Eigenvalues and Eigenvectors Questions with Solutions

Examples and questions on the eigenvalues and eigenvectors of square matrices along with their solutions are presented. The properties of the eigenvalues and their corresponding eigenvectors are also discussed and used in solving questions.

Definition of Eigenvalues and Eigenvectors

Let A be an n × n square matrix. If there exist a non trivial (not all zeroes) column vector X solution to the matrix equation $A X = \lambda X$; where λ is a scalar, then X is called the eigenvector of matrix A and the corresponding value of λ is called the eigenvalue of matrix A.

Let us rewrite the matrix equation in standard form:

$A X - \lambda X = 0$

Let I be the n × n identity matrix and substitute X by $I X$ in the above equation

$A X - \lambda I X = 0$

Rewrite as

$(A - \lambda I) X = 0$

The above matrix equation has non trivial solutions if and only if the determinant of the matrix $(A - \lambda I)$ is equal to zero. Det $(A - \lambda I) = 0$ is called the characteristic equation of A.

If A is an n by n matrix, when $(A - \lambda I)$ is expanded, it is a polynomial of degree n and therefore $(A - \lambda I)$ is called the characteristic polynomial of A.

Examples with Solutions on Eigenvalues and Eigenvectors

Example 1

Find all eigenvalues and eigenvectors of matrix

$$\begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix}$$

Solution

We first calculate the eigenvalues and then the eigenvectors.

Find Eigenvalues

We substitute in the matrix $A - \lambda I$ as follows

$$A - \lambda I = \begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2-\lambda & 1 \\ 12 & -3-\lambda \end{bmatrix}$$

Solve the equation $Det(A - \lambda I) = 0$

Calculate the determinant and substitute in the above equation

$$(-2-\lambda)(-3-\lambda) - (1)(12) = 0$$

Expand and rewrite as

$\lambda^2 + 5\lambda - 6 = 0$

Solve the above quadratic equation to find two eigenvalues

$\lambda = 1$ and $\lambda = -6$

Find Eigenvectors

Eigenvectors for $\lambda = 1$

Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$

$$\left(\begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) X = 0$$

Simplify the above

$$\begin{bmatrix} -3 & 1 \\ 12 & -4 \end{bmatrix} X = 0$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and rewrite the above matrix equation as

$$\begin{bmatrix} -3 & 1 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Multiply the top equation by 4 and add it to the second equation and rewrite the system of

equations as follows

$$\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

A solution for x_2 could be written as $x_2 = t$ where t takes all real numbers.

Use the top equation $-3x_1 + x_2 = 0$

to find x_1 as follows

$$x_1 = \frac{x_2}{3}$$

substitute x_2 by t to obtain

$$x_1 = \frac{1}{3}t$$

Hence the eigenvector X corresponding the eigenvalue $\lambda = 1$ may be written as

$$X = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Eigenvectors for $\lambda = -6$

Substitute λ by 6 in the matrix equation $(A - \lambda I)X = 0$

$$\left(\begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - (-6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) X = 0$$

which may be simplified to

$$\left(\begin{bmatrix} 4 & 1 \\ 12 & 3 \end{bmatrix} \right) X = 0$$

Subtract 3 times the top row from the second row to obtain

$$\left(\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \right) X = 0$$

A solution for x_2 could be written as $x_2 = t$ where t takes all real numbers.

Use the top equation $4x_1 + x_2 = 0$

Example 2

Find all eigenvalues and eigenvectors of matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Solution

Find Eigenvalues

We first find the matrix $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ -2 & 2 & 1-\lambda \end{bmatrix}$$

Write the characteristic equation.

$$Det(A - \lambda I) = (1-\lambda)(-\lambda(1-\lambda)) - 1(2-2\lambda) = 0$$

factor and rewrite the equation as

$$(1-\lambda)(\lambda-2)(\lambda+1) = 0$$

which gives 3 solutions

$$\lambda = -1, \lambda = 1, \lambda = 2$$

Find Eigenvectors

Eigenvectors for $\lambda = -1$

Substitute λ by - 1 in the matrix equation $(A - \lambda I)X = 0$ with $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = t/2, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = -2$ is given by

$$X = t \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = 1$

Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$.

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = 0, x_2 = t, x_1 = t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 1$ is given by

$$X = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvectors for $\lambda = 2$

Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$.

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = -t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 2$ is given by

$$X = t \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

Differential equation with first order and higher degrees than one

remember that : $\frac{dy}{dx} - \frac{x}{6y} = \frac{x^4}{3y^2}$ is first order first degree

but $\left(\frac{dy}{dx}\right)^2 - \frac{x}{y} = \frac{x^4}{3y^2}$ is first orde, second degree

Example **solve the following D.E**

$$xy\left(\frac{dy}{dx}\right)^2 + (x+y)\frac{dy}{dx} + 1 = 0$$

Solution : Let $p = \frac{dy}{dx}$

then $xyp^2 + (x+y)p + 1 = 0$

نلاحظ ان الطرف الايسر قابل للتجزئة

$$(xp+1)(yp+1) = 0$$

وبذلك نحصل على المعادلتين

$$yp + 1 = 0 \quad \text{---} \; (1)$$

$$xp + 1 = 0 \quad \text{---} \; (2)$$

من المعادلة رقم ① نحصل على

$$yp + 1 = 0$$

$$y\frac{dy}{dx} + 1 = 0$$

$$y\frac{dy}{dx} = -1$$

$$y\,dy = -dx \qquad \text{seperable equation}$$

$$\frac{y^2}{2} = -x + c_1$$

$$y^2 = -2(x - c_1) \quad \text{——— (3)}$$

من المعادلة رقم ② نحصل على

$$xp + 1 = 0$$

$$x\frac{dy}{dx} = -1$$

$$x\,dy = -dx$$

$$dy = \frac{-dx}{x} \qquad \text{seperable eq.}$$

$$y = -\ln x + \ln c_2$$

$$y = -\ln(c_2 x) \quad \text{——— (4)}$$

وبذلك يكون الحل العام للمعادلة هو كل من العلاقتين (3) و (4) واذا اردنا معرفة الحل الخاص للمعادلة ، عند توفر الشرط الاولي ، فيمكننا استخراجه من احدى العلاقتين اعلاه أو من كليهما

Example 2

Solve the following D.E

$$y^2\left(\frac{dy}{dx}\right)^2 - a^2 + y^2 = 0$$

a is constant

Solution:

let $p = \frac{dy}{dx}$

$$y^2 p^2 - a^2 + y^2 = 0$$

$$y^2 p^2 = a^2 - y^2$$

بجذر الطرفين

$$yp = \mp\sqrt{a^2 - y^2}$$

$$yp = +\sqrt{a^2 - y^2} \quad \text{——} (1)$$

$$yp = -\sqrt{a^2 - y^2} \quad \text{——} (2)$$

من المعادلة ① نحصل على

$$yp = +\sqrt{a^2 - y^2}$$

$$y\frac{dy}{dx} = \sqrt{a^2 - y^2}$$

$$\frac{y\,dy}{\sqrt{a^2 - y^2}} = dx$$

$$y(a^2-y^2)^{-\frac{1}{2}}\,dy = dx$$

$$-\frac{2}{2}y(a^2-y^2)^{-\frac{1}{2}}\,dy = dx$$ بالتكامل

$$\frac{-\frac{1}{2}(a^2-y^2)^{\frac{1}{2}}}{\frac{1}{2}} + C_1 = x$$

$$x = C_1 - \sqrt{a^2-y^2} \quad \text{—— (3)}$$

ومن المعادلة رقم (2) نحصل على

$$x = C_2 + \sqrt{a^2-y^2} \quad \text{—— (4)}$$

H.W1

Solve the following

$$x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0$$

Second Order Differential equations

The second order Linear nonhomogenous D.E. is written in the general form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

or

$$a\bar{\bar{y}} + b\bar{y} + cy = f(x)$$

where a, b, c are constant,

$f(x)$ is function of (x)

If $f(x) = 0$ then the D.E. is called homogenous.

The complete solution $y_g = y_c + y_p$

where: y_c is complementary solution.

y_p is particular solution.

1- Homogenous 2nd orde D.E.

$$a\bar{\bar{y}} + b\bar{y} + cy = 0$$

let $y = e^{mx} \rightarrow \bar{y} = me^{mx} \rightarrow \bar{\bar{y}} = m^2e^{mx}$

$\therefore a(m^2e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$

$e^{mx}(am^2 + bm + c) = 0$

$e^{mx} \neq 0$, then

$am^2 + bm + c = 0$ (characteristic eq)

$$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

1- If $m_1 \neq m_2 \implies y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

2- If $m_1 = m_2 = m \implies y_c = c_1 e^{mx} + c_2 x e^{mx}$

3- If m_1, m_2 = Imaginary roots $m_{1,2} = \alpha + \beta i$

$\implies y_c = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x)$

Example : Solve $y'' + 3y' - 4y = 0$

$m^2 + 3m - 4 = 0$

$$m_{1,2} = \frac{-3 \mp \sqrt{3^2 + 4 \times 4}}{2} \implies m_1 = 1, \quad m_2 = -4$$

$m_1 \neq m_2$

$\therefore y_c = c_1 e^{1x} + c_2 e^{-4x}$

Second Order Differential equations

The second order linear nonhomogenous D.E. is written in the general form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

or

$$a\bar{\bar{y}} + b\bar{y} + cy = f(x)$$

where a, b, c are constant,
$f(x)$ is function of (x)

If $f(x) = 0$ then the D.E. is called homogenous.

The complete solution $y_g = y_c + y_p$

where: y_c is complementary solution.
y_p is particular solution.

1- Homogenous 2nd order D.E.

$$a\bar{\bar{y}} + b\bar{y} + cy = 0$$

let $y = e^{mx} \rightarrow \bar{y} = m e^{mx} \rightarrow \bar{\bar{y}} = m^2 e^{mx}$

$\therefore a(m^2 e^{mx}) + b(m e^{mx}) + c(e^{mx}) = 0$

$e^{mx}(am^2 + bm + c) = 0$

$e^{mx} \neq 0$, then

$am^2 + bm + c = 0$ (characteristic eq)

$$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

1- If $m_1 \neq m_2 \Rightarrow y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

2- If $m_1 = m_2 = m \Rightarrow y_c = c_1 e^{mx} + c_2 x e^{mx}$

3- If m_1, m_2 = Imaginary roots $m_{1,2} = \alpha + \beta i$

$\Rightarrow y_c = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x)$

Example :

Solve $y'' + 3y' - 4y = 0$

$m^2 + 3m - 4 = 0$

$$m_{1,2} = \frac{-3 \mp \sqrt{3^2 + 4 \times 4}}{2} \Rightarrow m_1 = 1, \quad m_2 = -4$$

$m_1 \neq m_2$

$\therefore y_c = c_1 e^{1x} + c_2 e^{-4x}$

The Method of Variation of Parameters

There are two main method to solve equations like

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x)$$

where $P(x)$, $Q(X)$ and $f(X)$ are function of (x)

1- Undetermined Coefficient : which only works when $f(x)$ is a polynomial, exponential, sine, cosine or linear combination of those.

2- Variation of Parameters : which works on a wide range of function. in this method the equation in

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$$

where p and q are constanst
$f(x)$ is non-zero function of x.

The complete solution to such an equation can be found by combining to type of solution.

a- Coplementary solution of the homogenous equation $y'' + py' + qy = 0$

b- particular solution of non-homogenous equation $y'' + py' + qy = f(x)$

note: $f(x)$ could be a single function or sum of two or more functions

Example : Solve the following D.E

$$\frac{d^2y}{dx^2} + y = \csc(x)$$

solution

$$\bar{\bar{y}} + y = 0$$

$$m^2 + 1 = 0$$

$$m = \mp i$$

$$y_c = C_1 \underbrace{\sin x}_{y_1} + C_2 \underbrace{\cos x}_{y_2}$$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

$$u_1 = \int \frac{\cos x \cdot \csc x}{-1} dx = -\int \frac{\cos x}{\sin x} dx = -\ln(\sin x)$$

$$u_2 = \int \frac{\sin x \cdot \csc x}{-1} dx = -\int \frac{\sin x}{\sin x} dx = -x$$

$$y_p = (\ln(\sin x) \cdot \sin x) - (x \cdot \cos x)$$

$$y_t = C_1 \sin(x) + C_2 \cos x + (\ln(\sin x) \cdot \sin x) - x\cos x$$

Cauchy-Euler differential Equations

for second-order Cauchy-Euler D.E. has the form:

$$a_2 x^2 \bar{\bar{y}} + a_1 x \bar{y} + a_0 y = f(x)$$

where:

a_2, a_1, a_0 are constant (numbers)

x^2, x, are non-constant (variable), $x > 0$

$f(x)$ is function of (x)

If $f(x) = 0$ the equation is homogenous

or $f(x) \neq 0$ " " " non homogenous

Note: the exponent of x equal to the number of the derivative.

Solution of homogenous, second order Cauchy-Euler

$$a_2 x^2 \bar{\bar{y}} + a_1 x \bar{y} + a_0 y = 0 \qquad x > 0$$

Let $y = x^m$

$$\bar{y} = m x^{m-1}$$

$$\bar{\bar{y}} = m(m-1) x^{m-2}$$

sub. in equation above

$$a_2 x^2 \left(m(m-1) x^{m-2}\right) + a_1 x \left(m x^{m-1}\right) + a_0 x^m = 0$$

$$a_2 \left(m(m-1) x^m\right) + a_1 m x^m + a_0 x^m = 0$$

$$x^m \left[a_2 m(m-1) + a_1 m + a_0\right] = 0 \qquad x > 0$$

$$a_2 m(m-1) + a_1 m + a_0 = 0$$

$$a_2 m^2 - a_2 m + a_1 m + a_0 = 0$$

$$a_2 m^2 + \underbrace{(a_1 - a_2)}_{\text{Constant}} m + a_0 = 0$$

that is quadratic equation Called characterstic equation of homogenous S. eq.

According to the Solution m_1 and m_2 can take three case

Higher Order Linear D.E.

The general form of non-homogenous linear equation with Constant coefficient is

$$a_n y^{n} + a_{n-1} y^{n-1} + \cdots + a_1 \bar{y} + a_0 y = f(x)$$

$n: 3, 4, \cdots, n$

a_n, a_{n-1}, a_1 and a_0 are constant (numbers)

$f(x)$ is function of x, the solution of this equation solved depended on $f(x)$ to

1- undetermined coefficient method

2- Variation of parameters

for non-homogenous linear equation with variable cofficient the general form is

$$a_n x^{n} y^{n} + a_{n-1} x^{n-1} y^{n-1} + \cdots + a_1 x \bar{y} + a_0 y = f(x)$$

this equation solved with Cauchy-Euler method to find x_1 as follows

$$x_1 = -\frac{x_2}{4}$$

substitute x_2 by t to obtain

$$x_1 = -\frac{1}{4}t$$

Hence the eigenvector X corresponding the eigenvalue $\lambda = -6$ may be written as

$$X = t\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Example 2

Find all eigenvalues and eigenvectors of matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Solution

Find Eigenvalues

We first find the matrix $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ -2 & 2 & 1-\lambda \end{bmatrix}$$

Write the characteristic equation.

$$Det(A - \lambda I) = (1-\lambda)(-\lambda(1-\lambda)) - 1(2-2\lambda) = 0$$

factor and rewrite the equation as

$$(1-\lambda)(\lambda-2)(\lambda+1) = 0$$

which gives 3 solutions

$$\lambda = -1, \lambda = 1, \lambda = 2$$

Find Eigenvectors

Eigenvectors for $\lambda = -1$

Substitute λ by - 1 in the matrix equation $(A - \lambda I)X = 0$ with $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = t/2, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = -2$ is given by

$$X = t \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = 1$

Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$.

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = 0, x_2 = t, x_1 = t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 1$ is given by

$$X = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvectors for $\lambda = 2$

Substitute λ by 1 in the matrix equation $(A - \lambda I)X = 0$.

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Row reduce to echelon form gives

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to the above system and are given by

$$x_3 = t, x_2 = -t/2, x_1 = -t, t \in \mathbb{R}$$

Hence the eigenvector corresponding to the eigenvalue $\lambda = 2$ is given by

$$X = t \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$