DG} by summing moments about B:

$M_B = -L(2F) - (2L)T_{DG} = 0$

$T_{DG} = -F$



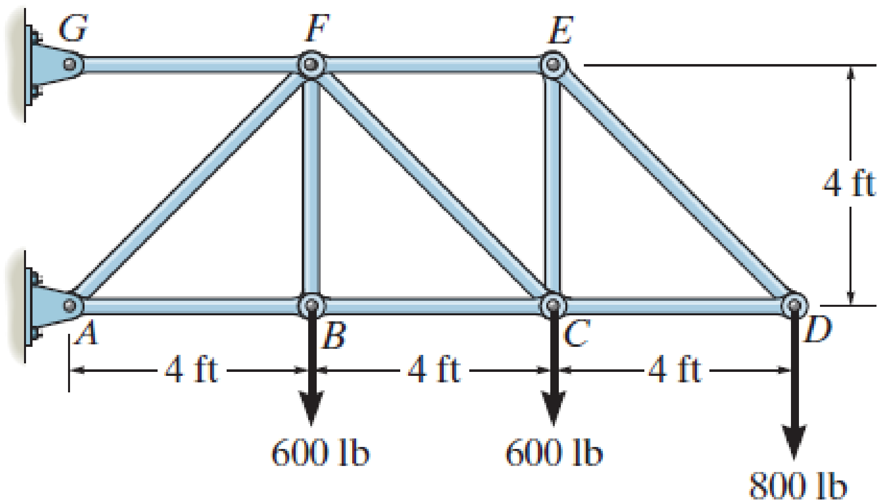
Then, from the equilibrium equation:

$\Sigma F_x = T_{DG} + T_{BE} = 0$

We see that $T_{BE} = -T_{DG} = F$ Member DG is in compression, and member BE is in tension.

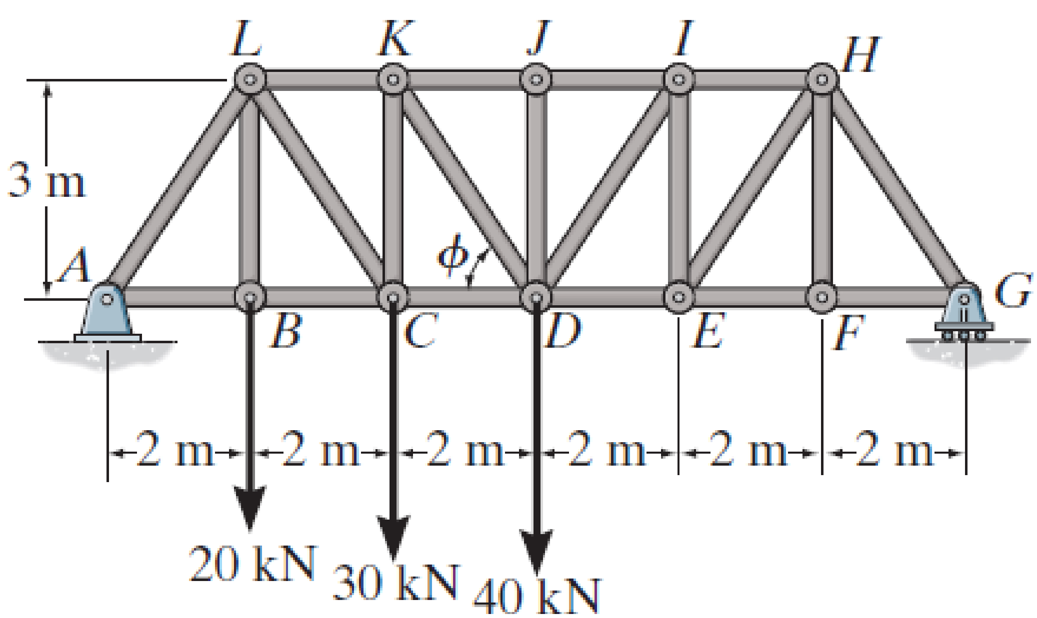
Homework

1. Determine the force in members BC, CF, and FE. State if the members are in tension or compression.



2. Determine the force in members:
    - ✓ LK, KC, and CD
    - ✓ KJ, KD, and CD

State if the members are in tension or compression.



3. For the truss shown in figure below:
   - ➢ Determine the force in members BC, HC, and HG. After the truss is sectioned, use a single equation of equilibrium for the calculation of each force. State if these members are in tension or compression.
   - ➢ Determine the force in members CD, CF, and CG and state if these members are in tension or compression.