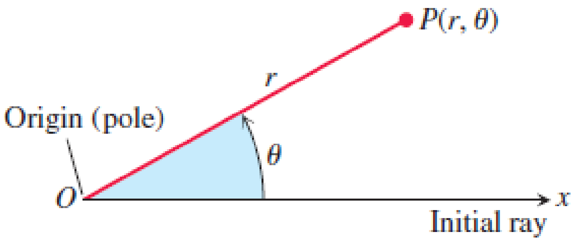
Polar Coordinates

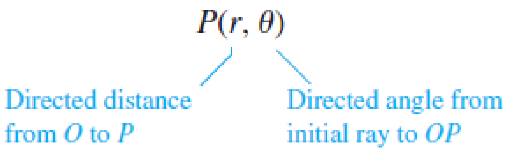
In this section we study polar coordinates and their relation to Cartesian coordinates. You will see that polar coordinates are very useful for calculating many multiple integrals. They are also useful in describing the paths of planets and satellites.



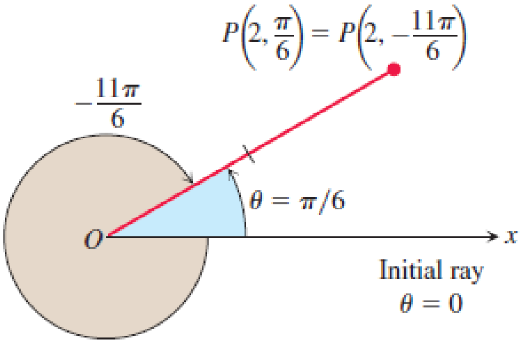
To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (See Figure). Usually the positive x-axis is chosen as the initial ray. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as

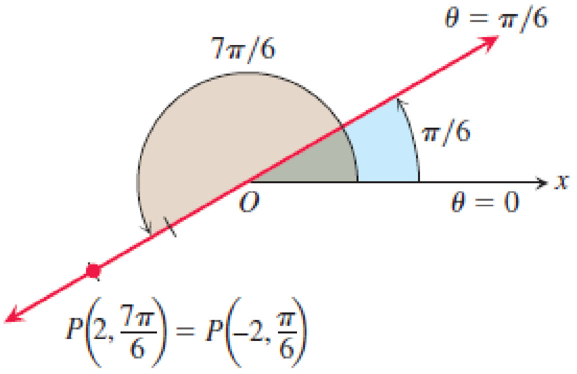


As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$.



Polar coordinates are not unique.

In some situations we allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (See Figure). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.



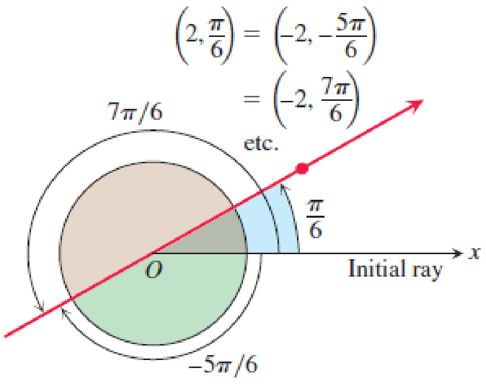
Polar coordinates can have negative r-values.

EXAMPLE Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (See Figure). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$



The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

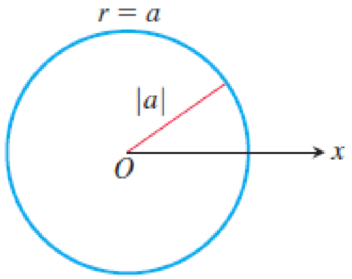
and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O.



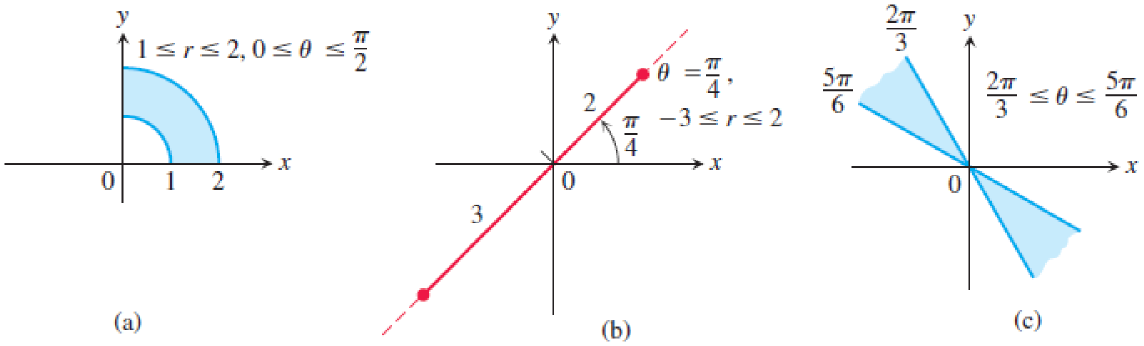
The polar equation for a circle is $r = a$.

EXAMPLE Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)



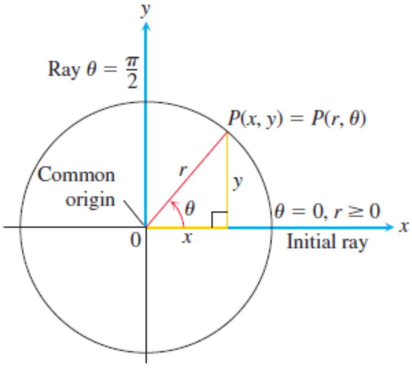
The graphs of typical inequalities in r and θ

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis (**See Figure**). The two coordinate systems are then related by the following equations.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$



The usual way to relate polar and Cartesian coordinates.

EXAMPLE Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

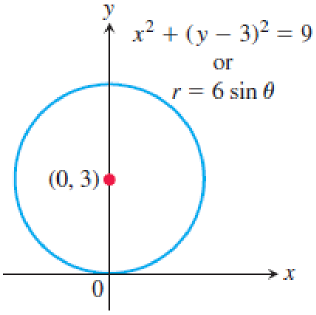
Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not. ■

EXAMPLE Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$\begin{aligned} x^2 + (y-3)^2 &= 9 \\ x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\ x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\ r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2,\ y = r\sin\theta \\ r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\ r &= 6\sin\theta && \text{Includes both possibilities} \end{aligned}$$



EXAMPLE Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation:

$$r\cos\theta = -4$$

$$x = -4 \qquad \text{Substitution}$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$\begin{aligned} r^2 &= 4r\cos\theta \\ x^2 + y^2 &= 4x && \text{Substitution} \\ x^2 - 4x + y^2 &= 0 \\ x^2 - 4x + 4 + y^2 &= 4 && \text{Completing the square} \\ (x-2)^2 + y^2 &= 4 && \text{Factoring} \end{aligned}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$\begin{aligned} r(2\cos\theta - \sin\theta) &= 4 \\ 2r\cos\theta - r\sin\theta &= 4 && \text{Multiplying by } r \\ 2x - y &= 4 && \text{Substitution} \\ y &= 2x - 4 && \text{Solve for } y. \end{aligned}$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ■

Graphing Polar Coordinate Equations

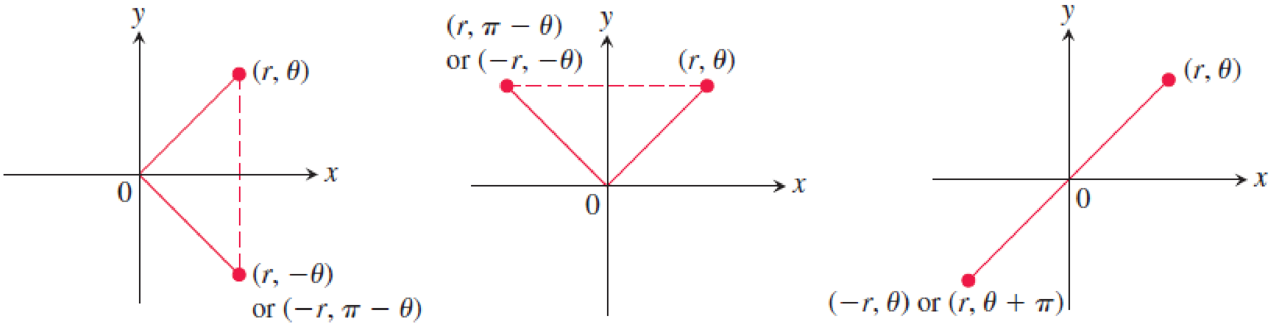
It is often helpful to graph an equation expressed in polar coordinates in the Cartesian xy-plane. This section describes some techniques for graphing these equations using symmetries and tangents to the graph.

Symmetry

Figure below illustrates the standard polar coordinate tests for symmetry. The following summary says how the symmetric points are related.

Symmetry Tests for Polar Graphs in the Cartesian *xy*-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure c).



(a) About the *x*-axis (b) About the *y*-axis (c) About the origin

Three tests for symmetry in polar coordinates.

EXAMPLE Graph the curve $r = 1 - \cos\theta$ in the Cartesian *xy*-plane.

Solution The curve is symmetric about the *x*-axis because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\
&\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\
&\Rightarrow (r, -\theta) \text{ on the graph.}
\end{aligned}
$$

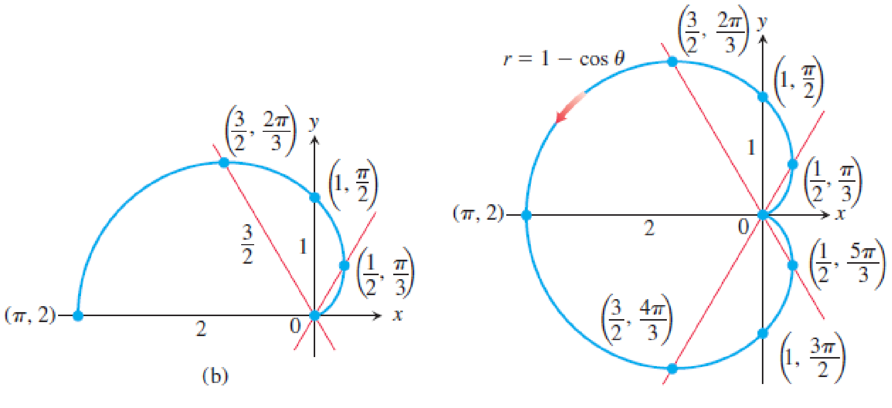
As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the *x*-axis to complete the graph (See Figure). The curve is called a *cardioid* because of its heart shape. ■

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b) (c)

The steps in graphing the cardioid $r = 1 - \cos\theta$. The arrow shows the direction of increasing θ.

EXAMPLE Graph the curve $r^2 = 4\cos\theta$ in the Cartesian xy-plane.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

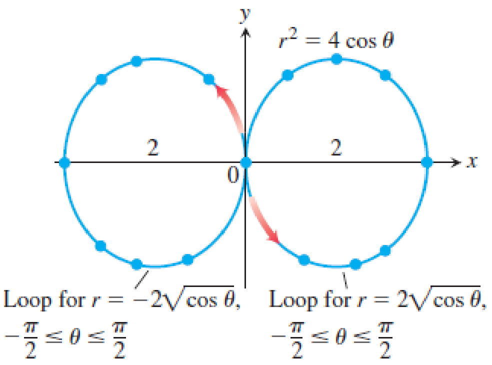
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

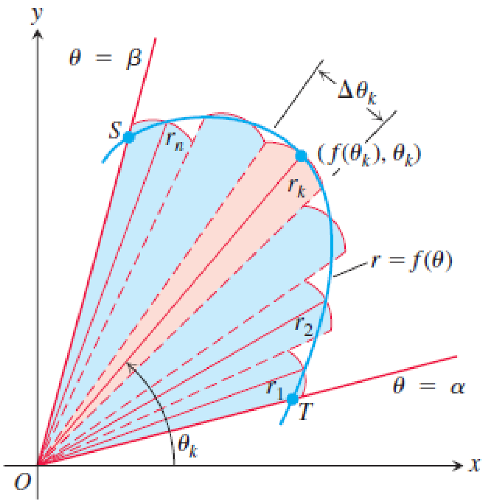
The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded

Areas and Lengths in Polar Coordinates

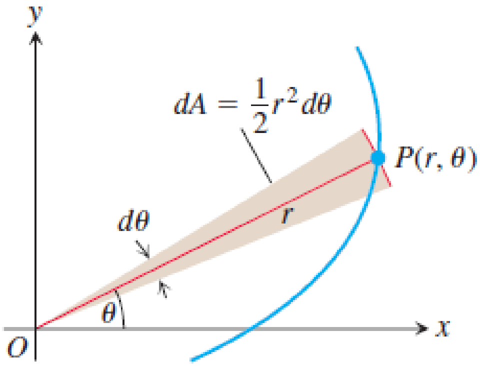
This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

Area in the Plane

The region *OTS* in the Figure. is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle *TOS*.



To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.



The area differential dA for the curve $r = f(\theta)$.

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

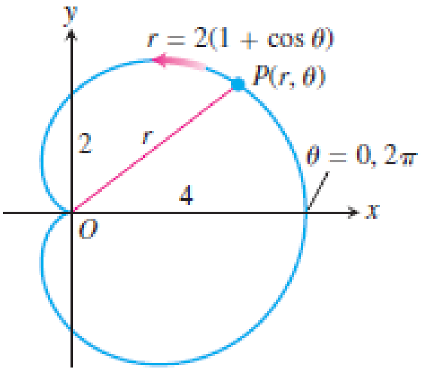
$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid shown in Figure and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$
\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2\, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1+\cos\theta)^2\, d\theta \\
&= \int_0^{2\pi} 2(1+2\cos\theta+\cos^2\theta)\, d\theta \\
&= \int_0^{2\pi} \left(2+4\cos\theta+2\,\frac{1+\cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)\, d\theta \\
&= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .
\end{aligned}
$$



EXAMPLE 2 Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1 .$$

Solution After sketching the curve (see Figure), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

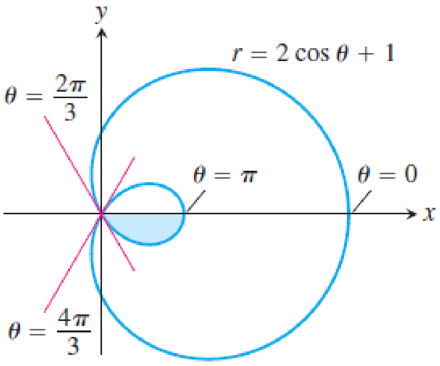
$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2\, d\theta = \int_{2\pi/3}^{\pi} r^2\, d\theta .$$

Since

$$
\begin{aligned}
r^2 &= (2\cos\theta+1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\
&= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\
&= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\
&= 3 + 2\cos 2\theta + 4\cos\theta ,
\end{aligned}
$$

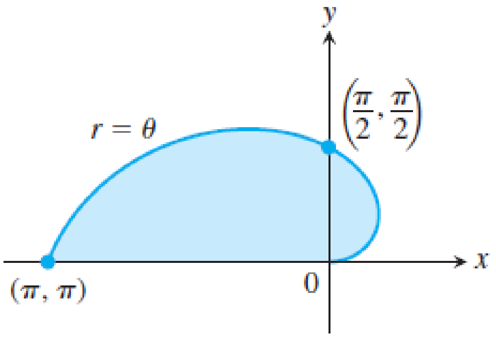
we have

$$
\begin{aligned}
A &= \int_{2\pi/3}^{\pi} (3+2\cos 2\theta+4\cos\theta)\, d\theta \\
&= \left[3\theta + \sin 2\theta + 4\sin\theta\right]_{2\pi/3}^{\pi} \\
&= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\
&= \pi - \frac{3\sqrt{3}}{2} .
\end{aligned}
$$



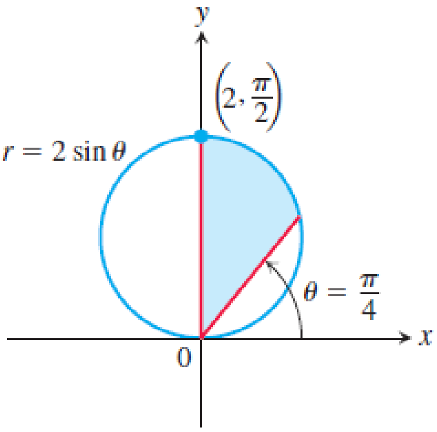
Examples: Find the areas of the regions in the following examples

1. Bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$



Solution: $A = \int_0^{\pi} \frac{1}{2}\theta^2 \, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{\pi} = \frac{\pi^3}{6}$

2. Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$

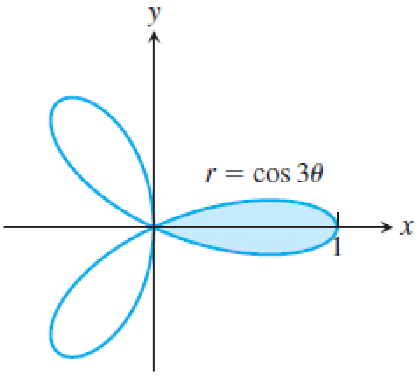


Solution:

$$A = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\sin\theta)^2 \, d\theta = 2\int_{\pi/4}^{\pi/2} \sin^2\theta \, d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1-\cos 2\theta}{2} \, d\theta = \int_{\pi/4}^{\pi/2} (1-\cos 2\theta) d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}$$

3. Inside one leaf of the three-leaved rose $r = \cos 3\theta$



Solution:

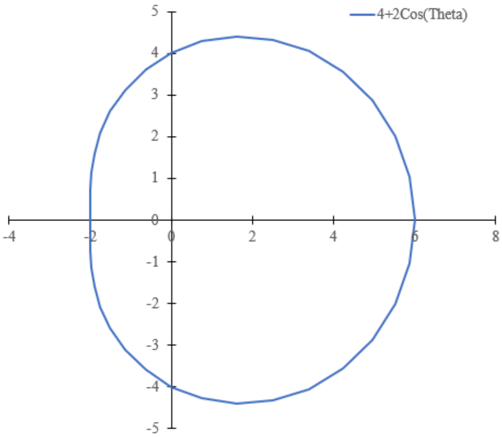
$$A=\int_{-\pi/6}^{\pi/6}\frac{1}{2}(\cos 3\theta)^2\,d\theta=\frac{1}{2}\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/6}^{\pi/6}\frac{1+\cos 6\theta}{2}\,d\theta=\frac{1}{4}\int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\,d\theta$$

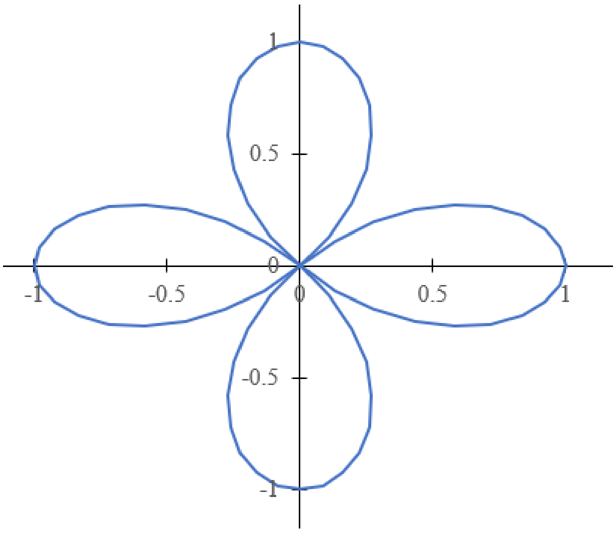
$$=\frac{1}{4}\left[\theta+\frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6}=\frac{1}{4}\left(\frac{\pi}{6}+0\right)-\frac{1}{4}\left(-\frac{\pi}{6}+0\right)=\frac{\pi}{12}$$

Homework 1: Find the areas of the regions in the following:

1. Inside the oval limaçon $r = 4 + 2\cos\theta$

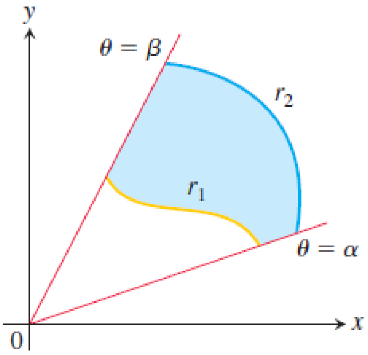


2. Inside one leaf of the four-leaved rose $r = \cos 2\theta$



Area between Polar Curves

To find the area of a region like the one shown in Figure, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2){r_1}^2\,d\theta$ from the integral of $(1/2){r_2}^2\,d\theta$. This leads to the following formula.



The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

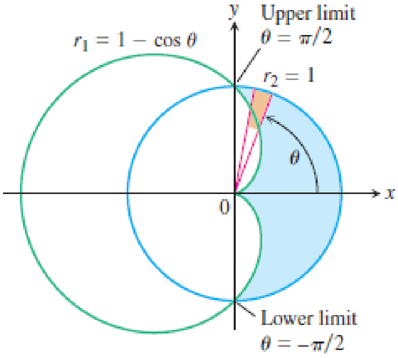
Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} {r_2}^2\,d\theta - \int_{\alpha}^{\beta} \frac{1}{2} {r_1}^2\,d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (See Figure). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is



$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \\ &= 2\int_{0}^{\pi/2} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \qquad \text{Symmetry} \\ &= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\,d\theta \\ &= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\,d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$

■

Example: Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

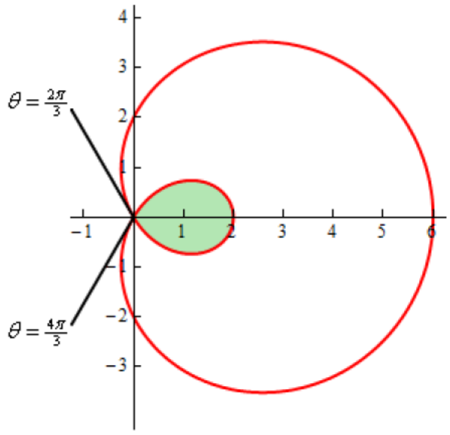
Solution

We graphed this function back when we first started looking at polar coordinates. For this problem we'll also need to know the values of θ where the curve goes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Here is the sketch of this curve with the inner loop shaded in.



Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.

So, the area is then,

$$A = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2 + 4\cos\theta\right)^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4 + 16\cos\theta + 16\cos^2\theta\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 8\cos\theta + 4\left(1 + \cos\left(2\theta\right)\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6 + 8\cos\theta + 4\cos\left(2\theta\right) d\theta$$

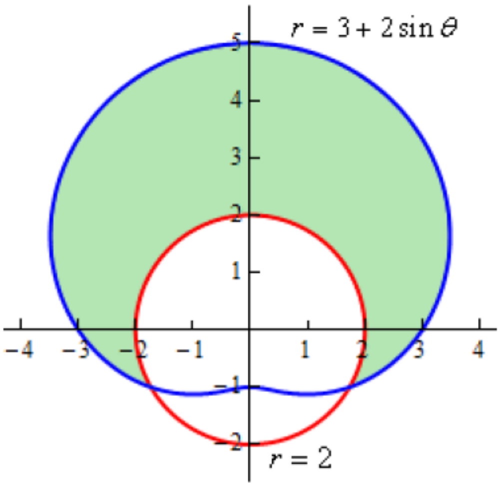
$$= \left(6\theta + 8\sin\theta + 2\sin\left(2\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= 4\pi - 6\sqrt{3} = 2.174$$

Example: Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution

Here is a sketch of the region that we are after.

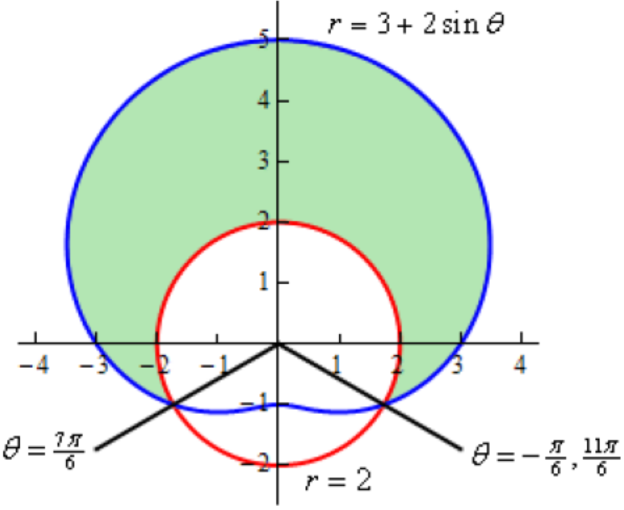


To determine this area, we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However, if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

So, the area is then,

$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(\left(3 + 2\sin\theta\right)^2 - \left(2\right)^2\right) d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(5+12\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(7+12\sin\theta-2\cos(2\theta)\right)d\theta$$

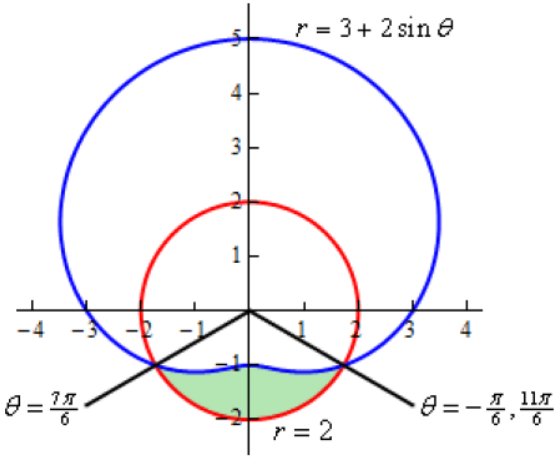
$$=\frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$=\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}=24.187$$

Example: Determine the area of the region outside $r=3+2\sin\theta$ and inside $r=2$.

Solution

This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$A=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left((2)^2-(3+2\sin\theta)^2\right)d\theta$$

$$=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left(-5-12\sin\theta-4\sin^2\theta\right)d\theta$$

$$=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left(-7-12\sin\theta+2\cos(2\theta)\right)d\theta$$

$$=\frac{1}{2}\left(-7\theta+12\cos\theta+\sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

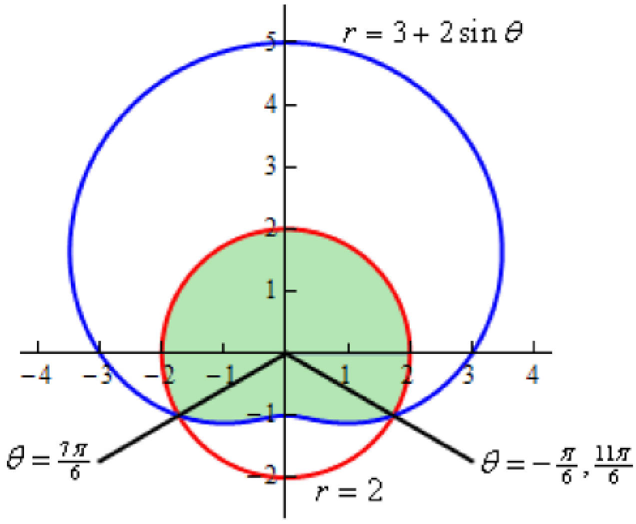
$$=\frac{11\sqrt{3}}{2}-\frac{7\pi}{3}=2.196$$

Notice that for this area the "outer" and "inner" function were opposite!

Example: Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution

Here is the sketch for this example.



We are not going to be able to do this problem in the same fashion that we did the previous two. There is no set of limits that will allow us to enclose this area as we increase from one to the other. Remember that as we increase θ the area we're after must be enclosed. However, the only two ranges for θ that we can work with enclose the area from the previous two examples and not this region.

In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1

In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned}\text{Area} &= \text{Area of Circle} - \text{Area from Example 3}\\ &= \pi(2)^2 - 2.196\\ &= 10.370\end{aligned}$$

Solution 2

In this case we do pretty much the same thing except this time we'll think of the area as the other portion of the limacon than the portion that we were dealing with in Example 2. We'll also need to actually compute the area of the limacon in this case.

So, the area using this approach is then,

$$\begin{aligned}\text{Area} &= \text{Area of Limacon} - \text{Area from Example 2}\\ &= \int_0^{2\pi} \frac{1}{2}(3 + 2\sin\theta)^2\, d\theta - 24.187\\ &= \int_0^{2\pi} \frac{1}{2}\left(9 + 12\sin\theta + 4\sin^2\theta\right) d\theta - 24.187\end{aligned}$$

$$= \int_0^{2\pi} \frac{1}{2}\left(11 + 12\sin\theta - 2\cos(2\theta)\right)d\theta - 24.187$$

$$= \frac{1}{2}\left(11\theta - 12\cos(\theta) - \sin(2\theta)\right)\Big|_0^{2\pi} - 24.187$$

$$= 11\pi - 24.187$$

$$= 10.370$$

A slightly longer approach, but sometimes we are forced to take this longer approach.

As this last example has shown we will not be able to get all areas in polar coordinates straight from an integral.

Examples: Find the areas of the regions in the following Exercises:

1. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

Solution:

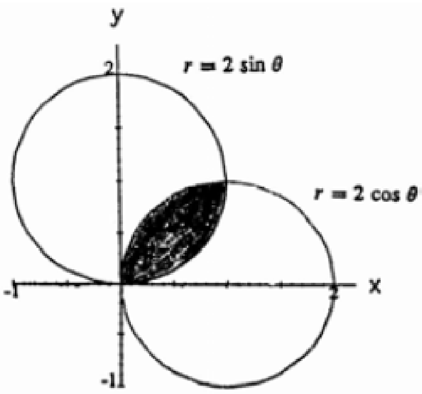
$$r = 2\cos\theta \text{ and } r = 2\sin\theta \Rightarrow 2\cos\theta = 2\sin\theta$$

$$\Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \tfrac{\pi}{4}; \text{ therefore}$$

$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2 d\theta = \int_0^{\pi/4} 4\sin^2\theta \, d\theta$$

$$= \int_0^{\pi/4} 4\left(\tfrac{1-\cos 2\theta}{2}\right) d\theta = \int_0^{\pi/4} (2 - 2\cos 2\theta)\, d\theta$$

$$= \left[2\theta - \sin 2\theta\right]_0^{\pi/4} = \tfrac{\pi}{2} - 1$$



2. Shared by the circles $r = 1$ and $r = 2\sin\theta$

Solution:

$$r = 1 \text{ and } r = 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \tfrac{1}{2}$$

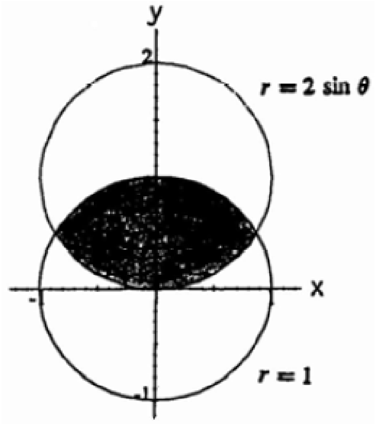
$$\Rightarrow \theta = \tfrac{\pi}{6} \text{ or } \tfrac{5\pi}{6.}; \text{ therefore}$$

$$A = \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}[(2\sin\theta)^2 - 1^2]\, d\theta$$

$$= \pi - \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \tfrac{1}{2}\right) d\theta = \pi - \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - \tfrac{1}{2}\right) d\theta$$

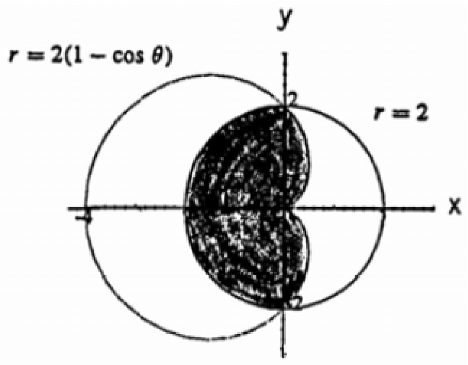
$$= \pi - \int_{\pi/6}^{5\pi/6} \left(\tfrac{1}{2} - \cos 2\theta\right) d\theta = \pi - \left[\tfrac{1}{2}\theta - \tfrac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$$

$$= \pi - \left(\tfrac{5\pi}{12} - \tfrac{1}{2}\sin\tfrac{5\pi}{3}\right) + \left(\tfrac{\pi}{12} - \tfrac{1}{2}\sin\tfrac{\pi}{3}\right) = \tfrac{4\pi - 3\sqrt{3}}{6}$$



3. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

Solution:



$r = 2$ and $r = 2(1-\cos\theta) \Rightarrow 2 = 2(1-\cos\theta) \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pm\frac{\pi}{2}$; therefore

$A = 2\int_0^{\pi/2} \frac{1}{2}[2(1-\cos\theta)]^2\, d\theta + \frac{1}{2}$ area of the circle

$= \int_0^{\pi/2} 4\left(1-2\cos\theta+\cos^2\theta\right)d\theta + \left(\frac{1}{2}\pi\right)(2)^2$

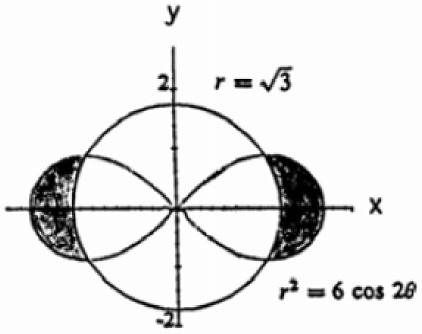
$= \int_0^{\pi/2} 4\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta + 2\pi$

$= \int_0^{\pi/2} \left(4-8\cos\theta+2+2\cos 2\theta\right)d\theta + 2\pi$

$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi = 5\pi - 8$

4. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$

Solution:



$r = \sqrt{3}$ and $r^2 = 6\cos 2\theta \Rightarrow 3 = 6\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$

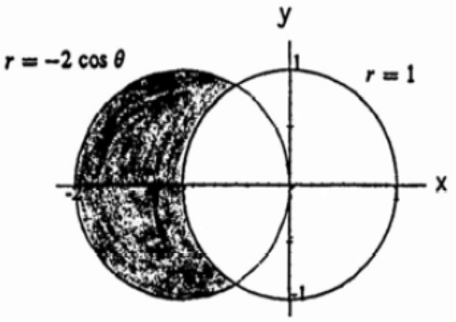
$\Rightarrow \theta = \frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph

to find the area, so $A = 4\int_0^{\pi/6}\left[\frac{1}{2}\left(6\cos 2\theta\right) - \frac{1}{2}\left(\sqrt{3}\right)^2\right]d\theta$

$= 2\int_0^{\pi/6}\left(6\cos 2\theta - 3\right)d\theta = 2\left[3\sin 2\theta - 3\theta\right]_0^{\pi/6} = 3\sqrt{3} - \pi$

5. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$

Solution:



$r = 1$ and $r = -2\cos\theta \Rightarrow 1 = -2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$ in quadrant II; therefore

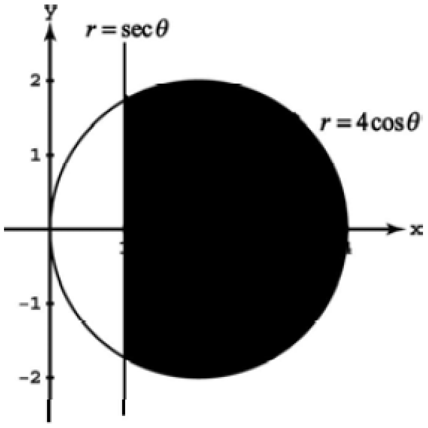
$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}\left[\left(-2\cos\theta\right)^2 - 1^2\right]d\theta = \int_{2\pi/3}^{\pi}\left(4\cos^2\theta - 1\right)d\theta$

$= \int_{2\pi/3}^{\pi}\left[2\left(1+\cos 2\theta\right) - 1\right]d\theta = \int_{2\pi/3}^{\pi}\left(1+2\cos 2\theta\right)d\theta$

$= \left[\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

6. Inside the circle $r = 4\cos\theta$ and to the right of the vertical line $r = \sec\theta$

Solution:



$r=\sec\theta$ and $r=4\cos\theta \Rightarrow 4\cos\theta=\sec\theta \Rightarrow \cos^2\theta=\frac{1}{4}$

$\Rightarrow \theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore

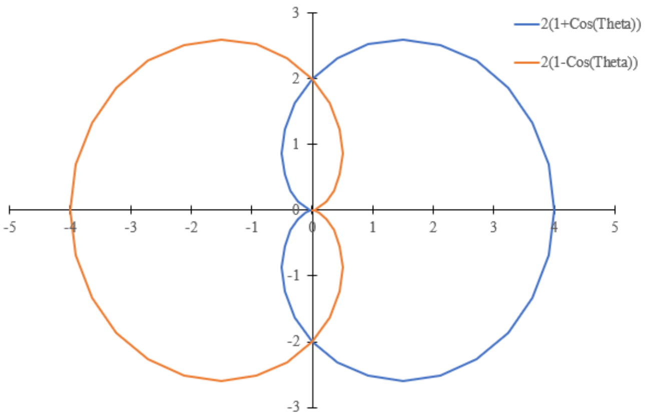
$$A=2\int_0^{\pi/3}\frac{1}{2}\left(16\cos^2\theta-\sec^2\theta\right)d\theta$$

$$=\int_0^{\pi/3}\left(8+8\cos 2\theta-\sec^2\theta\right)d\theta=\left[8\theta+4\sin 2\theta-\tan\theta\right]_0^{\pi/3}$$

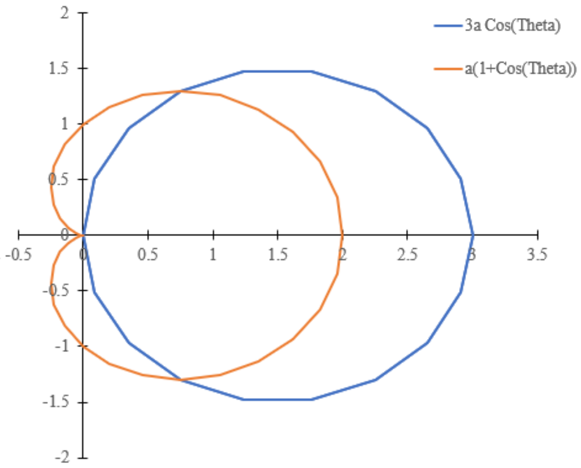
$$=\left(\frac{8\pi}{3}+2\sqrt{3}-\sqrt{3}\right)-(0+0-0)=\frac{8\pi}{3}+\sqrt{3}$$

Homework 2: Find the areas of the regions in the following Exercises:

1. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$



2. Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta), a > 0$



Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \leq \theta \leq \beta.$$

The parametric length formula, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

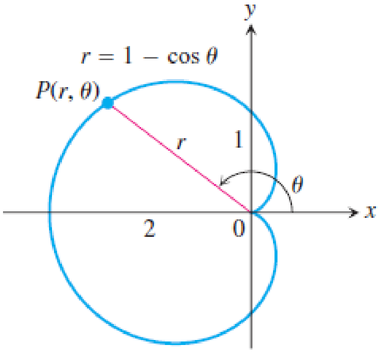
This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration· The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.



With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$

$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

and

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_{0}^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_{0}^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_{0}^{2\pi} = 4 + 4 = 8.$$

Examples: Find the lengths of the curves in the following Exercises

1. The spiral $r=\theta^2, \quad 0\le\theta\le\sqrt{5}$

Solution:

$$r=\theta^2, 0\le\theta\le\sqrt{5} \Rightarrow \frac{dr}{d\theta}=2\theta; \text{ therefore Length } =\int_0^{\sqrt5}\sqrt{\left(\theta^2\right)^2+(2\theta)^2}\ d\theta=\int_0^{\sqrt5}\sqrt{\theta^4+4\theta^2}\ d\theta$$

$$=\int_0^{\sqrt5}|\,\theta\,|\sqrt{\theta^2+4}\ d\theta=\int_0^{\sqrt5}\theta\sqrt{\theta^2+4}\ d\theta; \quad (\text{since } \theta\ge0)$$

$$\left[u=\theta^2+4\Rightarrow\tfrac{1}{2}\,du=\theta\ d\theta;\ \theta=0\Rightarrow u=4, \theta=\sqrt5\Rightarrow u=9\right]\rightarrow\int_4^9\tfrac{1}{2}\sqrt{u}\ du=\tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_4^9=\tfrac{19}{3}$$

2. The spiral $r=e^{\theta}/\sqrt{2}, \quad 0\le\theta\le\pi$

Solution:

$$r=\frac{e^\theta}{\sqrt2}, 0\le\theta\le\pi\Rightarrow\frac{dr}{d\theta}=\frac{e^\theta}{\sqrt2}; \text{ therefore Length } =\int_0^\pi\sqrt{\left(\frac{e^\theta}{\sqrt2}\right)^2+\left(\frac{e^\theta}{\sqrt2}\right)^2}\ d\theta=\int_0^\pi\sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\ d\theta$$

$$=\int_0^\pi e^\theta\ d\theta=\left[e^\theta\right]_0^\pi=e^\pi-1$$

3. The cardioid $r=1+\cos\theta$

Solution:

$$r=1+\cos\theta\Rightarrow\frac{dr}{d\theta}=-\sin\theta; \text{ therefore Length } =\int_0^{2\pi}\sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\ d\theta$$

$$=2\int_0^\pi\sqrt{2+2\cos\theta}\ d\theta=2\int_0^\pi\sqrt{\frac{4(1+\cos\theta)}{2}}\ d\theta=4\int_0^\pi\sqrt{\frac{1+\cos\theta}{2}}d\theta=4\int_0^\pi\cos\left(\frac{\theta}{2}\right)d\theta=4\left[2\sin\frac{\theta}{2}\right]_0^\pi=8$$

4. The curve $r=a\sin^2(\theta/2), \quad 0\le\theta\le\pi, \quad a>0$

Solution:

$$r=a\sin^2\frac{\theta}{2}, 0\le\theta\le\pi, a>0\Rightarrow\frac{dr}{d\theta}=a\sin\frac{\theta}{2}\cos\frac{\theta}{2}; \text{ therefore Length } =\int_0^\pi\sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2+\left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\ d\theta$$

$$=\int_0^\pi\sqrt{a^2\sin^4\frac{\theta}{2}+a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\ d\theta=\int_0^\pi a\left|\sin\frac{\theta}{2}\right|\sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}\ d\theta=a\int_0^\pi\sin\left(\frac{\theta}{2}\right)d\theta \quad (\text{since } 0\le\theta\le\pi)$$

$$=\left[-2a\cos\frac{\theta}{2}\right]_0^\pi=2a$$

Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, and apply the surface area equations for parametrized curves

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration (Figure 10.55a). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Equation (5) in stages. First,

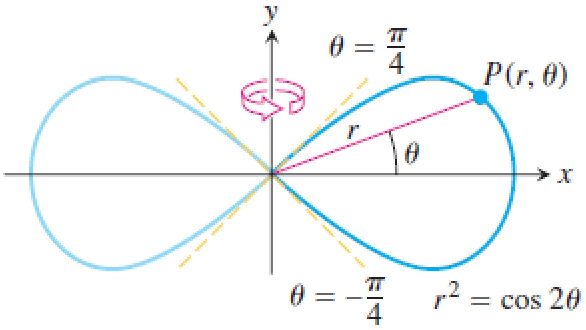
$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}.$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$



Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1 .$$

All together, we have

$$\begin{aligned} S &= \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos\theta \cdot (1)\, d\theta \\ &= 2\pi\Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

Examples: Find the areas of the surface generated by revolving the curves in the following Exercises about the indicated axes.

1. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad y\text{-axis}$

Solution:

$\mathrm{r} = \sqrt{\cos 2\theta}\,, 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(\cos 2\theta)^{-1/2}(-\sin 2\theta)(2) = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$; therefore Surface Area

$$= \int_0^{\pi/4} (2\pi \mathrm{r} \cos\theta) \sqrt{\left(\sqrt{\cos 2\theta}\right)^2 + \left(\frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta = \int_0^{\pi/4} \left(2\pi\sqrt{\cos 2\theta}\right)(\cos\theta)\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= \int_0^{\pi/4} \left(2\pi\sqrt{\cos 2\theta}\right)(\cos\theta)\sqrt{\frac{1}{\cos 2\theta}}\, d\theta = \int_0^{\pi/4} 2\pi \cos\theta\, d\theta = [2\pi \sin\theta]_0^{\pi/4} = \pi\sqrt{2}$$

2. $r = \sqrt{2}e^{\theta/2}, \quad 0 \le \theta \le \pi/2, \quad x\text{-axis}$

Solution:

$\mathrm{r} = \sqrt{2}e^{\theta/2}, 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \sqrt{2}\left(\frac{1}{2}\right)e^{\theta/2} = \frac{\sqrt{2}}{2}e^{\theta/2}$; therefore Surface Area

$$= \int_0^{\pi/2} \left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{\left(\sqrt{2}\,e^{\theta/2}\right)^2 + \left(\frac{\sqrt{2}}{2}e^{\theta/2}\right)^2}\, d\theta = \int_0^{\pi/2} \left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{2e^{\theta} + \frac{1}{2}e^{\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{\frac{5}{2}e^{\theta}}\, d\theta = \int_0^{\pi/2} \left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\left(\frac{\sqrt{5}}{\sqrt{2}}e^{\theta/2}\right) d\theta = 2\pi\sqrt{5}\int_0^{\pi/2} e^{\theta}\sin\theta\, d\theta$$

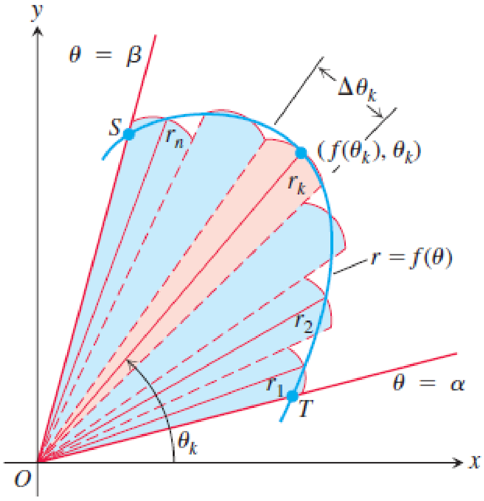
$= 2\pi\sqrt{5}\left[\frac{e^{\theta}}{2}(\sin\theta - \cos\theta)\right]_0^{\pi/2} = \pi\sqrt{5}\,(e^{\pi/2} + 1)$ where we integrated by parts

Areas and Lengths in Polar Coordinates

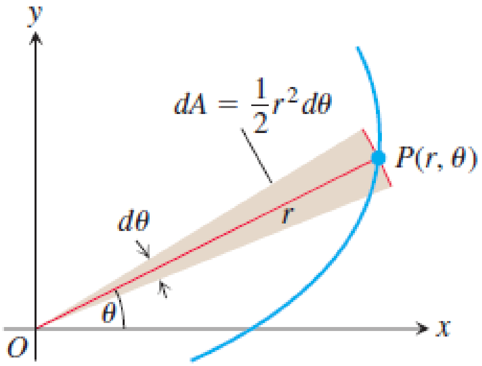
This section shows how to calculate areas of plane regions and lengths of curves in polar coordinates. The defining ideas are the same as before, but the formulas are different in polar versus Cartesian coordinates.

Area in the Plane

The region OTS in the Figure. is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS.



To derive a formula for the area of region OTS, we approximate the region with fan-shaped circular sectors.



The area differential dA for the curve $r = f(\theta)$.

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

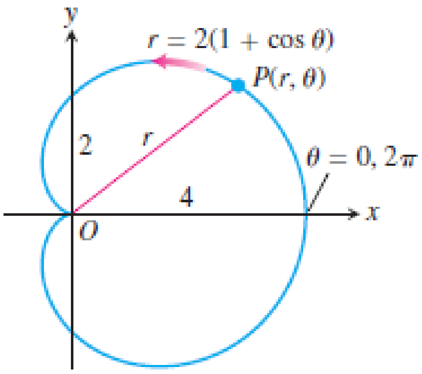
$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta)\right)^2 d\theta.$$

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid shown in Figure and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\begin{aligned}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2\, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1+\cos\theta)^2\, d\theta \\ &= \int_0^{2\pi} 2(1+2\cos\theta+\cos^2\theta)\, d\theta \\ &= \int_0^{2\pi} \left(2+4\cos\theta+2\,\frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)\, d\theta \\ &= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .\end{aligned}$$



EXAMPLE 2 Finding Area

Find the area inside the smaller loop of the limaçon

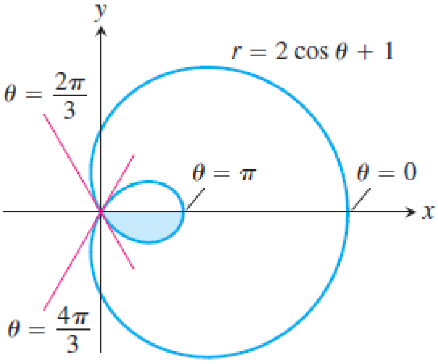
$$r = 2\cos\theta + 1 .$$

Solution After sketching the curve (see Figure), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2\, d\theta = \int_{2\pi/3}^{\pi} r^2\, d\theta .$$

Since

$$\begin{aligned} r^2 &= (2\cos\theta+1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\ &= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\ &= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\ &= 3 + 2\cos 2\theta + 4\cos\theta ,\end{aligned}$$

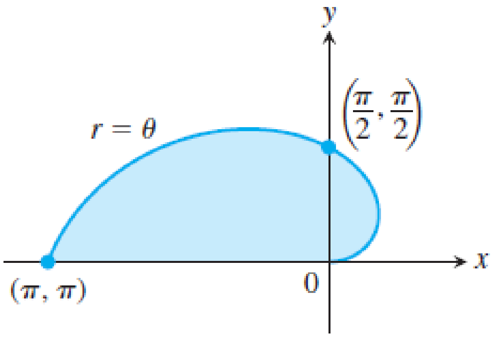


we have

$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \left[3\theta + \sin 2\theta + 4\sin\theta\right]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

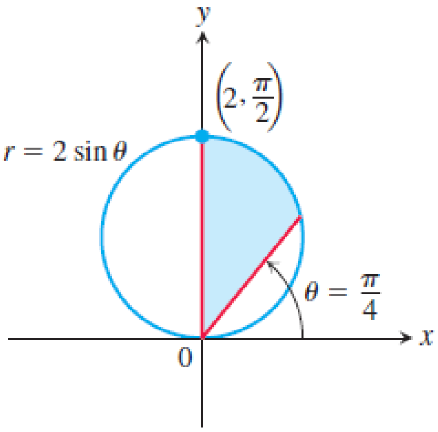
Examples: Find the areas of the regions in the following examples

1. Bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$



Solution: $A = \int_0^{\pi} \frac{1}{2}\theta^2 \, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{\pi} = \frac{\pi^3}{6}$

2. Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$

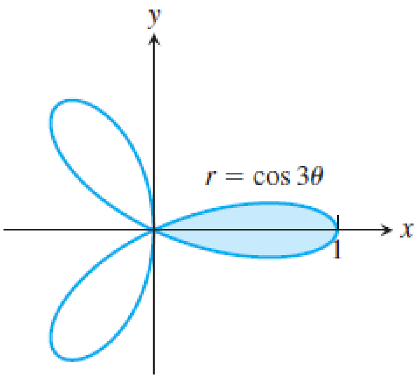


Solution:

$$A = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\sin\theta)^2 \, d\theta = 2\int_{\pi/4}^{\pi/2} \sin^2\theta \, d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1-\cos 2\theta}{2} \, d\theta = \int_{\pi/4}^{\pi/2} (1-\cos 2\theta) d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}$$

3. Inside one leaf of the three-leaved rose $r = \cos 3\theta$



Solution:

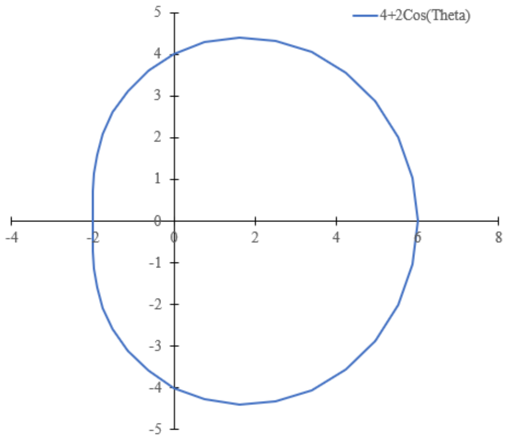
$$A=\int_{-\pi/6}^{\pi/6}\frac{1}{2}(\cos 3\theta)^2\ d\theta=\frac{1}{2}\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\ d\theta$$

$$=\frac{1}{2}\int_{-\pi/6}^{\pi/6}\frac{1+\cos 6\theta}{2}\ d\theta=\frac{1}{4}\int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\ d\theta$$

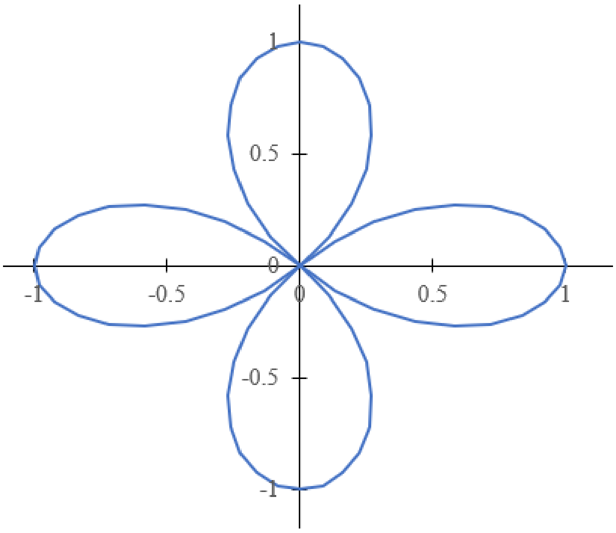
$$=\frac{1}{4}\left[\theta+\frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6}=\frac{1}{4}\left(\frac{\pi}{6}+0\right)-\frac{1}{4}\left(-\frac{\pi}{6}+0\right)=\frac{\pi}{12}$$

Homework 1: Find the areas of the regions in the following:

1. Inside the oval limaçon $r = 4 + 2\cos\theta$

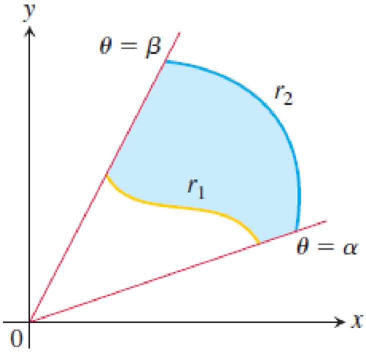


2. Inside one leaf of the four-leaved rose $r = \cos 2\theta$



Area between Polar Curves

To find the area of a region like the one shown in Figure, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2\,d\theta$ from the integral of $(1/2)r_2^2\,d\theta$. This leads to the following formula.



The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

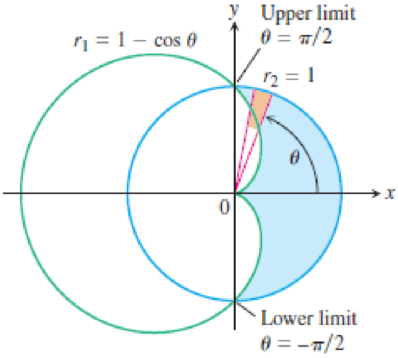
Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\,d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\,d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right)d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (See Figure). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right)d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right)d\theta \qquad \text{Symmetry}$$

$$= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\,d\theta$$

$$= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\,d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}. \qquad \blacksquare$$

Example: Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

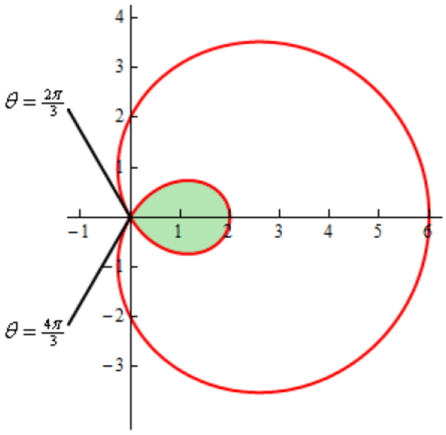
Solution

We graphed this function back when we first started looking at polar coordinates. For this problem we'll also need to know the values of θ where the curve goes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Here is the sketch of this curve with the inner loop shaded in.



Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.

So, the area is then,

$$A = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2 + 4\cos\theta\right)^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4 + 16\cos\theta + 16\cos^2\theta\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 8\cos\theta + 4\left(1 + \cos\left(2\theta\right)\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6 + 8\cos\theta + 4\cos\left(2\theta\right) d\theta$$

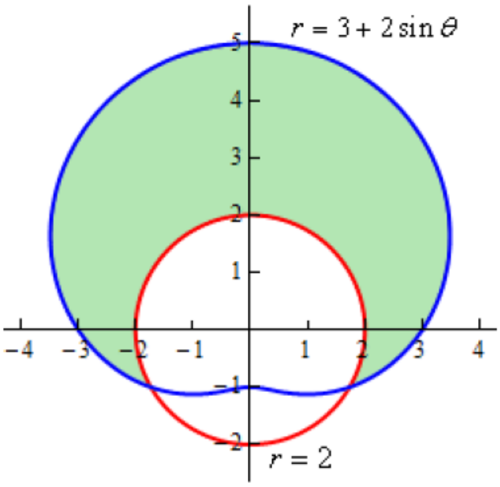
$$= \left(6\theta + 8\sin\theta + 2\sin\left(2\theta\right)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= 4\pi - 6\sqrt{3} = 2.174$$

Example: Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution

Here is a sketch of the region that we are after.

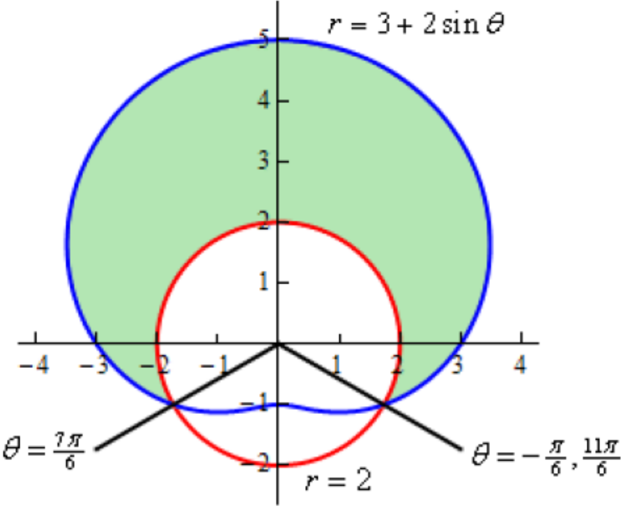


To determine this area, we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However, if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

So, the area is then,

$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3 + 2\sin\theta)^2 - (2)^2\right) d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(5+12\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(7+12\sin\theta-2\cos(2\theta)\right)d\theta$$

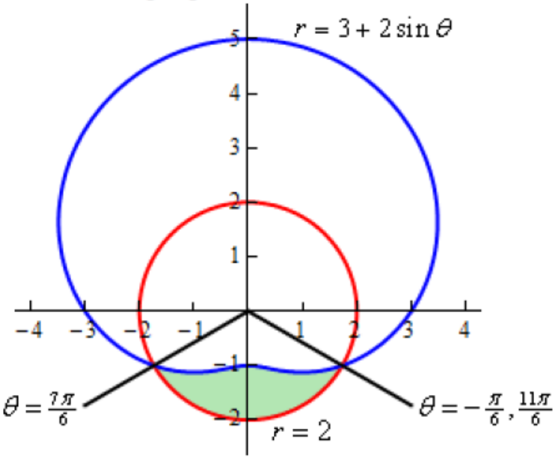
$$=\frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$=\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}=24.187$$

Example: Determine the area of the region outside $r=3+2\sin\theta$ and inside $r=2$.

Solution

This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$A=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left((2)^2-(3+2\sin\theta)^2\right)d\theta$$

$$=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left(-5-12\sin\theta-4\sin^2\theta\right)d\theta$$

$$=\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}\frac{1}{2}\left(-7-12\sin\theta+2\cos(2\theta)\right)d\theta$$

$$=\frac{1}{2}\left(-7\theta+12\cos\theta+\sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

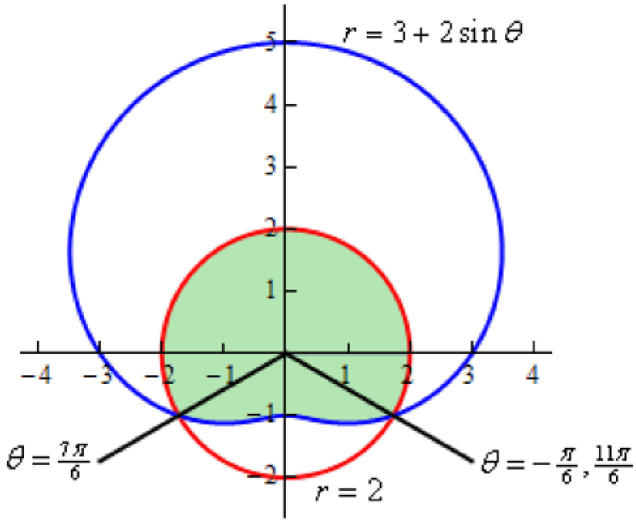
$$=\frac{11\sqrt{3}}{2}-\frac{7\pi}{3}=2.196$$

Notice that for this area the "outer" and "inner" function were opposite!

Example: Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution

Here is the sketch for this example.



We are not going to be able to do this problem in the same fashion that we did the previous two. There is no set of limits that will allow us to enclose this area as we increase from one to the other. Remember that as we increase θ the area we're after must be enclosed. However, the only two ranges for θ that we can work with enclose the area from the previous two examples and not this region.

In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1

In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned}\text{Area} &= \text{Area of Circle} - \text{Area from Example 3}\\ &= \pi(2)^2 - 2.196\\ &= 10.370\end{aligned}$$

Solution 2

In this case we do pretty much the same thing except this time we'll think of the area as the other portion of the limacon than the portion that we were dealing with in Example 2. We'll also need to actually compute the area of the limacon in this case.

So, the area using this approach is then,

$$\begin{aligned}\text{Area} &= \text{Area of Limacon} - \text{Area from Example 2}\\ &= \int_0^{2\pi} \frac{1}{2}(3+2\sin\theta)^2\,d\theta - 24.187\\ &= \int_0^{2\pi} \frac{1}{2}\left(9+12\sin\theta+4\sin^2\theta\right)d\theta - 24.187\end{aligned}$$

$$=\int_0^{2\pi}\frac{1}{2}\left(11+12\sin\theta-2\cos(2\theta)\right)d\theta-24.187$$

$$=\frac{1}{2}\left(11\theta-12\cos(\theta)-\sin(2\theta)\right)\Big|_0^{2\pi}-24.187$$

$$=11\pi-24.187$$

$$=10.370$$

A slightly longer approach, but sometimes we are forced to take this longer approach.

As this last example has shown we will not be able to get all areas in polar coordinates straight from an integral.

Examples: Find the areas of the regions in the following Exercises:

1. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

Solution:

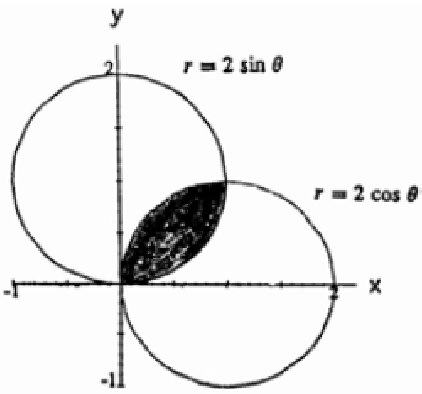
$$r=2\cos\theta \text{ and } r=2\sin\theta \Rightarrow 2\cos\theta=2\sin\theta$$

$$\Rightarrow \cos\theta=\sin\theta \Rightarrow \theta=\tfrac{\pi}{4}; \text{ therefore}$$

$$A=2\int_0^{\pi/4}\tfrac{1}{2}(2\sin\theta)^2 d\theta=\int_0^{\pi/4}4\sin^2\theta\, d\theta$$

$$=\int_0^{\pi/4}4\left(\tfrac{1-\cos 2\theta}{2}\right)d\theta=\int_0^{\pi/4}(2-2\cos 2\theta)\, d\theta$$

$$=\left[2\theta-\sin 2\theta\right]_0^{\pi/4}=\tfrac{\pi}{2}-1$$



2. Shared by the circles $r = 1$ and $r = 2\sin\theta$

Solution:

$$r=1 \text{ and } r=2\sin\theta \Rightarrow 2\sin\theta=1 \Rightarrow \sin\theta=\tfrac{1}{2}$$

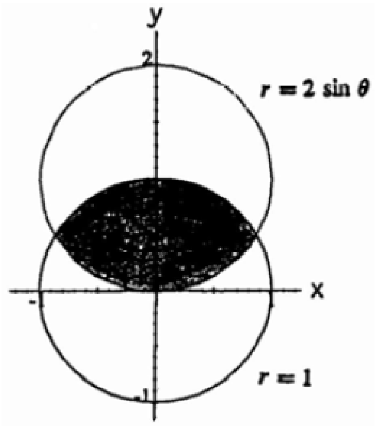
$$\Rightarrow \theta=\tfrac{\pi}{6} \text{ or } \tfrac{5\pi}{6.}; \text{ therefore}$$

$$A=\pi(1)^2-\int_{\pi/6}^{5\pi/6}\tfrac{1}{2}[(2\sin\theta)^2-1^2]\, d\theta$$

$$=\pi-\int_{\pi/6}^{5\pi/6}\left(2\sin^2\theta-\tfrac{1}{2}\right)d\theta=\pi-\int_{\pi/6}^{5\pi/6}\left(1-\cos 2\theta-\tfrac{1}{2}\right)d\theta$$

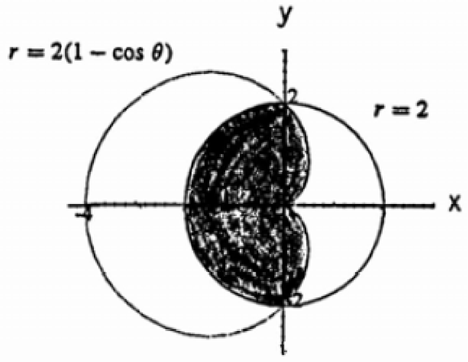
$$=\pi-\int_{\pi/6}^{5\pi/6}\left(\tfrac{1}{2}-\cos 2\theta\right)d\theta=\pi-\left[\tfrac{1}{2}\theta-\tfrac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$$

$$=\pi-\left(\tfrac{5\pi}{12}-\tfrac{1}{2}\sin\tfrac{5\pi}{3}\right)+\left(\tfrac{\pi}{12}-\tfrac{1}{2}\sin\tfrac{\pi}{3}\right)=\tfrac{4\pi-3\sqrt{3}}{6}$$



3 . Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$

Solution:



$r = 2$ and $r = 2(1-\cos\theta) \Rightarrow 2 = 2(1-\cos\theta) \Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pm\frac{\pi}{2}$; therefore

$A = 2\int_0^{\pi/2} \frac{1}{2}[2(1-\cos\theta)]^2\, d\theta + \frac{1}{2}$ area of the circle

$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \left(\frac{1}{2}\pi\right)(2)^2$

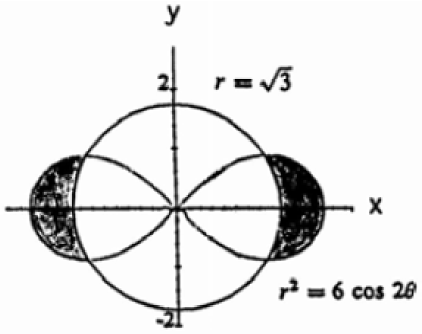
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta + 2\pi$

$= \int_0^{\pi/2} \left(4 - 8\cos\theta + 2 + 2\cos 2\theta\right) d\theta + 2\pi$

$= [6\theta - 8\sin\theta + \sin 2\theta]_0^{\pi/2} + 2\pi = 5\pi - 8$

4 . Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$

Solution:



$r = \sqrt{3}$ and $r^2 = 6\cos 2\theta \Rightarrow 3 = 6\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$

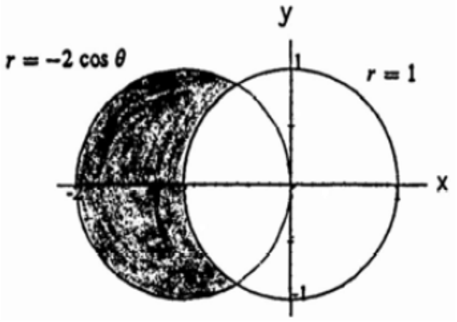
$\Rightarrow \theta = \frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph

to find the area, so $A = 4\int_0^{\pi/6}\left[\frac{1}{2}(6\cos 2\theta) - \frac{1}{2}\left(\sqrt{3}\right)^2\right] d\theta$

$= 2\int_0^{\pi/6}(6\cos 2\theta - 3)\, d\theta = 2[3\sin 2\theta - 3\theta]_0^{\pi/6} = 3\sqrt{3} - \pi$

5 . Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$

Solution:



$r = 1$ and $r = -2\cos\theta \Rightarrow 1 = -2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$ in quadrant II; therefore

$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}\left[(-2\cos\theta)^2 - 1^2\right] d\theta = \int_{2\pi/3}^{\pi}\left(4\cos^2\theta - 1\right) d\theta$

$= \int_{2\pi/3}^{\pi}\left[2(1+\cos 2\theta) - 1\right] d\theta = \int_{2\pi/3}^{\pi}(1 + 2\cos 2\theta)\, d\theta$

$= [\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

6. Inside the circle $r = 4\cos\theta$ and to the right of the vertical line $r = \sec\theta$

Solution:

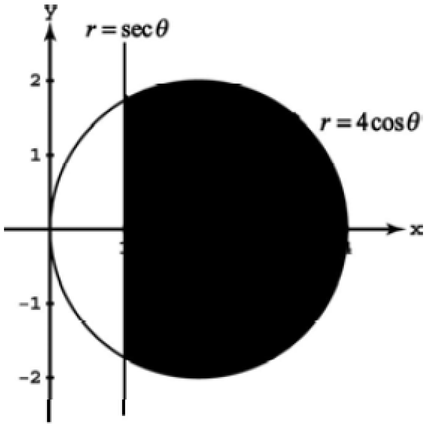
$r = \sec\theta$ and $r = 4\cos\theta \Rightarrow 4\cos\theta = \sec\theta \Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore

$$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left(16\cos^2\theta - \sec^2\theta\right) d\theta$$

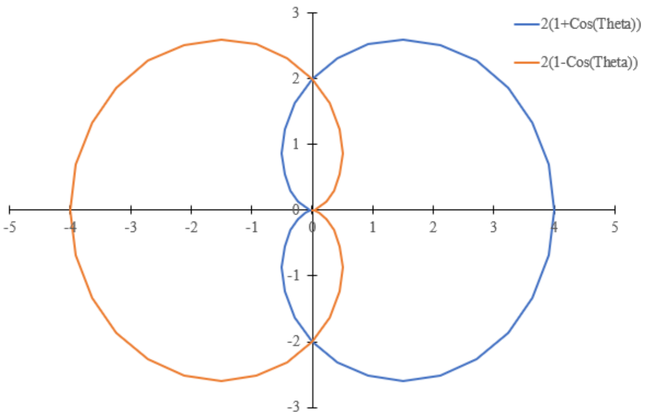
$$= \int_0^{\pi/3}\left(8 + 8\cos 2\theta - \sec^2\theta\right) d\theta = \left[8\theta + 4\sin 2\theta - \tan\theta\right]_0^{\pi/3}$$

$$= \left(\tfrac{8\pi}{3} + 2\sqrt{3} - \sqrt{3}\right) - (0+0-0) = \tfrac{8\pi}{3} + \sqrt{3}$$

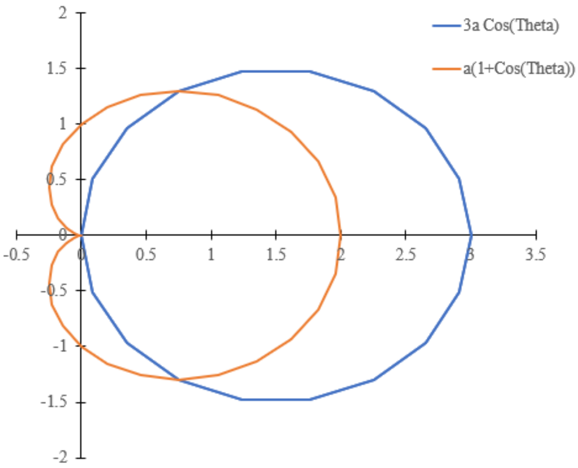


Homework 2: Find the areas of the regions in the following Exercises:

1. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$



2. Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta), a > 0$



Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta.$$

The parametric length formula, then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

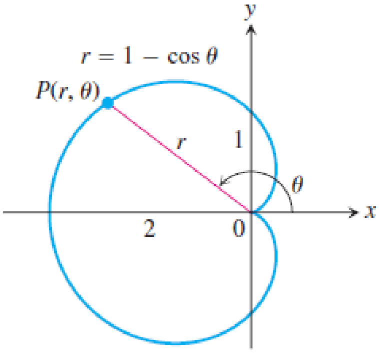
This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration·
The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.



With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8. \end{aligned}$$

Examples: Find the lengths of the curves in the following Exercises

1. The spiral $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$

Solution:

$$r=\theta^2, 0\le\theta\le\sqrt{5} \Rightarrow \tfrac{dr}{d\theta}=2\theta; \text{ therefore Length } = \int_0^{\sqrt{5}} \sqrt{\left(\theta^2\right)^2+(2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4+4\theta^2}\, d\theta$$

$$= \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2+4}\, d\theta = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta; \quad (\text{since } \theta \ge 0)$$

$$\left[u=\theta^2+4 \Rightarrow \tfrac{1}{2}\, du = \theta\, d\theta; \theta=0 \Rightarrow u=4, \theta=\sqrt{5} \Rightarrow u=9\right] \rightarrow \int_4^9 \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_4^9 = \tfrac{19}{3}$$

2. The spiral $r = e^{\theta}/\sqrt{2}, \quad 0 \le \theta \le \pi$

Solution:

$$r=\frac{e^{\theta}}{\sqrt{2}}, 0\le\theta\le\pi \Rightarrow \frac{dr}{d\theta}=\frac{e^{\theta}}{\sqrt{2}}; \text{ therefore Length } = \int_0^{\pi} \sqrt{\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2+\left(\frac{e^{\theta}}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\, d\theta$$

$$= \int_0^{\pi} e^{\theta}\, d\theta = \left[e^{\theta}\right]_0^{\pi} = e^{\pi}-1$$

3. The cardioid $r = 1 + \cos\theta$

Solution:

$$r=1+\cos\theta \Rightarrow \frac{dr}{d\theta}=-\sin\theta; \text{ therefore Length } = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{\frac{4(1+\cos\theta)}{2}}\, d\theta = 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}} d\theta = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8$$

4. The curve $r = a\sin^2(\theta/2), \quad 0 \le \theta \le \pi, \quad a > 0$

Solution:

$$r=a\sin^2\frac{\theta}{2}, 0\le\theta\le\pi, a>0 \Rightarrow \frac{dr}{d\theta}=a\sin\frac{\theta}{2}\cos\frac{\theta}{2}; \text{ therefore Length } = \int_0^{\pi} \sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2+\left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{a^2\sin^4\frac{\theta}{2}+a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta = \int_0^{\pi} a\left|\sin\frac{\theta}{2}\right| \sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}\, d\theta = a\int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta \quad (\text{since } 0\le\theta\le\pi)$$

$$= \left[-2a\cos\frac{\theta}{2}\right]_0^{\pi} = 2a$$

Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, and apply the surface area equations for parametrized curves

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis $(y \geq 0)$:

$$S = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

2. Revolution about the y-axis $(x \geq 0)$:

$$S = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

EXAMPLE Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration (Figure 10.55a). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Equation (5) in stages. First,

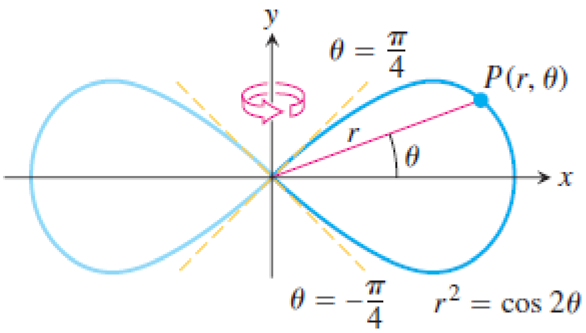
$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}.$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$



Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1 .$$

All together, we have

$$\begin{aligned} S &= \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos\theta \cdot (1)\, d\theta \\ &= 2\pi \Big[\sin\theta \Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

Examples: Find the areas of the surface generated by revolving the curves in the following Exercises about the indicated axes.

1. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad y\text{-axis}$

Solution:

$r = \sqrt{\cos 2\theta}\,, 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(\cos 2\theta)^{-1/2}(-\sin 2\theta)(2) = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$; therefore Surface Area

$= \int_0^{\pi/4} (2\pi r \cos\theta) \sqrt{\left(\sqrt{\cos 2\theta}\right)^2 + \left(\frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta = \int_0^{\pi/4} \left(2\pi\sqrt{\cos 2\theta}\right)(\cos\theta)\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta$

$= \int_0^{\pi/4} \left(2\pi\sqrt{\cos 2\theta}\right)(\cos\theta)\sqrt{\frac{1}{\cos 2\theta}}\, d\theta = \int_0^{\pi/4} 2\pi\cos\theta\, d\theta = [2\pi\sin\theta]_0^{\pi/4} = \pi\sqrt{2}$

2. $r = \sqrt{2}e^{\theta/2}, \quad 0 \le \theta \le \pi/2, \quad x\text{-axis}$

Solution:

$r = \sqrt{2}e^{\theta/2}, 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \sqrt{2}\left(\frac{1}{2}\right)e^{\theta/2} = \frac{\sqrt{2}}{2}e^{\theta/2}$; therefore Surface Area

$= \int_0^{\pi/2} \left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{\left(\sqrt{2}\,e^{\theta/2}\right)^2 + \left(\frac{\sqrt{2}}{2}e^{\theta/2}\right)^2}\, d\theta = \int_0^{\pi/2}\left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{2e^{\theta} + \frac{1}{2}e^{\theta}}\, d\theta$

$= \int_0^{\pi/2}\left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\sqrt{\frac{5}{2}e^{\theta}}\, d\theta = \int_0^{\pi/2}\left(2\pi\sqrt{2}\,e^{\theta/2}\right)(\sin\theta)\left(\frac{\sqrt{5}}{\sqrt{2}}e^{\theta/2}\right) d\theta = 2\pi\sqrt{5}\int_0^{\pi/2} e^{\theta}\sin\theta\, d\theta$

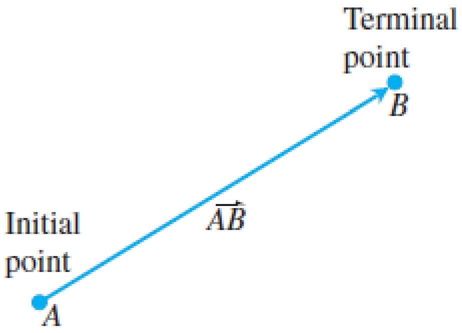
$= 2\pi\sqrt{5}\left[\frac{e^{\theta}}{2}(\sin\theta - \cos\theta)\right]_0^{\pi/2} = \pi\sqrt{5}\left(e^{\pi/2} + 1\right)$ where we integrated by parts

Vectors

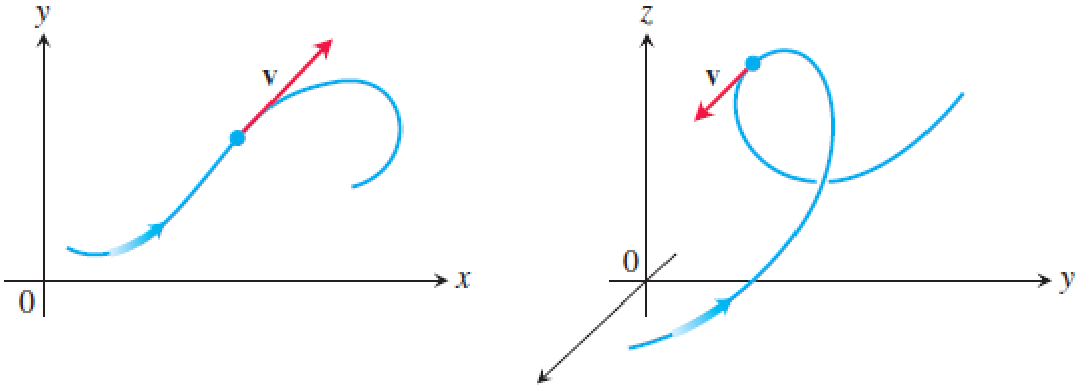
To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. In this section we show how to represent things that have both magnitude and direction in the plane or in space.

Component Form

A quantity such as force, displacement, or velocity is called a **vector** and is represented by a **directed line segment** as shown in the figure.



The directed line segment $\overrightarrow{AB}$ is called a vector.

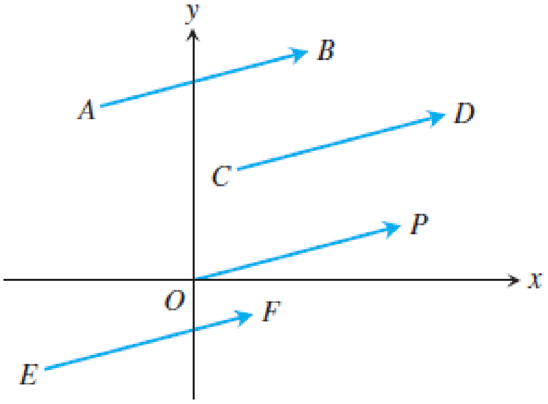


(a) two dimensions (b) three dimensions

The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

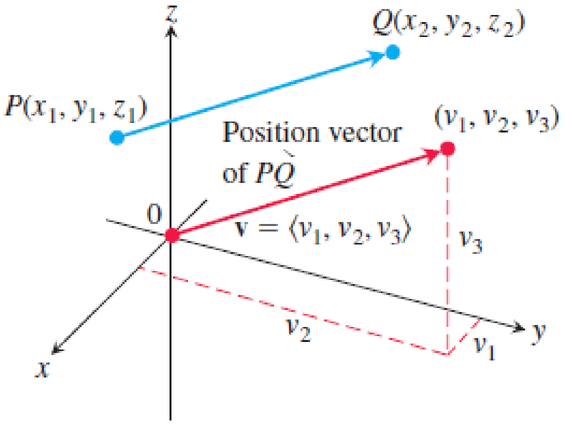
DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

The arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction (see Figure) regardless of the initial point.



The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}.$$



A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

DEFINITION If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \end{aligned}$$

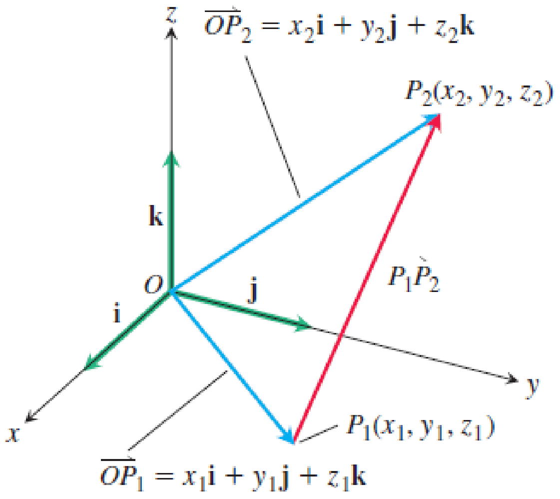
We call the scalar (or number) v_1 the **i-component** of the vector $\mathbf{v}$, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

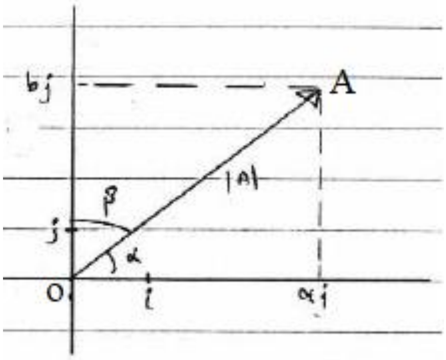
$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.



The vector from P_1 to P_2 is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$



$\vec{A} = \overline{OA} = ai + bj \qquad (Vector)$

$|\vec{A}| = \sqrt{a^2 + b^2} \qquad (Length\ of\ vector)$

i, j are the fundamental unit vectors

$$Unit\ vector = \vec{u} = \frac{\vec{A}}{|A|} = \frac{ai + bj}{|A|}$$

EXAMPLE 1 Find the (**a**) component form and (**b**) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(**a**) The standard position vector **v** representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(**b**) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

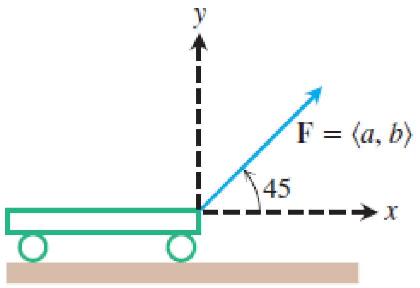
■

EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force **F** making a 45° angle to the floor (see Figure). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45° = (20)\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ lb}.$$

Notice that **F** is a two-dimensional vector. ■



The force pulling the cart forward is represented by the vector **F** whose horizontal component is the effective force (Example 2).

Vector Algebra Operations

Two principal operations involving vectors are *vector addition* and *scalar multiplication*. A **scalar** is simply a real number and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero and are used to "scale" a vector by multiplication.

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

The **difference** $\mathbf{u} - \mathbf{v}$ of two vectors is defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$ ■

EXAMPLE 4 Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underset{\substack{\text{Length} \\ \text{(speed)}}}{5}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$ ■

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$ ■

Midpoint of a Line Segment

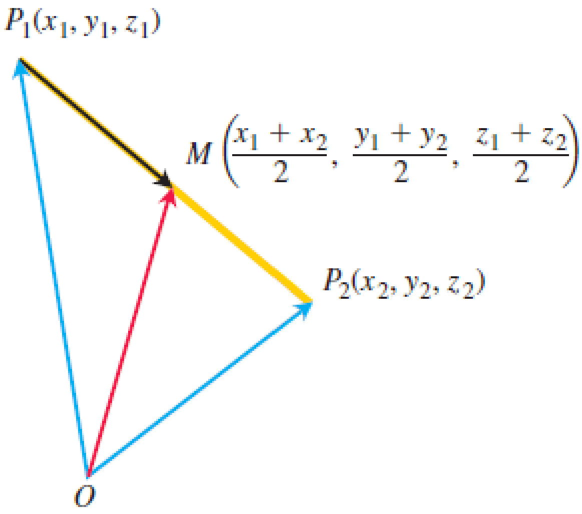
Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

To see why, observe (see Figure) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$



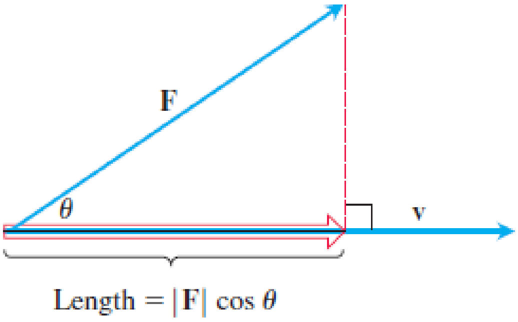
The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$ ■

The Dot Product

Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another



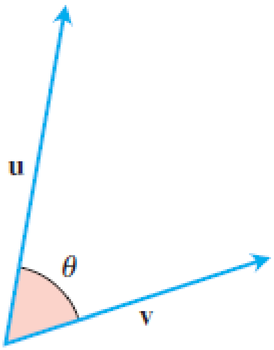
The magnitude of the force **F** in the direction of vector **v** is the length $|\mathbf{F}|\cos\theta$ of the projection of **F** onto **v**.

DEFINITION The **dot product u · v** (**"u dot v"**) of vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is the scalar

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right).$$



The angle between **u** and **v**.

The Angle Between Two Nonzero Vectors u and v

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

EXAMPLE 1 We illustrate the definition.

$$\text{(a)}\quad \langle 1, -2, -1\rangle \cdot \langle -6, 2, -3\rangle = (1)(-6) + (-2)(2) + (-1)(-3)$$
$$= -6 - 4 + 3 = -7$$

$$\text{(b)}\quad \left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right)\cdot(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$$ ■

EXAMPLE 2 Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4\\
|\mathbf{u}| &= \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3\\
|\mathbf{v}| &= \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7\\
\theta &= \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians or } 100.98^\circ.
\end{aligned}$$ ■

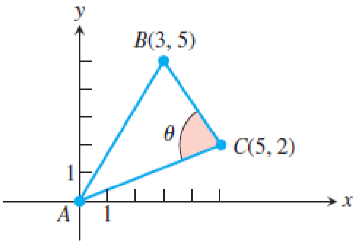
EXAMPLE 3 Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (see Figure).

Solution The angle θ is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

$$\overrightarrow{CA} = \langle -5, -2\rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3\rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\begin{aligned}
\overrightarrow{CA}\cdot\overrightarrow{CB} &= (-5)(-2) + (-2)(3) = 4\\
|\overrightarrow{CA}| &= \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}\\
|\overrightarrow{CB}| &= \sqrt{(-2)^2 + (3)^2} = \sqrt{13}
\end{aligned}$$



The triangle in Example 3.

Then applying the angle formula, we have

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right)$$

$$= \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$

$$\approx 78.1^\circ \text{ or } 1.36 \text{ radians.}$$ ■

Orthogonal Vectors

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u} \cdot \mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$. The following definition also allows for one or both of the vectors to be the zero vector.

DEFINITION Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 To determine if two vectors are orthogonal, calculate their dot product.

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0. \end{aligned}$$ ■

Dot Product Properties and Vector Projections

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

5. $\mathbf{0} \cdot \mathbf{u} = 0$.

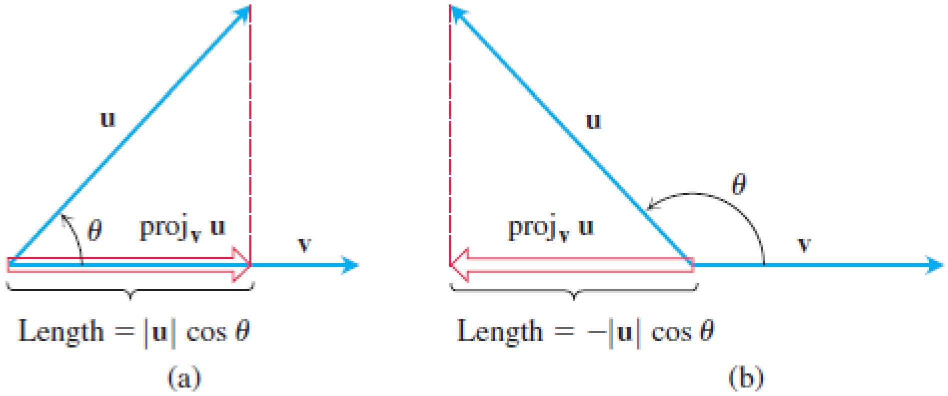
The vector projection

The vector projection of **u** onto **v** is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}. \qquad (1)$$

The scalar component of **u** in the direction of **v** is the scalar

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \qquad (2)$$



The length of $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is (a) $|\mathbf{u}| \cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{u}| \cos\theta$ if $\cos\theta < 0$.

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of **u** in the direction of **v**.

Solution We find $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ from Equation (1):

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$$= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

We find the scalar component of **u** in the direction of **v** from Equation (2):

$$|\mathbf{u}| \cos\theta = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$$

$$= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.$$

■

EXAMPLE 6 Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$.

Solution The vector projection is

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\ &= \frac{5-6}{1+9}(\mathbf{i} - 3\mathbf{j}) = -\frac{1}{10}(\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

The scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$ is

$$|\mathbf{F}|\cos\theta = \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{5-6}{\sqrt{1+9}} = -\frac{1}{\sqrt{10}}.$$

Example:

Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.

Solution:

$\overrightarrow{AB} = \langle 3, 1\rangle$, $\overrightarrow{BC} = \langle -1, -3\rangle$, and $\overrightarrow{AC} = \langle 2, -2\rangle$. $\overrightarrow{BA} = \langle -3, -1\rangle$, $\overrightarrow{CB} = \langle 1, 3\rangle$, $\overrightarrow{CA} = \langle -2, 2\rangle$.

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right| = \sqrt{10}$, $\left|\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\right| = \sqrt{10}$, $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{CA}\right| = 2\sqrt{2}$,

$$\text{Angle at } A = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$$

$$\text{Angle at } B = \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{\left(\sqrt{10}\right)\left(\sqrt{10}\right)}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°, \text{ and}$$

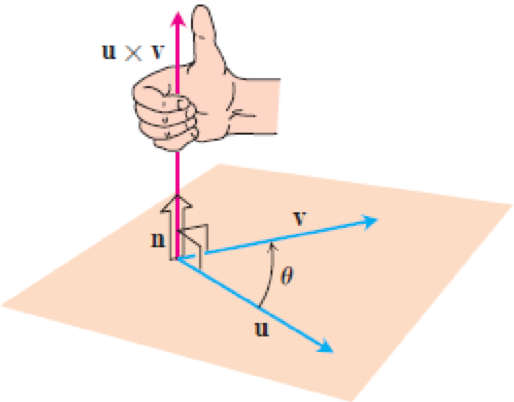
$$\text{Angle at } C = \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$$

The Cross Product

DEFINITION The **cross product** $\mathbf{u} \times \mathbf{v}$ ("**u cross v**") is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta)\, \mathbf{n}.$$

Unlike the dot product, the cross product is a vector. For this reason it's also called the **vector product** of $\mathbf{u}$ and $\mathbf{v}$, and applies *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because it is a scalar multiple of $\mathbf{n}$.



Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The cross product obeys the following laws.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$

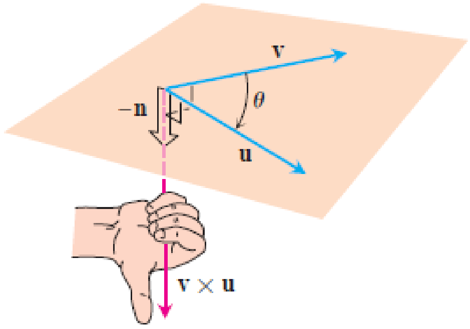
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$



$$\mathbf{i}\times\mathbf{j} = -(\mathbf{j}\times\mathbf{i}) = \mathbf{k}$$

$$\mathbf{j}\times\mathbf{k} = -(\mathbf{k}\times\mathbf{j}) = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{k}) = \mathbf{j}$$

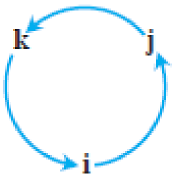


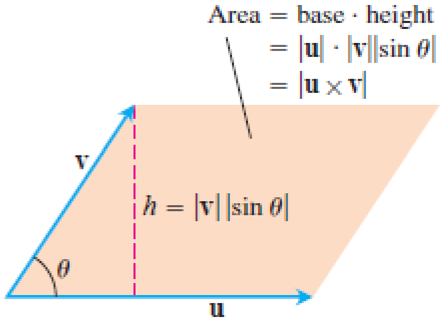
Diagram for recalling these products

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}.$$

$|\mathbf{u}\times\mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u}\times\mathbf{v}$ is

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\;|\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$



EXAMPLE Calculating Cross Products with Determinants

Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Solution

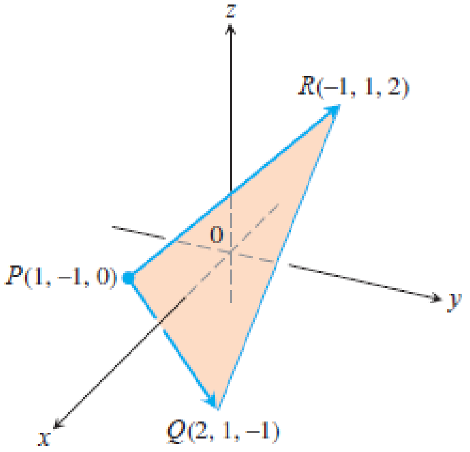
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

EXAMPLE Finding Vectors Perpendicular to a Plane

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$



Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

EXAMPLE Finding the Area of a Triangle

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}|$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE Finding a Unit Normal to a Plane

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Note:

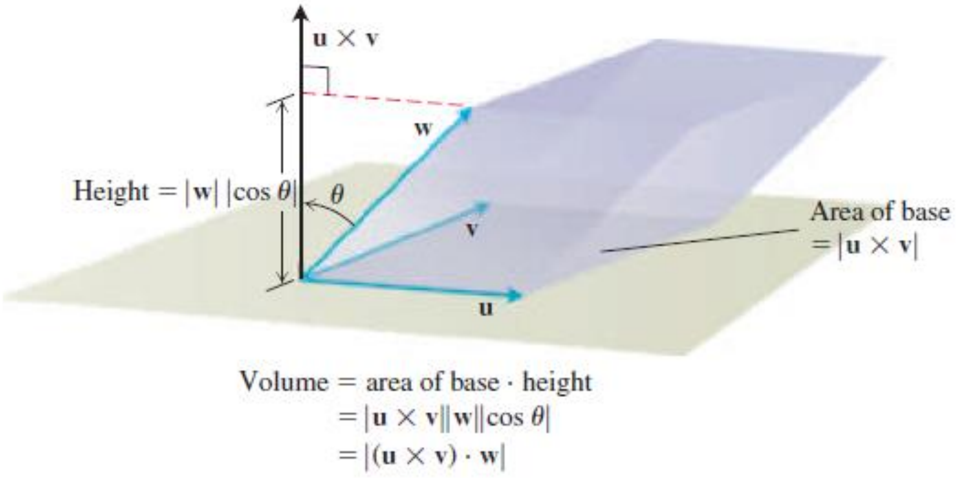
For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w** (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos\theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by **u**, **v**, and **w** (See Figure). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base parallelogram. The number $|\mathbf{w}||\cos\theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of **u**, **v**, and **w**.



The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

By treating the planes of **v** and **w** and of **w** and **u** as the base planes of the parallelepiped determined by **u**, **v**, and **w**, we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{v}=-2\mathbf{i}+3\mathbf{k}$, and $\mathbf{w}=7\mathbf{j}-4\mathbf{k}$.

Solution Using the rule for calculating a 3 × 3 determinant, we find

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} 1 & 2 & -1\\ -2 & 0 & 3\\ 0 & 7 & -4\end{vmatrix}=(1)\begin{vmatrix} 0 & 3\\ 7 & -4\end{vmatrix}-(2)\begin{vmatrix} -2 & 3\\ 0 & -4\end{vmatrix}+(-1)\begin{vmatrix} -2 & 0\\ 0 & 7\end{vmatrix}=-23.$$

The volume is $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=23$ units cubed. ■

Examples:

1. **a.** Find the area of the triangle determined by the points P, Q, and R.

   b. Find a unit vector perpendicular to plane PQR.

   $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$

Solution:

$$\text{(a)}\quad \overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k}\Rightarrow \text{Area }=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{64+16+16}=2\sqrt{6}$$

$$\text{(b)}\quad \mathbf{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$$

2. find the volume of the parallelepiped (box) determined by **u**, **v**, and **w**.

u	**v**	**w**
$\mathbf{i}-\mathbf{j}+\mathbf{k}$	$2\mathbf{i}+\mathbf{j}-2\mathbf{k}$	$-\mathbf{i}+2\mathbf{j}-\mathbf{k}$

Solution:

$$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix} 1 & -1 & 1\\ 2 & 1 & -2\\ -1 & 2 & -1\end{vmatrix}=4$$

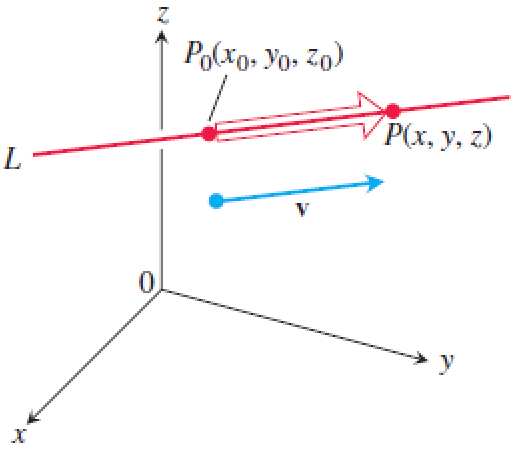
Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$



A point P lies on L through P_0 parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$. Rewriting Equation (2), we have

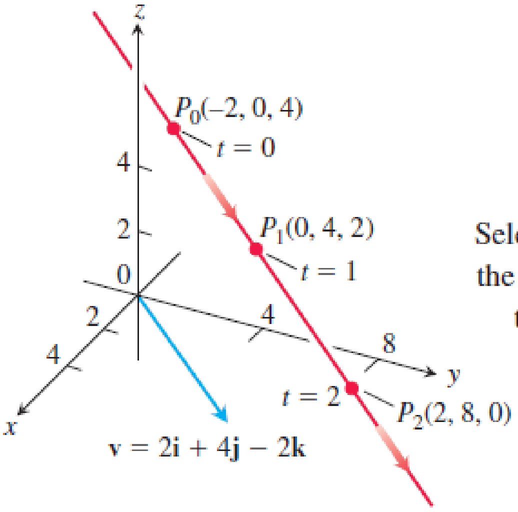
$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \underset{\text{Initial position}}{\mathbf{r}_0} + \underset{\text{Time}}{t}\underset{\text{Speed}}{|\mathbf{v}|}\underset{\text{Direction}}{\frac{\mathbf{v}}{|\mathbf{v}|}}. \end{aligned} \tag{4}$$

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (See Figure).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

■



Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing t.

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t. ■

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (see Figure).

Solution We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

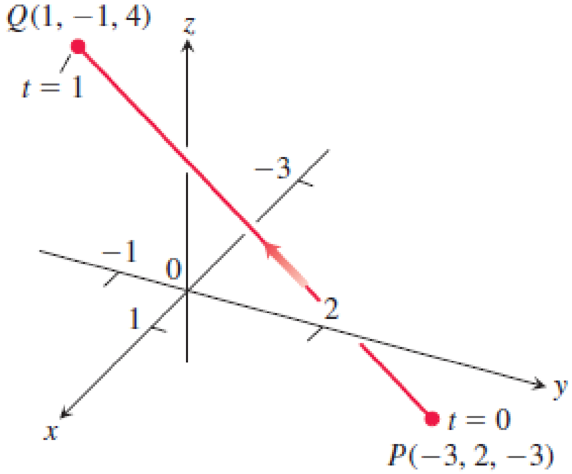
$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1. \qquad ■$$



Example 3 derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

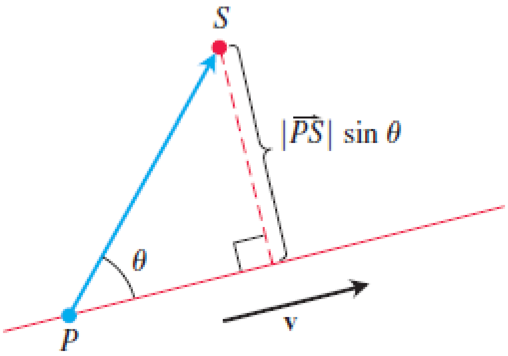
When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \left\langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \right\rangle.\end{aligned}$$

After 10 sec of flight from the origin toward (1, 1, 1), the helicopter is located at the point $\left(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\right)$ in space. It has traveled a distance of (60 ft/sec)(10 sec) = 600 ft, which is the length of the vector $\mathbf{r}(10)$. ■

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line (See Figure). In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}|\sin\theta$, which is $\dfrac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$.



The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$ ■

An Equation for a Plane in Space

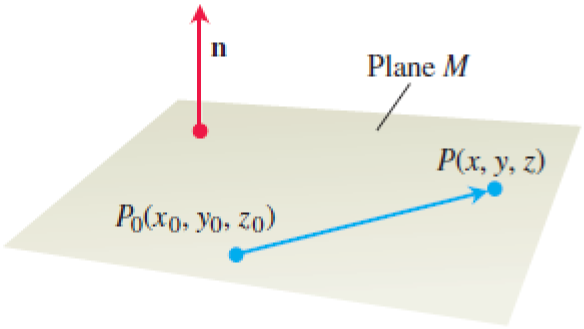
A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (see Figure). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0,$$

so the plane M consists of the points (x, y, z) satisfying

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$\begin{aligned} 5x + 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$

■

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x, y, and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$

■

PARTIAL DERIVATIVES

OVERVIEW Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and its height, so it is a function $V(r, h)$ of two variables r and h.

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a “partial” derivative. We will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions, so they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

$If\ z = f(x,y)\quad then$

First – order partial derivative:

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y$$

Second – order partial derivative:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(z_x)$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(z_y)$$

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(z_y)$$

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(z_x)$$

$$z_{xy} = z_{yx}$$

Example: If $w = (xy)^z$ find w_z, w_x, w_y

Solution:

$$w_z = \frac{\partial w}{\partial z} = (xy)^z \ \ln(x.y)$$

$$w_x = \frac{\partial w}{\partial x} = z(x)^{z-1}y^z$$

$$w_y = \frac{\partial w}{\partial y} = z(y)^{z-1}x^z$$

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$ ■

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With x held constant, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}. \quad ■$$

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

With y constant, $\frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}$.

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}. \quad ■$$

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5)

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4. \quad ■$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x \sin (y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x \sin (y + 3z)] = x \frac{\partial}{\partial z} \sin (y + 3z)$$

$$= x \cos (y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos (y + 3z).$$ ■

Examples: Find f_x, f_y, f_z

1. $f(x, y, z) = \sec^{-1}(x + yz)$

$$f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$$

2. $f(x, y, z) = \ln (x + 2y + 3z)$

$$f_x = \frac{1}{x+2y+3z}, \quad f_y = \frac{2}{x+2y+3z}, \quad f_z = \frac{3}{x+2y+3z}$$

Homework 3:

1. $If\ w = \tan^{-1}\left(\frac{y}{x}\right)$

   $Find\ \ w_x, w_y$

2. $If\ z = x^3 + y^5 + 3x^2y^4 + \sin(\frac{y}{x})$

   $Find\ \ z_x, z_y$

3. $If\ z = \frac{y}{x}$ Then show that.

   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$\frac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$ ■

EXAMPLE 10 Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x\partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y\partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x\partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0). ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x\partial y^2} = f_{yyx},$$

$$\frac{\partial^4 f}{\partial x^2\partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$ ■

Examples: Find all the second-order partial derivatives of the following functions:

1. $f(x, y) = \sin xy$

$$\frac{\partial f}{\partial x} = y\cos xy, \quad \frac{\partial f}{\partial y} = x\cos xy,$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy, \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$$

2. $s(x, y) = \tan^{-1}(y/x)$

$$\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2},$$

$$\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2},$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{\left(x^2+y^2\right)^2} = -\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{\left(x^2+y^2\right)(-1)+y(2y)}{\left(x^2+y^2\right)} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

3. Verify that $w_{xy} = w_{yx}$

$$w = \ln(2x + 3y)$$

$$\frac{\partial w}{\partial x} = \frac{2}{2x+3y}, \quad \frac{\partial w}{\partial y} = \frac{3}{2x+3y},$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}, \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

4. Verify that $w_{xy} = w_{yx}$

$$w = xy^2 + x^2y^3 + x^3y^4$$

$$\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4, \quad \frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3,$$

$$\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3, \text{ and}$$

$$\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$$

5. Show that the following function satisfies a Laplace equation ($\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$)

$$f(x, y, z) = e^{3x+4y} \cos 5z$$

$$\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z, \quad \frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z, \quad \frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z;$$

$$\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z,$$

$$\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z,$$

$$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$$

Homework 4:

$\boldsymbol{If\ w = \tan^{-1}\left(\frac{y}{x}\right)}$, ***show that***

i) $xw_x + yw_y = 0$

ii) $w_{xy} = w_{yx}$

iii) $w_{xx} + w_{yy} = 0$

The Chain Rule

The Chain Rule for functions of a single variable
$w = f(x)$ was a differentiable function of x and $x = g(t)$ was a differentiable function of t, w became a differentiable function of t and dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\cdot\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\cdot\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2} \sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2} \sin 2t\right) = \frac{1}{2} \cdot 2 \cos 2t = \cos 2t.$$

In either case, at the given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1. \quad ■$$

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for the intermediate variables.

so

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2. \quad ■$$

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

If f is a function of two intermediate variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

If f is a function of a single intermediate variable x, our equations are even simpler.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

In this case, we use the ordinary (single-variable) derivative, dw/dx.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\ &= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r\end{aligned}$$

Substitute for intermediate variable z.

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.\end{aligned}$$

■

EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r-s) + 2(r+s) \\ &= 4r\end{aligned} \qquad \begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r-s) + 2(r+s) \\ &= 4s\end{aligned}$$

Substitute for the intermediate variables.

■

Example:

In Exercise below , (a) express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then (b) evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

$$z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v; \quad (u, v) = (2, \pi/4)$$

(a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(4e^X \ln y\right)\left(\frac{\cos v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$

$$= \frac{4(u\cos v)\ln(u\sin v)}{u} + \frac{4(u\cos v)(\sin v)}{u\sin v} = (4\cos v)\ln(u\sin v) + 4\cos v;$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(4e^X \ln y\right)\left(\frac{-u\sin v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(u\cos v) = -\left(4e^X \ln y\right)(\tan v) + \frac{4e^x u\cos}{y}$$

$$= \left[-4(u\cos v)\ln(u\sin v)\right](\tan v) + \frac{4(u\cos v)(u\cos v)}{u\sin v} = (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$$

$$z = 4e^X \ln y = 4(u\cos v)\ln(u\sin v) \Rightarrow \frac{\partial z}{\partial u} = (4\cos v)\ln(u\sin v) + 4(u\cos v)\left(\frac{\sin v}{u\sin v}\right)$$

$$= (4\cos v)\ln(u\sin v) + 4\cos v; \text{ also } \frac{\partial z}{\partial v} = (-4u\sin v)\ln(u\sin v) + 4(u\cos v)\left(\frac{u\cos v}{u\sin v}\right)$$

$$= (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4} = 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2);$

$$\frac{\partial z}{\partial v} = (-4)(2)\sin\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)} = -4\sqrt{2}\ln\sqrt{2} + 4\sqrt{2} = -2\sqrt{2}\ln 2 + 4\sqrt{2}$$

H.W. 5 . **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

1 . $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u\sin v;$
$(u, v) = (1.3, \pi/6)$

Directional Derivatives and Gradient Vectors

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f. Using the gradient notation, we restate Equation (3) as a theorem.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

EXAMPLE Finding the Directional Derivative Using the Gradient

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$
$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}} f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u}$$
$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1\,.$$

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

EXAMPLE We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f-g) = \nabla(x-4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

$= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j}$ $g\nabla f$ plus terms . . .

$= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j}$ simplified.

$= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g$ Rule 4 ■

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

so the properties listed earlier for functions of two variables extend to three variables. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

EXAMPLE

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy|_{(1,1,0)} = -2, \qquad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

Estimating Change in a Specific Direction

Estimating the Change in f in a Direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional} \\ \text{derivative}}} \underbrace{ds}_{\substack{\text{Distance} \\ \text{increment}}}$$

EXAMPLE Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit}.$$

Differentials

Recall from Section 3.9 that for a function of a single variable, $y = f(x)$, we defined the change in f as x changes from a to $a + \Delta x$ by

$$\Delta f = f(a + \Delta x) - f(a)$$

and the differential of f as

$$df = f'(a)\Delta x.$$

We now consider the differential of a function of two variables.

Suppose a differentiable function $f(x, y)$ and its partial derivatives exist at a point (x_0, y_0). If we move to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$, the change in f is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

A straightforward calculation from the definition of $L(x, y)$, using the notation $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, shows that the corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0)\\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.\end{aligned}$$

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the linearization of f is called the **total differential of** f.

EXAMPLE Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution To estimate the absolute change in $V = \pi r^2 h$, we use

$$\Delta V \approx dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh.$$

With $V_r = 2\pi rh$ and $V_h = \pi r^2$, we get

$$\begin{aligned} dV &= 2\pi r_0 h_0\,dr + \pi r_0{}^2\,dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63\text{ in}^3 \end{aligned}$$

■

EXAMPLE Estimating Percentage Error

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Solution We are told that

$$\left|\frac{dr}{r} \times 100\right| \le 2 \qquad \text{and} \qquad \left|\frac{dh}{h} \times 100\right| \le 0.5.$$

Since

$$\frac{dV}{V} = \frac{2\pi rh\,dr + \pi r^2\,dh}{\pi r^2 h} = \frac{2\,dr}{r} + \frac{dh}{h},$$

we have

$$\begin{aligned} \left|\frac{dV}{V}\right| &= \left|2\frac{dr}{r} + \frac{dh}{h}\right| \\ &\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \\ &\le 2(0.02) + 0.005 = 0.045. \end{aligned}$$

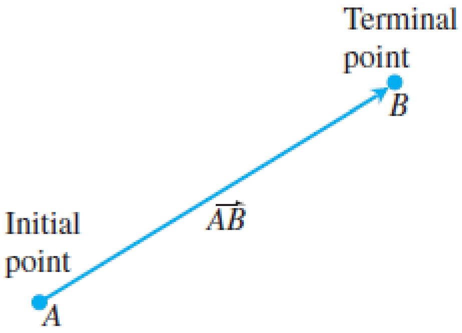
We estimate the error in the volume calculation to be at most 4.5%.

Vectors

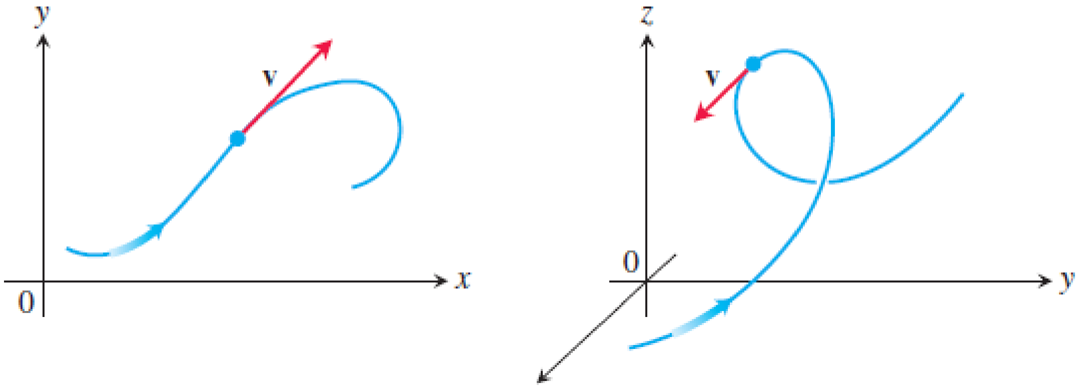
To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. In this section we show how to represent things that have both magnitude and direction in the plane or in space.

Component Form

A quantity such as force, displacement, or velocity is called a **vector** and is represented by a **directed line segment** as shown in the figure.



The directed line segment $\overrightarrow{AB}$ is called a vector.

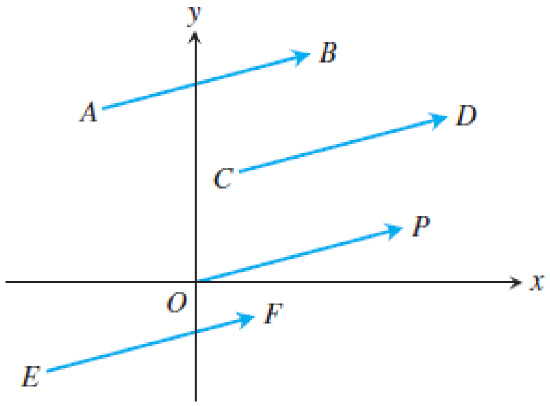


(a) two dimensions (b) three dimensions

The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

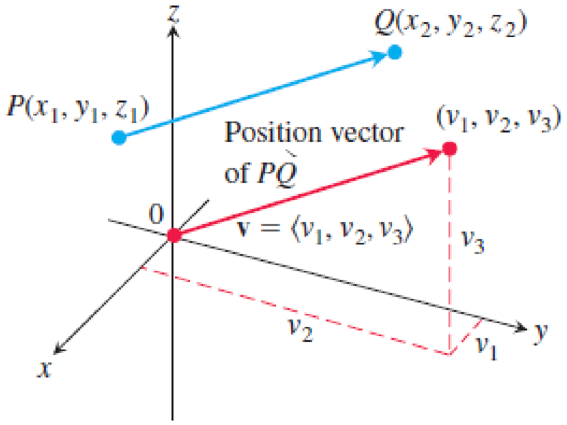
DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

The arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction (see Figure) regardless of the initial point.



The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}.$$



A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

DEFINITION If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Unit Vectors

A vector **v** of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} = \langle v_1, v_2, v_3 \rangle &= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \end{aligned}$$

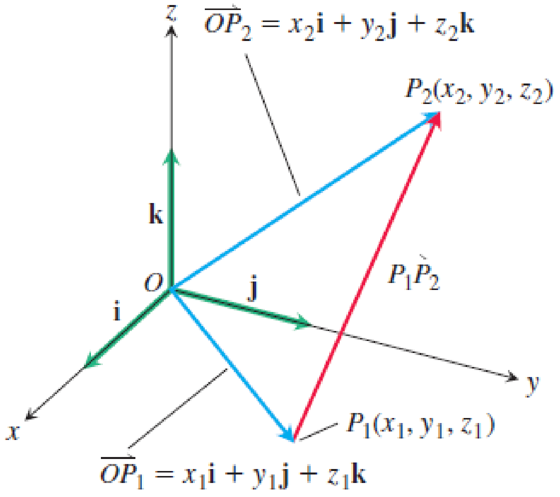
We call the scalar (or number) v_1 the **i-component** of the vector **v**, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

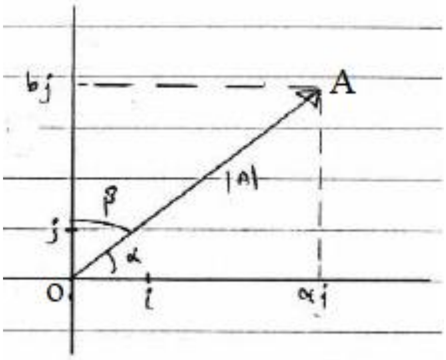
Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of **v**, called **the direction** of the nonzero vector **v**.



The vector from P_1 to P_2 is
$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$



$\vec{A} = \overline{OA} = ai + bj \qquad (Vector)$

$|\vec{A}| = \sqrt{a^2 + b^2} \qquad (Length\ of\ vector)$

i, j are the fundamental unit vectors

$$Unit\ vector = \vec{u} = \frac{\vec{A}}{|A|} = \frac{ai + bj}{|A|}$$

EXAMPLE 1 Find the (**a**) component form and (**b**) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(**a**) The standard position vector **v** representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle .$$

(**b**) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

■

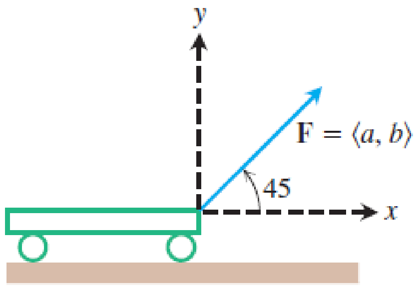
EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force **F** making a 45° angle to the floor (see Figure). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45° = (20)\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ lb}.$$

Notice that **F** is a two-dimensional vector.

■



The force pulling the cart forward is represented by the vector **F** whose horizontal component is the effective force (Example 2).

Vector Algebra Operations

Two principal operations involving vectors are *vector addition* and *scalar multiplication*. A **scalar** is simply a real number and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero and are used to "scale" a vector by multiplication.

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

The **difference** $\mathbf{u} - \mathbf{v}$ of two vectors is defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$ ■

EXAMPLE 4 Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underset{\substack{\text{Length}\\ \text{(speed)}}}{5}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

■

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 6 A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$

■

Midpoint of a Line Segment

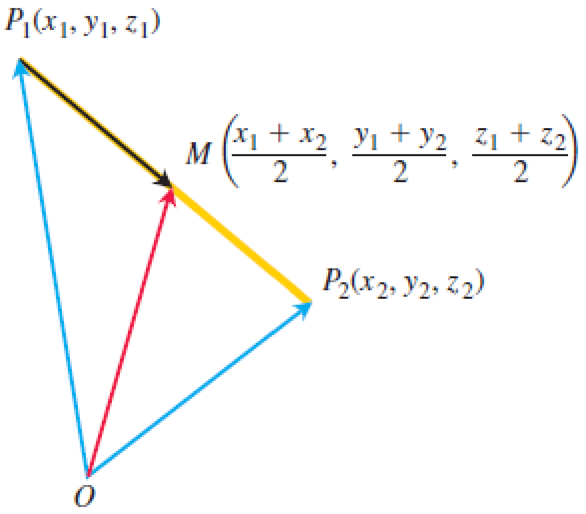
Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

To see why, observe (see Figure) that

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})\\ &= \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})\\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$



The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

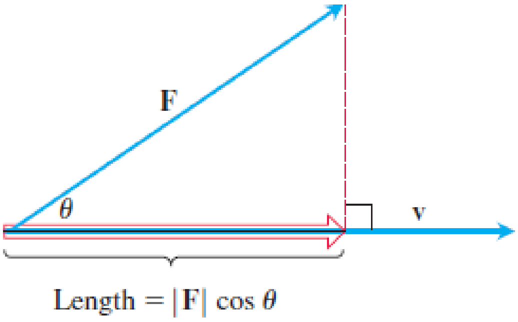
EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$

■

The Dot Product

Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another



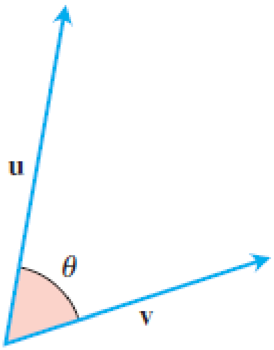
The magnitude of the force **F** in the direction of vector **v** is the length $|\mathbf{F}| \cos \theta$ of the projection of **F** onto **v**.

DEFINITION The **dot product $\mathbf{u} \cdot \mathbf{v}$** (“**u dot v**”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}||\mathbf{v}|}\right).$$



The angle between **u** and **v**.

The Angle Between Two Nonzero Vectors u and v

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

EXAMPLE 1 We illustrate the definition.

(a) $$\langle 1, -2, -1\rangle \cdot \langle -6, 2, -3\rangle = (1)(-6) + (-2)(2) + (-1)(-3) \\ = -6 - 4 + 3 = -7$$

(b) $$\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right)\cdot(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$$ ■

EXAMPLE 2 Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u}\cdot\mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians or } 100.98°.$$ ■

EXAMPLE 3 Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (see Figure).

Solution The angle θ is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

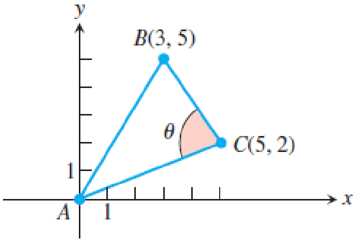
$$\overrightarrow{CA} = \langle -5, -2\rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3\rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\overrightarrow{CA}\cdot\overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$



The triangle in Example 3.

Then applying the angle formula, we have

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right)$$

$$= \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$

$$\approx 78.1^\circ \text{ or } 1.36 \text{ radians.}$$ ■

Orthogonal Vectors

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u}\cdot\mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$. The following definition also allows for one or both of the vectors to be the zero vector.

DEFINITION Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$.

EXAMPLE 4 To determine if two vectors are orthogonal, calculate their dot product.

(a) $\mathbf{u} = \langle 3, -2\rangle$ and $\mathbf{v} = \langle 4, 6\rangle$ are orthogonal because $\mathbf{u}\cdot\mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u}\cdot\mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned}\mathbf{0}\cdot\mathbf{u} &= \langle 0, 0, 0\rangle \cdot \langle u_1, u_2, u_3\rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0.\end{aligned}$$ ■

Dot Product Properties and Vector Projections

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

2. $(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$

3. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

4. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

5. $\mathbf{0}\cdot\mathbf{u} = 0.$

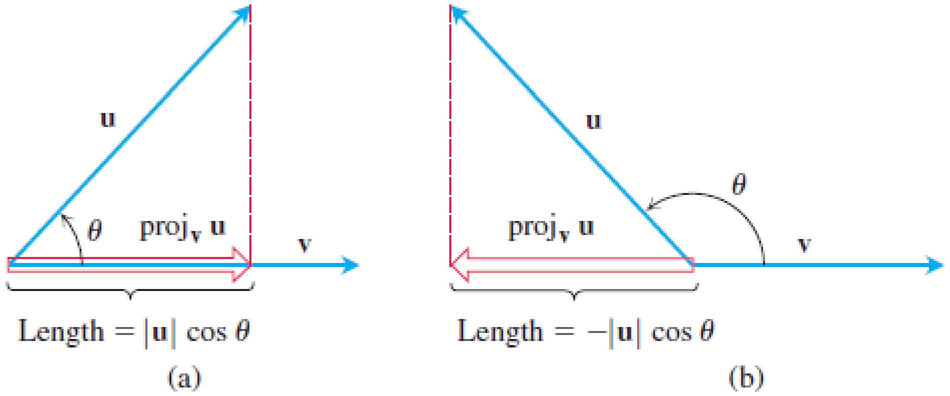
The vector projection

The vector projection of **u** onto **v** is the vector

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}. \qquad (1)$$

The scalar component of **u** in the direction of **v** is the scalar

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}. \qquad (2)$$



The length of $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is (a) $|\mathbf{u}|\cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{u}|\cos\theta$ if $\cos\theta < 0$.

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of **u** in the direction of **v**.

Solution We find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ from Equation (1):

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i}-2\mathbf{j}-2\mathbf{k})$$

$$= -\frac{4}{9}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

We find the scalar component of **u** in the direction of **v** from Equation (2):

$$|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$$

$$= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.$$

■

EXAMPLE 6 Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$.

Solution The vector projection is

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}\\ &= \frac{5-6}{1+9}(\mathbf{i}-3\mathbf{j}) = -\frac{1}{10}(\mathbf{i}-3\mathbf{j})\\ &= -\frac{1}{10}\mathbf{i} + \frac{3}{10}\mathbf{j}.\end{aligned}$$

The scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$ is

$$|\mathbf{F}|\cos\theta = \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{5-6}{\sqrt{1+9}} = -\frac{1}{\sqrt{10}}.$$

Example:

Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.

Solution:

$$\overrightarrow{AB} = \langle 3, 1\rangle,\ \overrightarrow{BC} = \langle -1, -3\rangle,\ \text{and}\ \overrightarrow{AC} = \langle 2, -2\rangle.\ \overrightarrow{BA} = \langle -3, -1\rangle,\ \overrightarrow{CB} = \langle 1, 3\rangle,\ \overrightarrow{CA} = \langle -2, 2\rangle.$$

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right| = \sqrt{10},\ \left|\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\right| = \sqrt{10},\ \left|\overrightarrow{AC}\right| = \left|\overrightarrow{CA}\right| = 2\sqrt{2},$$

$$\text{Angle at } A = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$$

$$\text{Angle at } B = \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{\left(\sqrt{10}\right)\left(\sqrt{10}\right)}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°,\ \text{and}$$

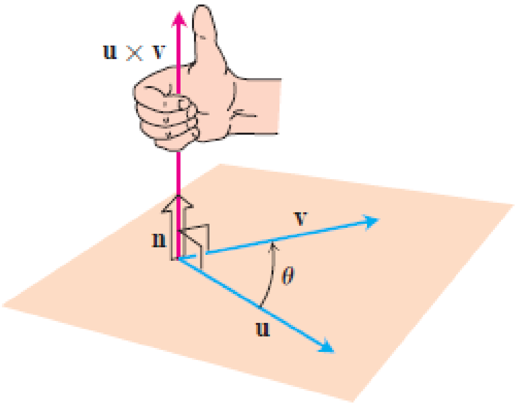
$$\text{Angle at } C = \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$$

The Cross Product

DEFINITION The **cross product** $\mathbf{u} \times \mathbf{v}$ (“**u cross v**”) is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta)\, \mathbf{n}.$$

Unlike the dot product, the cross product is a vector. For this reason it’s also called the **vector product** of **u** and **v**, and applies *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v** because it is a scalar multiple of **n**.



Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Parallel Vectors

Nonzero vectors **u** and **v** are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The cross product obeys the following laws.

Properties of the Cross Product

If **u**, **v**, and **w** are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$

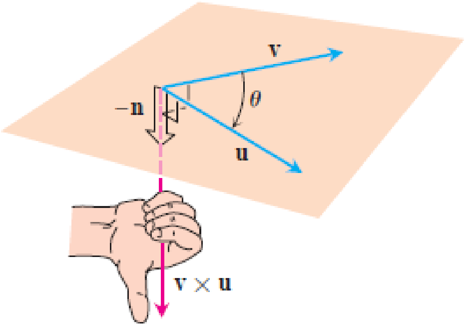
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$



$$\mathbf{i}\times\mathbf{j} = -(\mathbf{j}\times\mathbf{i}) = \mathbf{k}$$

$$\mathbf{j}\times\mathbf{k} = -(\mathbf{k}\times\mathbf{j}) = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{k}) = \mathbf{j}$$

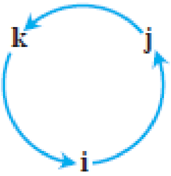


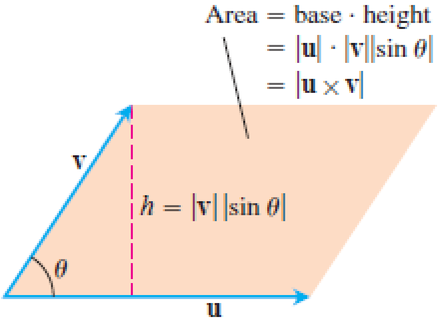
Diagram for recalling these products

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}.$$

$|\mathbf{u}\times\mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u}\times\mathbf{v}$ is

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\;|\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$



EXAMPLE Calculating Cross Products with Determinants

Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Solution

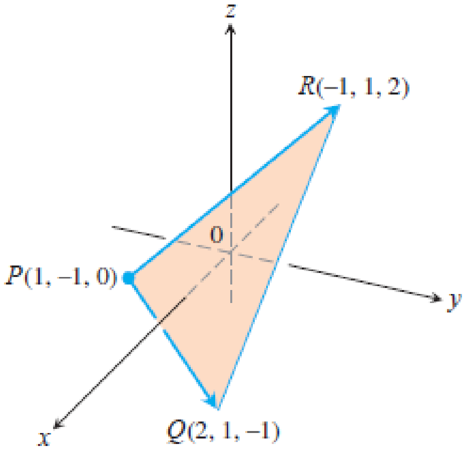
$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

EXAMPLE Finding Vectors Perpendicular to a Plane

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$



Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

EXAMPLE Finding the Area of a Triangle

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}|$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE Finding a Unit Normal to a Plane

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

Note:

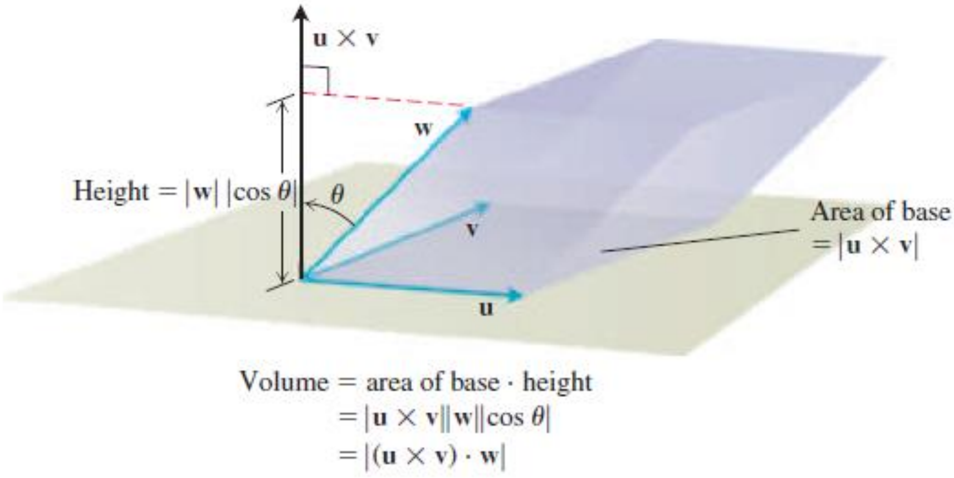
For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w** (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos\theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by **u**, **v**, and **w** (See Figure). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base parallelogram. The number $|\mathbf{w}||\cos\theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of **u**, **v**, and **w**.



The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

By treating the planes of **v** and **w** and of **w** and **u** as the base planes of the parallelepiped determined by **u**, **v**, and **w**, we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{v}=-2\mathbf{i}+3\mathbf{k}$, and $\mathbf{w}=7\mathbf{j}-4\mathbf{k}$.

Solution Using the rule for calculating a 3 × 3 determinant, we find

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}=(1)\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix}-(2)\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix}+(-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}=-23.$$

The volume is $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=23$ units cubed. ■

Examples:

1. a. Find the area of the triangle determined by the points P, Q, and R.

   b. Find a unit vector perpendicular to plane PQR.

   $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$

Solution:

(a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k}\Rightarrow$ Area $=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{64+16+16}=2\sqrt{6}$

(b) $\mathbf{u}=\dfrac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{1}{\sqrt{6}}\left(2\mathbf{i}+\mathbf{j}+\mathbf{k}\right)$

2. find the volume of the parallelepiped (box) determined by **u**, **v**, and **w**.

u	**v**	**w**
$\mathbf{i}-\mathbf{j}+\mathbf{k}$	$2\mathbf{i}+\mathbf{j}-2\mathbf{k}$	$-\mathbf{i}+2\mathbf{j}-\mathbf{k}$

Solution:

$$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|=abs\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix}=4$$

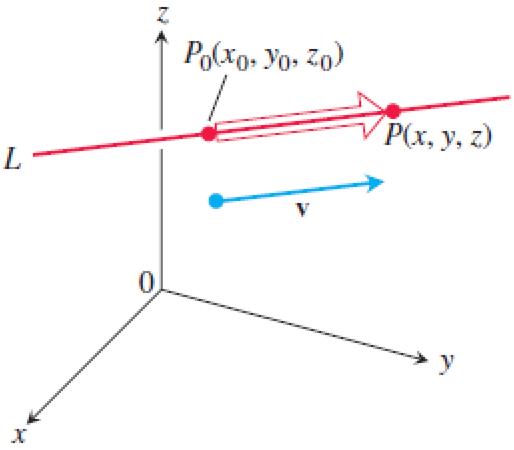
Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$



A point P lies on L through P_0 parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

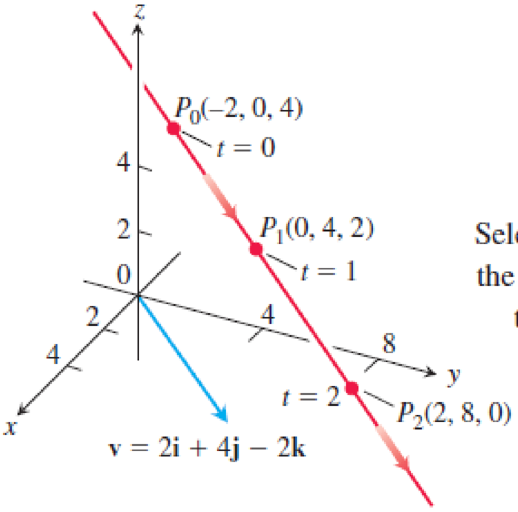
The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$. Rewriting Equation (2), we have

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v}\\ &= \underset{\substack{\nearrow\\ \text{Initial}\\ \text{position}}}{\mathbf{r}_0} + \underset{\substack{\nearrow\\ \text{Time}}}{t}\underset{\substack{\nwarrow\\ \text{Speed}}}{|\mathbf{v}|}\underset{\substack{\nwarrow\\ \text{Direction}}}{\frac{\mathbf{v}}{|\mathbf{v}|}}.\end{aligned} \tag{4}$$

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (See Figure).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t. \qquad ■$$



Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing t.

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t. ■

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (see Figure).

Solution We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

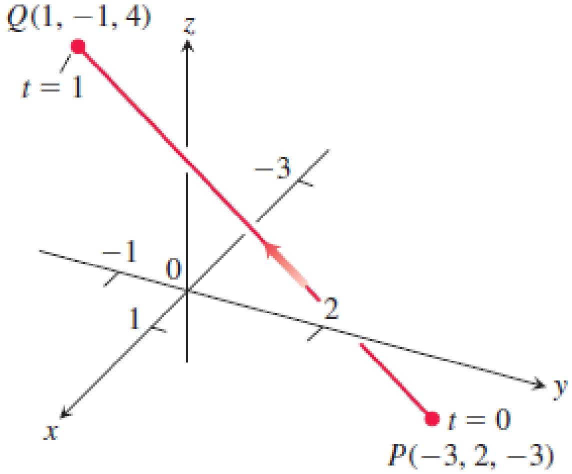
$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1. \quad ■$$



Example 3 derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u}\\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)\\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

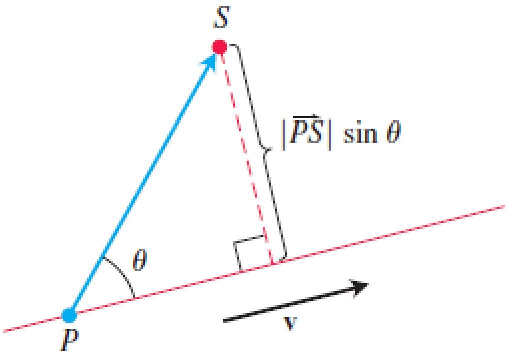
When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})\\ &= \left\langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\right\rangle.\end{aligned}$$

After 10 sec of flight from the origin toward (1, 1, 1), the helicopter is located at the point $\left(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\right)$ in space. It has traveled a distance of (60 ft/sec)(10 sec) = 600 ft, which is the length of the vector $\mathbf{r}(10)$. ■

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line (See Figure). In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}|\sin\theta$, which is $\dfrac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.



The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}. \quad \blacksquare$$

An Equation for a Plane in Space

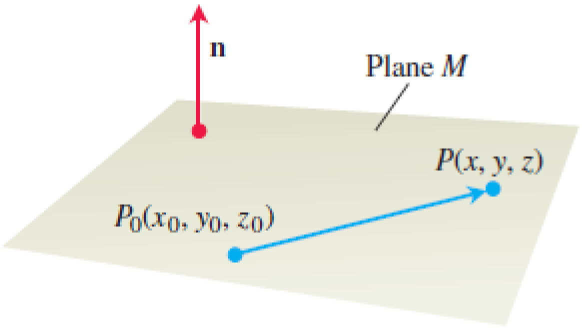
A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (see Figure). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0,$$

so the plane M consists of the points (x, y, z) satisfying

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$



The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D,$ where $D = Ax_0 + By_0 + Cz_0$

EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$\begin{aligned} 5x + 15 + 2y - z + 7 &= 0 \\ 5x + 2y - z &= -22. \end{aligned}$$ ■

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x, y, and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$ ■

PARTIAL DERIVATIVES

OVERVIEW Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and its height, so it is a function $V(r, h)$ of two variables r and h.

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. We will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions, so they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

$If\ z = f(x,y) \quad then$

First – order partial derivative:

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y$$

Second – order partial derivative:

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(z_x)$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(z_y)$$

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(z_y)$$

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(z_x)$$

$$z_{xy} = z_{yx}$$

Example: If $w = (xy)^z$ find w_z, w_x, w_y

Solution:

$$w_z = \frac{\partial w}{\partial z} = (xy)^z \ \ln(x.y)$$

$$w_x = \frac{\partial w}{\partial x} = z(x)^{z-1}y^z$$

$$w_y = \frac{\partial w}{\partial y} = z(y)^{z-1}x^z$$

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$ ■

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With x held constant, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$ ■

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

With y constant, $\frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}$.

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$ ■

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5)

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$ ■

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x \sin (y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x \sin (y + 3z)] = x \frac{\partial}{\partial z} \sin (y + 3z)$$

$$= x \cos (y + 3z) \frac{\partial}{\partial z}(y + 3z) = 3x \cos (y + 3z).$$ ■

Examples: Find f_x, f_y, f_z

1. $f(x, y, z) = \sec^{-1}(x + yz)$

$$f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}, \quad f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$$

2. $f(x, y, z) = \ln (x + 2y + 3z)$

$$f_x = \frac{1}{x+2y+3z}, \quad f_y = \frac{2}{x+2y+3z}, \quad f_z = \frac{3}{x+2y+3z}$$

Homework 3:

1. $If\ w = \tan^{-1}\left(\frac{y}{x}\right)$

   $Find\ \ w_x, w_y$

2. $If\ z = x^3 + y^5 + 3x^2y^4 + \sin(\frac{y}{x})$

   $Find\ \ z_x, z_y$

3. $If\ z = \frac{y}{x}$ Then show that.

   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$\dfrac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

EXAMPLE 10 Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0). ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx},$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

■

Examples: Find all the second-order partial derivatives of the following functions:

1. $f(x, y) = \sin xy$

$$\frac{\partial f}{\partial x} = y \cos xy, \quad \frac{\partial f}{\partial y} = x \cos xy,$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy, \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$$

2. $s(x, y) = \tan^{-1}(y/x)$

$$\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2},$$

$$\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2},$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{\left(x^2+y^2\right)^2} = -\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{\left(x^2+y^2\right)(-1)+y(2y)}{\left(x^2+y^2\right)} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

3. Verify that $w_{xy} = w_{yx}$

$w = \ln(2x + 3y)$

$$\frac{\partial w}{\partial x} = \frac{2}{2x+3y}, \quad \frac{\partial w}{\partial y} = \frac{3}{2x+3y},$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

4. Verify that $w_{xy} = w_{yx}$

$w = xy^2 + x^2y^3 + x^3y^4$

$$\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4, \quad \frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3,$$

$$\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3, \text{ and}$$

$$\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$$

5. Show that the following function satisfies a Laplace equation ($\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$)

$$f(x, y, z) = e^{3x+4y} \cos 5z$$

$$\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z, \quad \frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z, \quad \frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z;$$

$$\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z,$$

$$\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z,$$

$$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$$

Homework 4:

If $w = \tan^{-1}\left(\frac{y}{x}\right)$, ***show that***

i) $xw_x + yw_y = 0$

ii) $w_{xy} = w_{yx}$

iii) $w_{xx} + w_{yy} = 0$

The Chain Rule

The Chain Rule for functions of a single variable
$w = f(x)$ was a differentiable function of x and $x = g(t)$ was a differentiable function of t, w became a differentiable function of t and dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t), y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\cdot\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\cdot\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos 2t = \cos 2t.$$

In either case, at the given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1.$$

■

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for the intermediate variables.

so

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

■

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

If f is a function of two intermediate variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

If f is a function of a single intermediate variable x, our equations are even simpler.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

In this case, we use the ordinary (single-variable) derivative, dw/dx.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\ &= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r\end{aligned}$$

Substitute for intermediate variable z.

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.\end{aligned}$$

■

EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r\end{aligned} \qquad \begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s\end{aligned}$$

Substitute for the intermediate variables.

■

Example:

In Exercise below , (a) express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then (b) evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

$z = 4e^x \ln y, \quad x = \ln(u \cos v), \quad y = u \sin v; \quad (u, v) = (2, \pi/4)$

(a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(4e^x \ln y\right)\left(\frac{\cos v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$

$= \frac{4(u \cos v) \ln(u \sin v)}{u} + \frac{4(u \cos v)(\sin v)}{u \sin v} = (4 \cos v) \ln(u \sin v) + 4 \cos v;$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(4e^x \ln y\right)\left(\frac{-u \sin v}{u \cos v}\right) + \left(\frac{4e^x}{y}\right)(u \cos v) = -\left(4e^x \ln y\right)(\tan v) + \frac{4e^x u \cos}{y}$

$= \left[-4(u \cos v) \ln(u \sin v)\right](\tan v) + \frac{4(u \cos v)(u \cos v)}{u \sin v} = (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$

$z = 4e^x \ln y = 4(u \cos v) \ln(u \sin v) \Rightarrow \frac{\partial z}{\partial u} = (4 \cos v) \ln(u \sin v) + 4(u \cos v)\left(\frac{\sin v}{u \sin v}\right)$

$= (4 \cos v) \ln(u \sin v) + 4 \cos v;$ also $\frac{\partial z}{\partial v} = (-4u \sin v) \ln(u \sin v) + 4(u \cos v)\left(\frac{u \cos v}{u \sin v}\right)$

$= (-4u \sin v) \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4 \cos\frac{\pi}{4} \ln\left(2 \sin\frac{\pi}{4}\right) + 4 \cos\frac{\pi}{4} = 2\sqrt{2} \ln\sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2);$

$\frac{\partial z}{\partial v} = (-4)(2) \sin\frac{\pi}{4} \ln\left(2 \sin\frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)} = -4\sqrt{2} \ln\sqrt{2} + 4\sqrt{2} = -2\sqrt{2} \ln 2 + 4\sqrt{2}$

H.W. 5 . **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

1 . $z = \tan^{-1}(x/y), \quad x = u \cos v, \quad y = u \sin v;$
$(u, v) = (1.3, \pi/6)$

Directional Derivatives and Gradient Vectors

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f. Using the gradient notation, we restate Equation (3) as a theorem.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

EXAMPLE Finding the Directional Derivative Using the Gradient

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}} f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1 .$$

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

EXAMPLE We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f-g) = \nabla(x-4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$
$= 3y(\mathbf{i}-\mathbf{j}) + 3y\mathbf{j} + (3x-6y)\mathbf{j}$ $g\nabla f$ plus terms . . .
$= 3y(\mathbf{i}-\mathbf{j}) + (3x-3y)\mathbf{j}$ simplified.
$= 3y(\mathbf{i}-\mathbf{j}) + (x-y)3\mathbf{j} = g\nabla f + f\nabla g$ Rule 4 ■

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

so the properties listed earlier for functions of two variables extend to three variables. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

EXAMPLE

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy\big|_{(1,1,0)} = -2, \qquad f_z = -1\big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$ ■

Estimating Change in a Specific Direction

Estimating the Change in f in a Direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}} \underbrace{ds}_{\text{Distance increment}}$$

EXAMPLE Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

Differentials

Recall from Section 3.9 that for a function of a single variable, $y = f(x)$, we defined the change in f as x changes from a to $a + \Delta x$ by

$$\Delta f = f(a + \Delta x) - f(a)$$

and the differential of f as

$$df = f'(a)\Delta x.$$

We now consider the differential of a function of two variables.

Suppose a differentiable function $f(x, y)$ and its partial derivatives exist at a point (x_0, y_0). If we move to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$, the change in f is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

A straightforward calculation from the definition of $L(x, y)$, using the notation $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, shows that the corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0)\\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.\end{aligned}$$

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

in the linearization of f is called the **total differential of** f.

EXAMPLE Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution To estimate the absolute change in $V = \pi r^2 h$, we use

$$\Delta V \approx dV = V_r(r_0, h_0)\, dr + V_h(r_0, h_0)\, dh.$$

With $V_r = 2\pi rh$ and $V_h = \pi r^2$, we get

$$\begin{aligned} dV &= 2\pi r_0 h_0\, dr + \pi r_0{}^2\, dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in}^3 \end{aligned}$$

■

EXAMPLE Estimating Percentage Error

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Solution We are told that

$$\left|\frac{dr}{r} \times 100\right| \le 2 \qquad \text{and} \qquad \left|\frac{dh}{h} \times 100\right| \le 0.5.$$

Since

$$\frac{dV}{V} = \frac{2\pi rh\, dr + \pi r^2\, dh}{\pi r^2 h} = \frac{2\, dr}{r} + \frac{dh}{h},$$

we have

$$\begin{aligned} \left|\frac{dV}{V}\right| &= \left|2\frac{dr}{r} + \frac{dh}{h}\right| \\ &\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \\ &\le 2(0.02) + 0.005 = 0.045. \end{aligned}$$

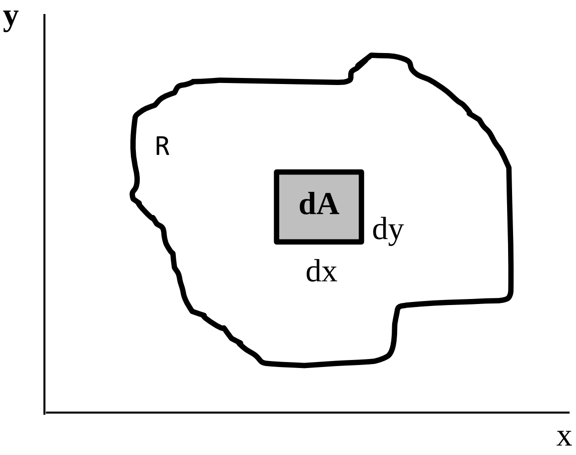
We estimate the error in the volume calculation to be at most 4.5%.

Multiple Integrals

1. Double Integrals

$$A = \iint_R dA = \iint_R dx\,dy$$

$$\text{Volume} = \iint_R f(x, y)\,dA = \iint_R Z\ dA$$



$$V = \iint_R z\,dA$$

$$= \iint_R f(x,y)\,dy\,dx\,.$$

$$= \iint_R f(x,y)\,dx\,dy\,.$$

EXAMPLE Evaluating a Double Integral

Calculate $\iint_R f(x, y)\,dA$ for

$$f(x, y) = 1 - 6x^2y \quad \text{and} \quad R:\ 0 \le x \le 2,\ \ -1 \le y \le 1.$$

Solution

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_{0}^{2}(1 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\left[x - 2x^3y\right]_{x=0}^{x=2}\,dy$$

$$= \int_{-1}^{1}(2 - 16y)\,dy = \left[2y - 8y^2\right]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1}(1 - 6x^2y)\,dy\,dx = \int_{0}^{2}\left[y - 3x^2y^2\right]_{y=-1}^{y=1}\,dx$$

$$= \int_{0}^{2}[(1 - 3x^2) - (-1 - 3x^2)]\,dx$$

$$= \int_{0}^{2} 2\,dx = 4.$$

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE Finding Volume

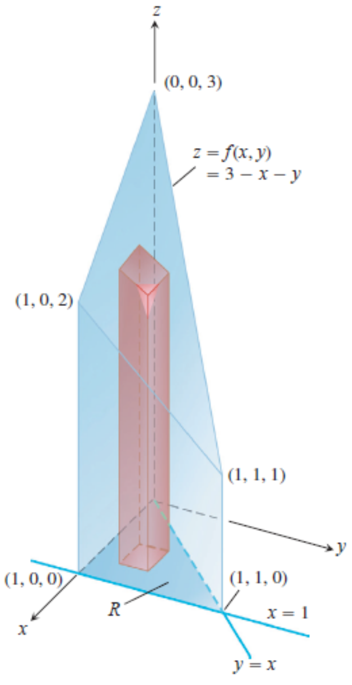
Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

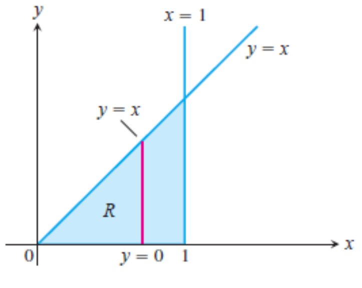
Solution For any x between 0 and 1, y may vary from $y = 0$ to $y = x$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

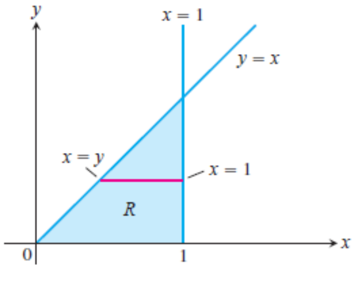
$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$



(a)



(b)



(c)

When the order of integration is reversed the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

EXAMPLE Evaluating a Double Integral

Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

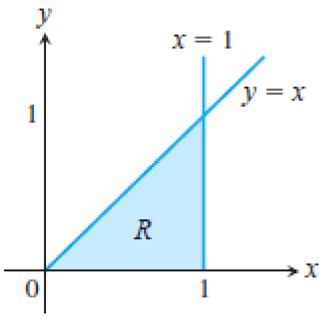
Solution The region of integration is shown in Figure . If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy\right) dx = \int_0^1 \left(y \frac{\sin x}{x}\right]_{y=0}^{y=x}\Bigg) dx = \int_0^1 \sin x\, dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

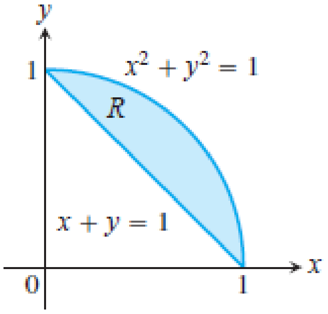


we run into a problem, because $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

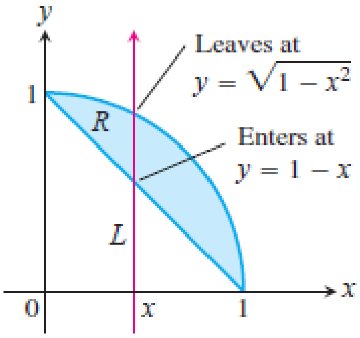
There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

Finding Limits of Integration

1. *Sketch*. Sketch the region of integration and label the bounding curves.



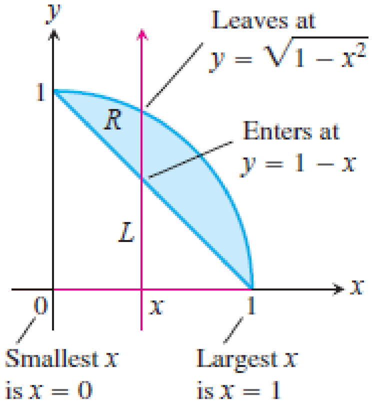
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).



3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral shown here is

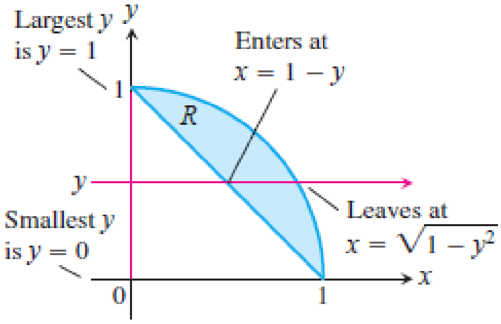
$$\iint_R f(x, y)\, dA =$$

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$



To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$



THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R:\ 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^1 \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^1 (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^1 = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^1 (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400. \quad ■$$

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.

Solution The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}. \qquad ■$$

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

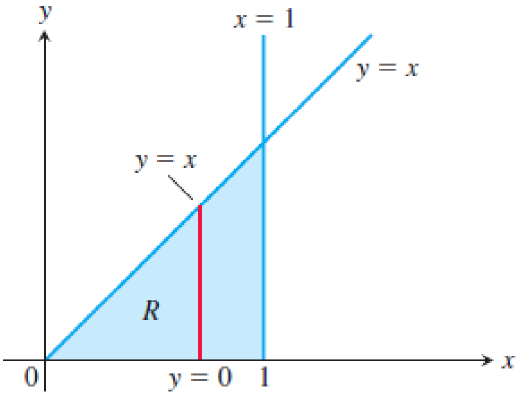
EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure . For any x between 0 and 1, y may vary from $y = 0$ to $y = x$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$



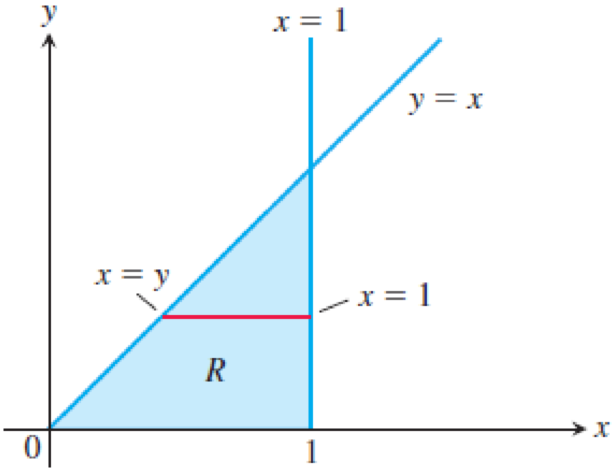
When the order of integration is reversed (**See** Figure), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be. ■



Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Examples of Double Integrals in Cartesian coordinates

EXAMPLE Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

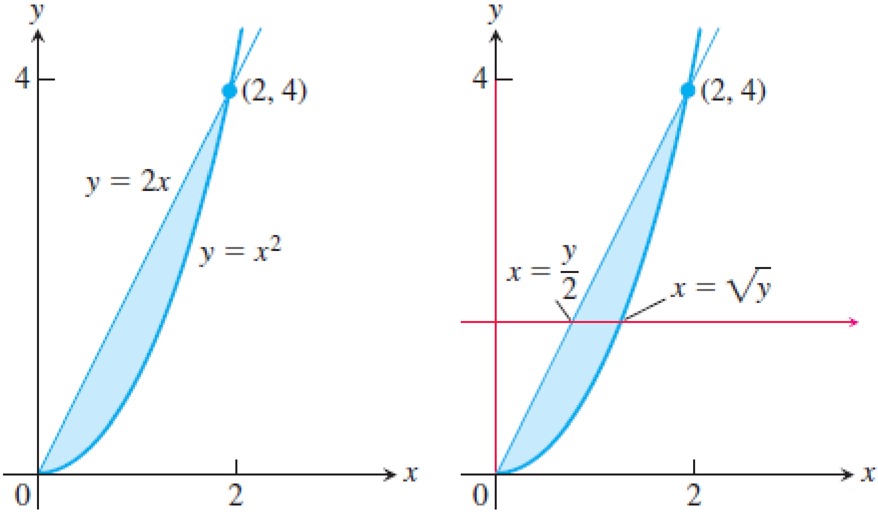
and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure .a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■



(a) (b)

EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\, dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\, dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

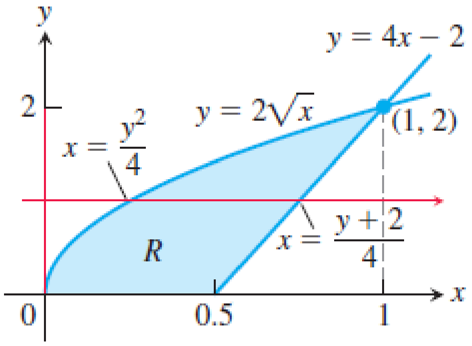
$$\iint_R (16 - x^2 - y^2)\, dA$$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2 \right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3 \cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344} \right]_0^2 = \frac{20803}{1680} \approx 12.4. \quad ■$$

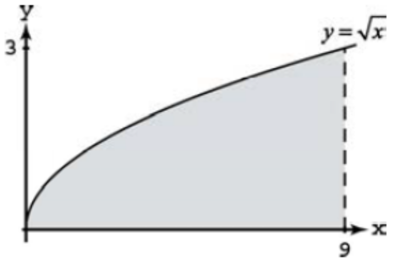


Examples: Write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

1. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$

(a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$

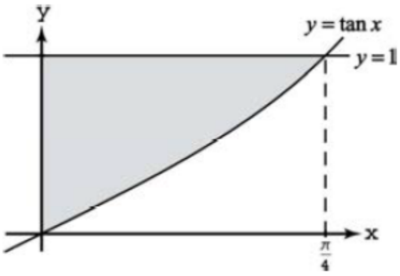
(b) $\int_0^3 \int_{y^2}^9 dx\, dy$



2 . Bounded by $y = \tan x$, $x = 0$, and $y = 1$

(a) $\int_0^{\pi/4} \int_{\tan x}^1 dy\, dx$

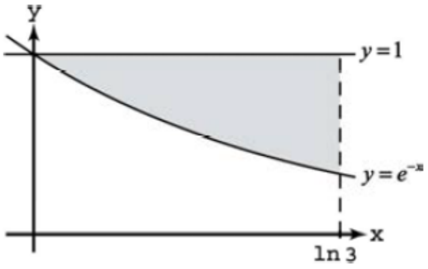
(b) $\int_0^1 \int_0^{\tan^{-1} y} dx\, dy$



3 . Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$

(a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy\, dx$

(b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx\, dy$



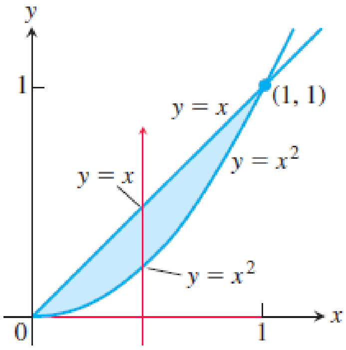
EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure), noting where the two curves intersect at the origin and (1, 1), and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

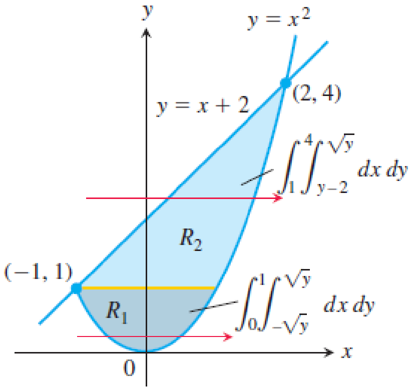
Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6. ■



EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$



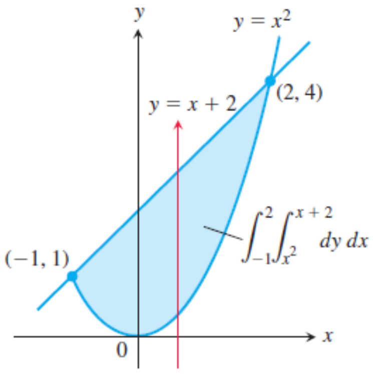
(a)

On the other hand, reversing the order of integration (Figure b) gives

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx.$$

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^{2}\Big[y\Big]_{x^2}^{x+2} dx = \int_{-1}^{2}(x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}. \qquad \blacksquare$$

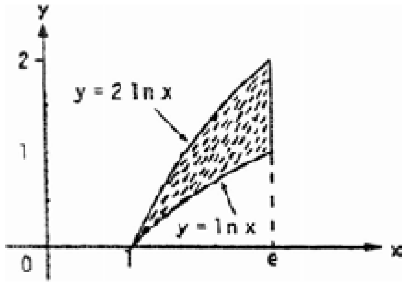


(b)

Examples: Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

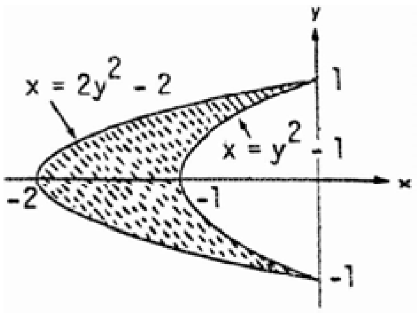
1. The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant

$$\int_1^e\int_{\ln x}^{2\ln x} dy\, dx = \int_1^e \ln x\, dx = [x\ln x - x]_1^e$$
$$= (e-e)-(0-1) = 1$$



2. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

$$\int_{-1}^{1}\int_{2y^2-2}^{y^2-1} dx\, dy = \int_{-1}^{1}\left(y^2 - 1 - 2y^2 + 2\right) dy$$
$$= \int_{-1}^{1}\left(1 - y^2\right) dy = \left[y - \frac{y^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$



Double Integrals in Polar Form

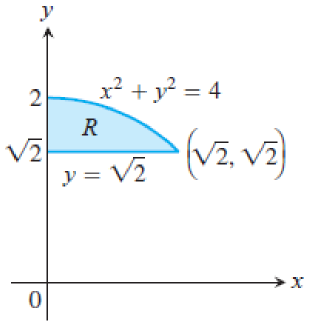
Double integrals are sometimes easier to evaluate if we change to polar coordinates.

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

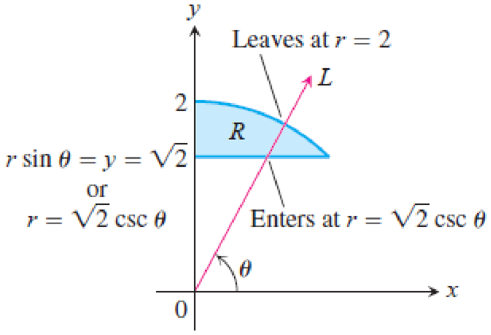
$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Finding Limits of Integration

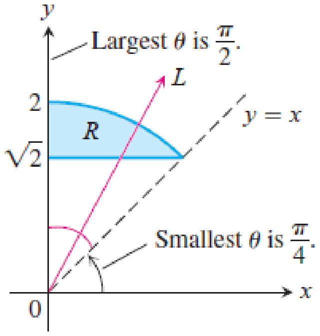
1. *Sketch*: Sketch the region and label the bounding curves.



2. *Find the r-limits of integration*: Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.



3. *Find the θ-limits of integration*: Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.



The integral is

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\,r\,dr\,d\theta.$$

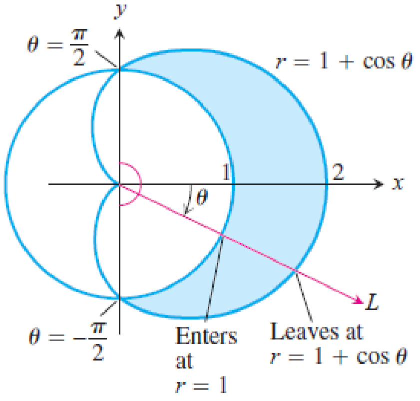
EXAMPLE Finding Limits of Integration

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves .
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta.$$



If $f(r,\theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

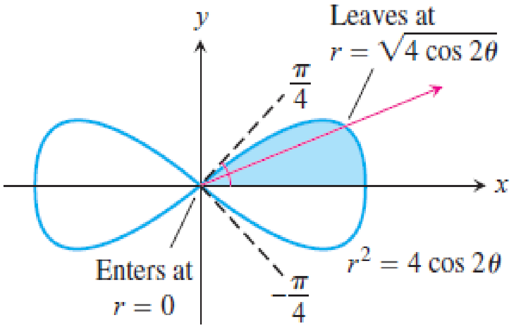
EXAMPLE Finding Area in Polar Coordinates

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

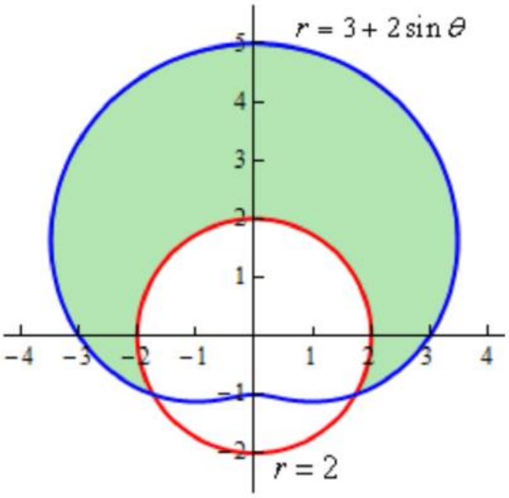
$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Hide Solution ▾

Here is a sketch of the region, D, that we want to determine the area of.



To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations equal and solving.

$$3 + 2\sin\theta = 2$$
$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.

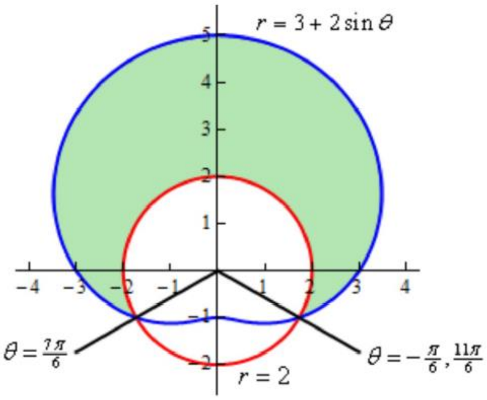
Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$ then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$
$$2 \leq r \leq 3 + 2\sin\theta$$

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region D is then,



$$\begin{aligned}
A &= \iint\limits_D dA \\
&= \int_{-\pi/6}^{7\pi/6} \int_2^{3+2\sin\theta} r \, dr d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \Big|_2^{3+2\sin\theta} d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta \, d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta) \, d\theta \\
&= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187
\end{aligned}$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

where G denotes the region of integration in polar coordinates.

Notice that $dx\, dy$ is not replaced by $dr\, d\theta$ but by $r\, dr\, d\theta$.

EXAMPLE 3 Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

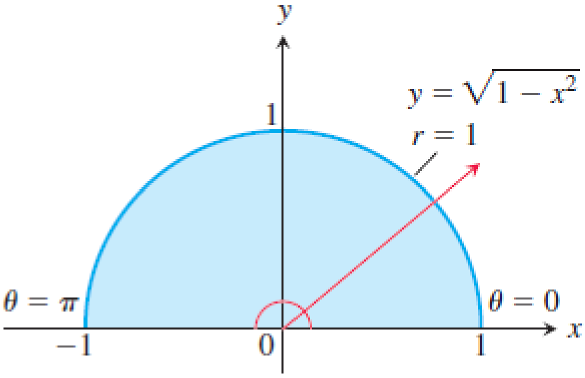
where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dy\, dx$ by $r\, dr\, d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$

The r in the $r\, dr\, d\theta$ was just what we needed to integrate e^{r^2}. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■



EXAMPLE Changing Cartesian Integrals to Polar Integrals

Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution We sketch the plate to determine the limits of integration . In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Integration with respect to y gives

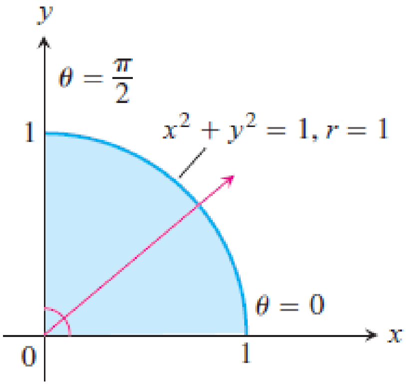
$$\int_0^1 \left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dx\,dy$ by $r\,dr\,d\theta$, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

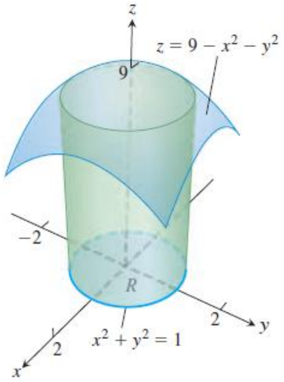
$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$



EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution The region of integration R is the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1$, $0 \leq \theta \leq 2\pi$. The solid region is shown in Figure The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$



$$= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta$$

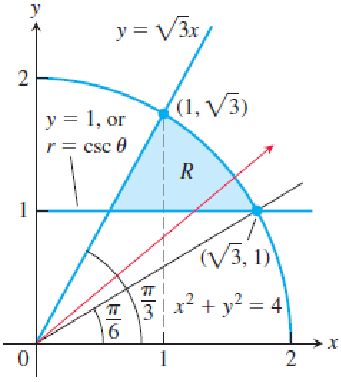
$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

■

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution A sketch of the region R is shown in Figure . First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.29.

Now, for the region R, as θ varies from $\pi/6$ to $\pi/3$, the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r \sin\theta$ for y in the equation for the horizontal line, we have $r \sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, r varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area then gives



$$\iint_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{1}{2} r^2 \right]_{r=\csc\theta}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2} [4 - \csc^2\theta]\, d\theta$$

$$= \frac{1}{2} \left[4\theta + \cot\theta \right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}} \right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3} \right) = \frac{\pi - \sqrt{3}}{3}.$$

■

Example Evaluate the following integrals by converting them into polar coordinates.

$\displaystyle\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

Solution

$\displaystyle\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between the two circles, so we will have the following inequality for r.

$$2 \le r \le 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}}\int_2^5 2\left(r\cos\theta\right)\left(r\sin\theta\right)r\,dr\,d\theta$$

Don't forget to do the conversions and to add in the extra r. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2x\,y\,dA &= \int_0^{\frac{\pi}{2}}\int_2^5 r^3\sin(2\theta)\,dr\,d\theta\\ &= \int_0^{\frac{\pi}{2}} \left.\frac{1}{4}r^4\sin(2\theta)\right|_2^5 d\theta\\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4}\sin(2\theta)\,d\theta\\ &= \left.-\frac{609}{8}\cos(2\theta)\right|_0^{\frac{\pi}{2}}\\ &= \frac{609}{4}\end{aligned}$$

Example: Evaluating a Double Integral by Converting from Rectangular Coordinates

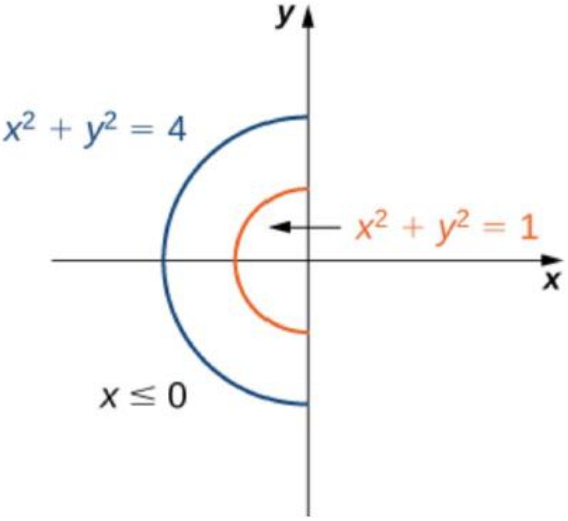
Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R = \{(x,y)\,|\,1 \le x^2+y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r,\theta)\,|\,1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).



Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

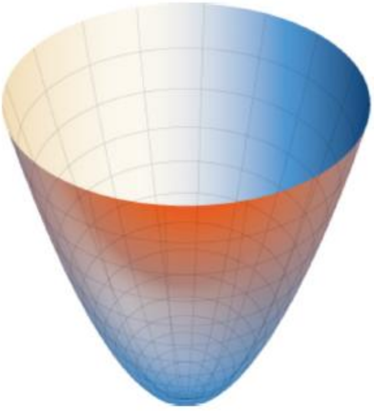
$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

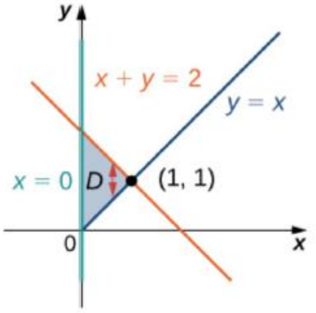
Example: Finding a Volume Using a Double Integral

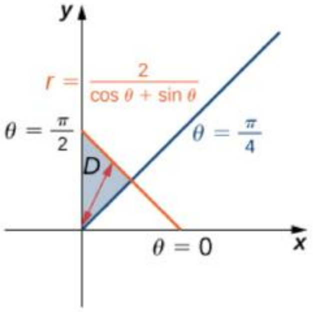
Find the volume of the region that lies under the paraboloid $z = x^2+y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.







The region D is $\{(x,y) \,|\, 0 \le x \le 1,\ x \le y \le 2-x\}$. Converting the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\ \theta + \sin\ \theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D = \{(r,\theta) \,|\, \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 2/(\cos\ \theta + \sin\ \theta)\}$.

Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned}
V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\
&= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\ \theta+\sin\ \theta)} r^2 r\, dr d\theta \\
&= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\ \theta+\sin\ \theta)} d\theta \\
&= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\
&= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\
&= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta.
\end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

$$\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Bigg|_0^1 \\
&= \frac{4}{3}\ \text{units}^3.
\end{aligned}$$

Triple Integrals in Rectangular Coordinates

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

Finding Limits of Integration in the Order *dz dy dx*

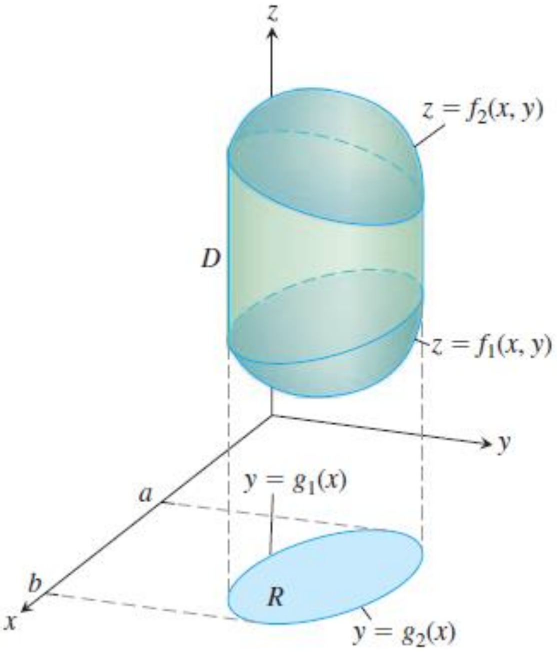
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.2) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these iterated integrals.

To evaluate

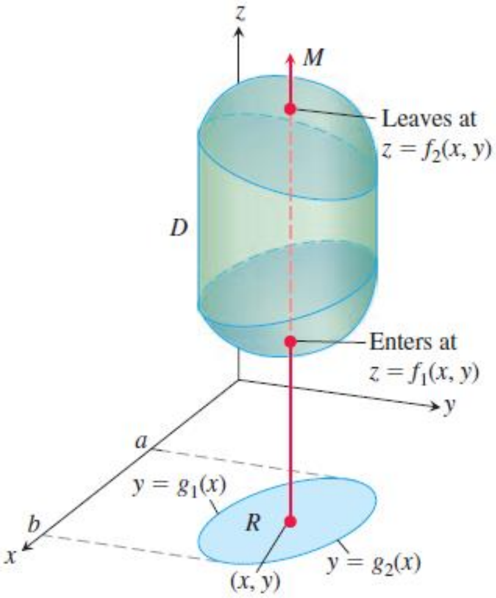
$$\iiint_D F(x, y, z)\, dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x. (You might choose a different order of integration, but the procedure is similar, as we illustrate in Example 2.)

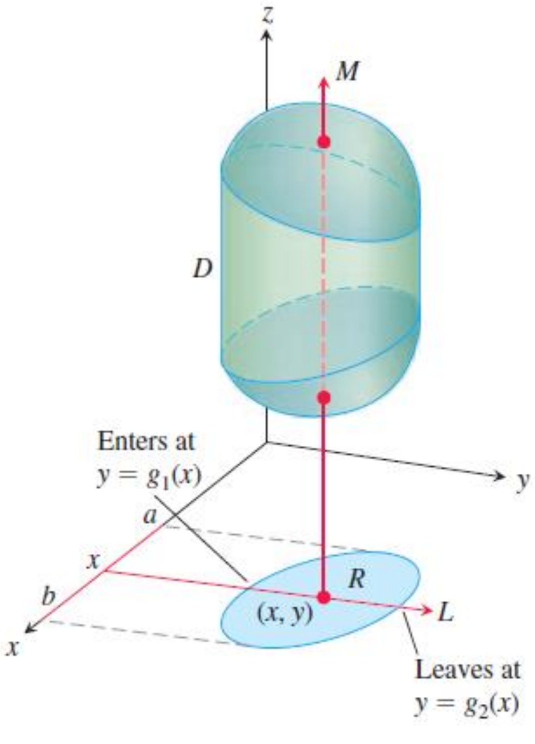
1. *Sketch.* Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x, y)}^{z=f_2(x, y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

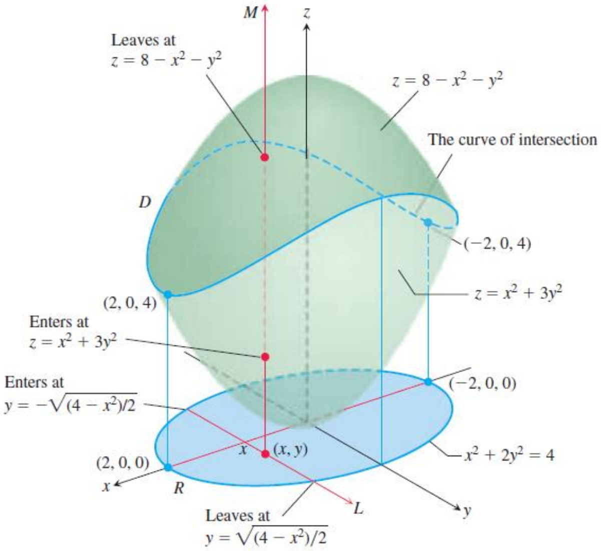
EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\,dy\,dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (**see** Figure) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.



Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$\begin{aligned} V &= \iiint_D dz\,dy\,dx \\ &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\ &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx \end{aligned}$$

$$= \int_{-2}^{2}\left[(8-2x^2)y-\frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}}dx$$

$$= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u$$ ■

EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dy\,dz\,dx$.

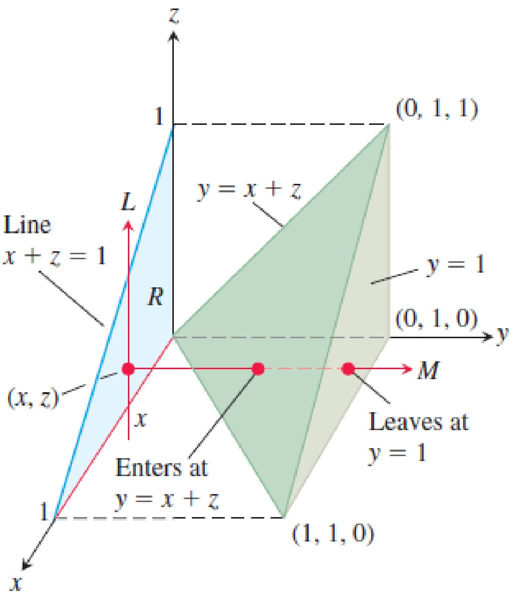
Solution We sketch D along with its "shadow" R in the xz-plane (**see** Figure). The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the y-limits of integration. The line through a typical point (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

Next we find the z-limits of integration. The line L through (x, z) parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. The integral is

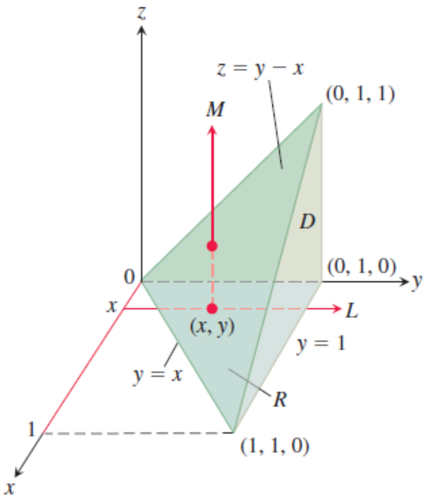
$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} F(x, y, z)\,dy\,dz\,dx.$$ ■



EXAMPLE 3 Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 2 in the order $dz\, dy\, dx$, and then integrate in the order $dy\, dz\, dx$.

Solution First we find the z-limits of integration. A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (see Figure).

Next we find the y-limits of integration. On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exits at $y = 1$ (see Figure).



Finally we find the x-limits of integration. As the line L parallel to the y-axis in the previous step sweeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$ (see Figure 15.33). The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

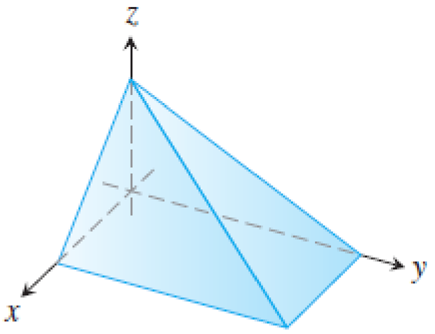
$$\begin{aligned} V &= \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx \\ &= \int_0^1 \int_x^1 (y - x)\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

We get the same result by integrating with the order $dy\,dz\,dx$. From Example 2,

$$\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\,dz\,dx \\
&= \int_0^1 \int_0^{1-x} (1 - x - z)\,dz\,dx \\
&= \int_0^1 \left[(1 - x)z - \frac{1}{2}z^2\right]_{z=0}^{z=1-x} dx \\
&= \int_0^1 \left[(1 - x)^2 - \frac{1}{2}(1 - x)^2\right] dx \\
&= \frac{1}{2}\int_0^1 (1 - x)^2\,dx \\
&= -\frac{1}{6}(1 - x)^3\Big]_0^1 = \frac{1}{6}.
\end{aligned}$$

■

Example: The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



Solution:

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\;dz\;dx = \int_0^1 \int_0^{1-x} (2-2z)\;dz\;dx$$

$$= \int_0^1 \left[2z - z^2\right]_0^{1-x} dx = \int_0^1 \left(1 - x^2\right) dx$$

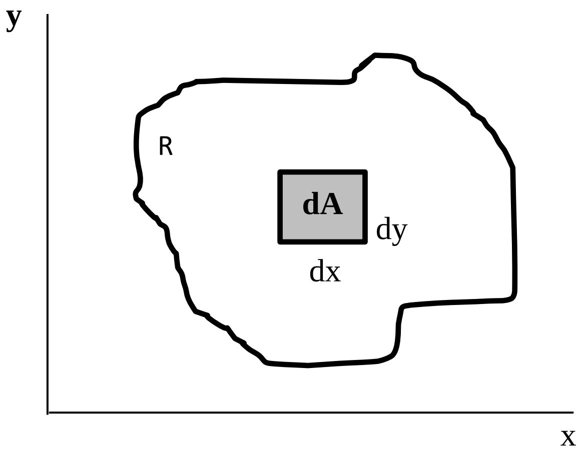
$$= \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

Multiple Integrals

1. Double Integrals

$$A = \iint_R dA = \iint_R dx\,dy$$

$$\text{Volume} = \iint_R f(x, y)\, dA = \iint_R Z\ dA$$



$$V = \iint_R z\, dA$$

$$= \iint_R f(x,y)\, dy\, dx \,.$$

$$= \iint_R f(x,y)\, dx\, dy \,.$$

EXAMPLE Evaluating a Double Integral

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 1 - 6x^2y \qquad \text{and} \qquad R:\quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2} (1 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \left[x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy = \left[2y - 8y^2\right]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1} (1 - 6x^2y)\, dy\, dx = \int_0^2 \left[y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)]\, dx$$

$$= \int_0^2 2\, dx = 4.$$

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE Finding Volume

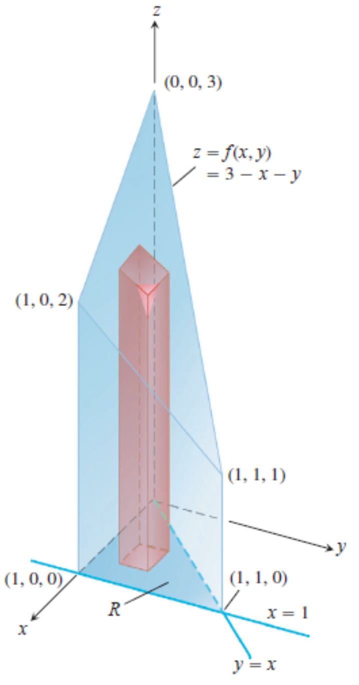
Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

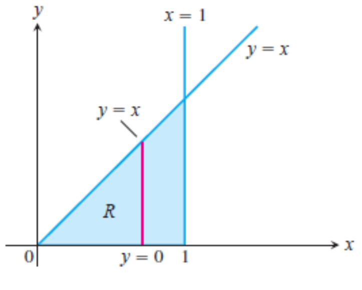
Solution For any x between 0 and 1, y may vary from $y = 0$ to $y = x$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

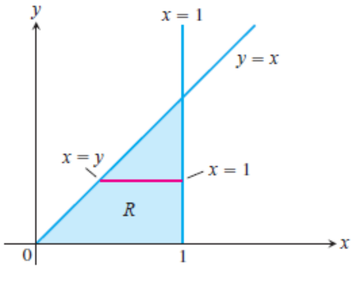
$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$



(a)



(b)



(c)

When the order of integration is reversed the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

EXAMPLE Evaluating a Double Integral

Calculate

$$\iint_R \frac{\sin x}{x}\,dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

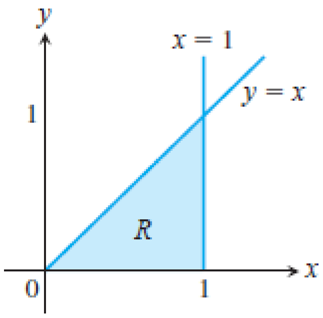
Solution The region of integration is shown in Figure . If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy\right) dx = \int_0^1 \left(y\frac{\sin x}{x}\right]_{y=0}^{y=x}\right) dx = \int_0^1 \sin x\,dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy,$$

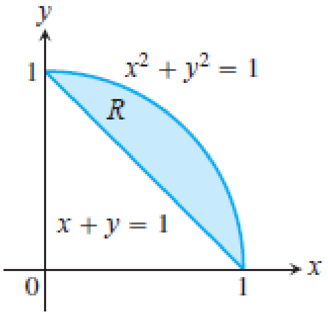


we run into a problem, because $\int((\sin x)/x)\,dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

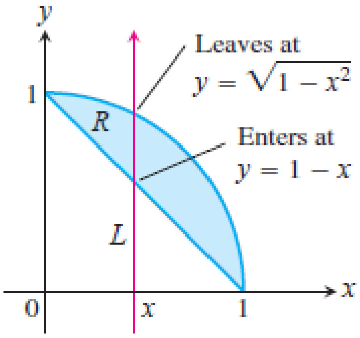
There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

Finding Limits of Integration

1. *Sketch*. Sketch the region of integration and label the bounding curves.



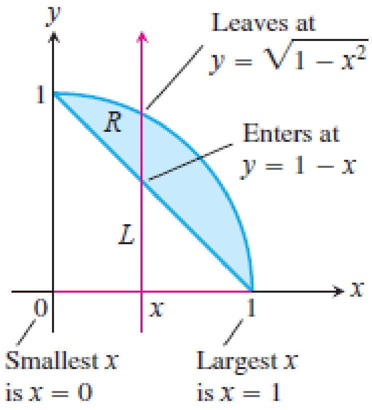
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).



3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral shown here is

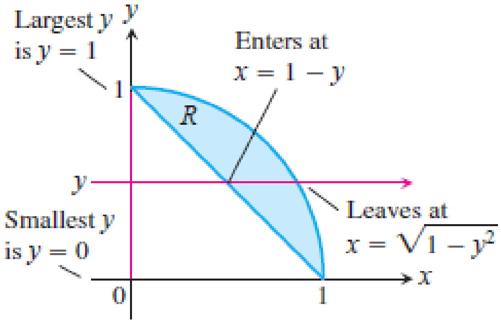
$$\iint_R f(x, y)\, dA =$$

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$



To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$



THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R:\ 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^1 \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^1 (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^1 = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^1 (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400. \quad ■$$

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.

Solution The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.$$

■

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

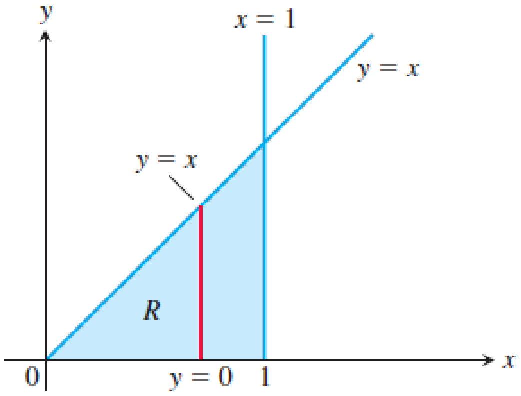
EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure . For any x between 0 and 1, y may vary from $y = 0$ to $y = x$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$



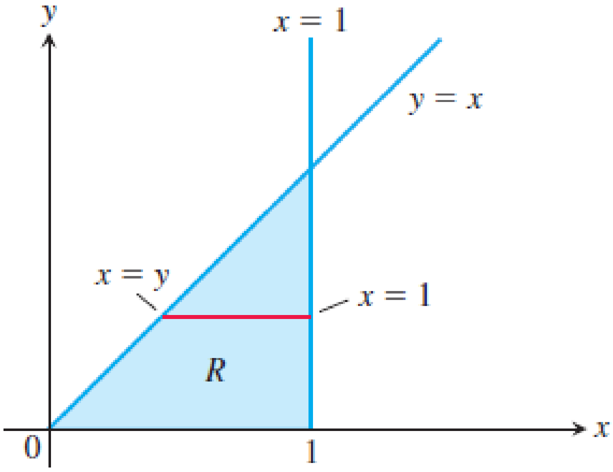
When the order of integration is reversed (**See** Figure), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be. ■



Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Examples of Double Integrals in Cartesian coordinates

EXAMPLE Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

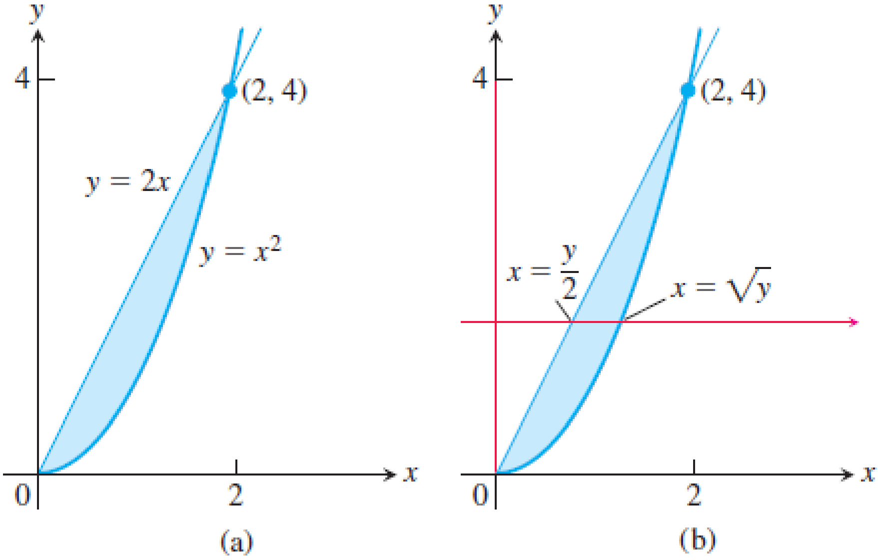
and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure .a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■



EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\, dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\, dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

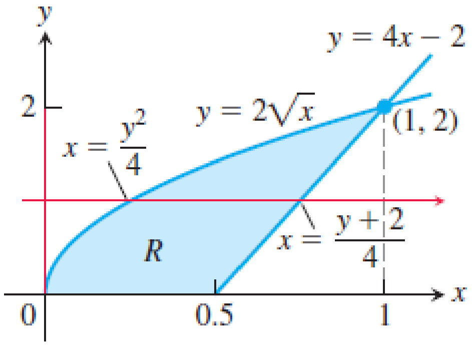
$$\iint_R (16 - x^2 - y^2)\, dA$$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.$$ ■

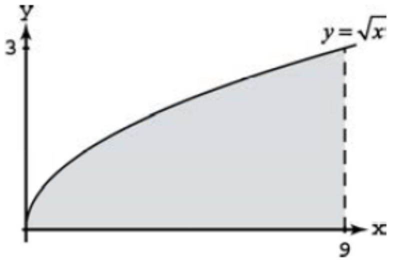


Examples: Write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

1. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$

(a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$

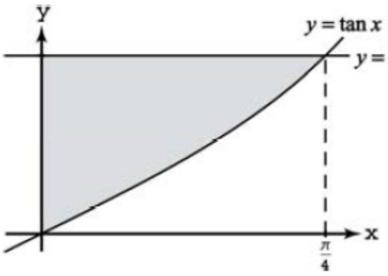
(b) $\int_0^3 \int_{y^2}^9 dx\, dy$



2 . Bounded by $y = \tan x$, $x = 0$, and $y = 1$

(a) $\int_0^{\pi/4} \int_{\tan x}^1 dy\, dx$

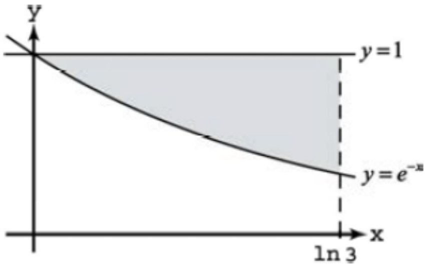
(b) $\int_0^1 \int_0^{\tan^{-1} y} dx\, dy$



3 . Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$

(a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy\, dx$

(b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx\, dy$



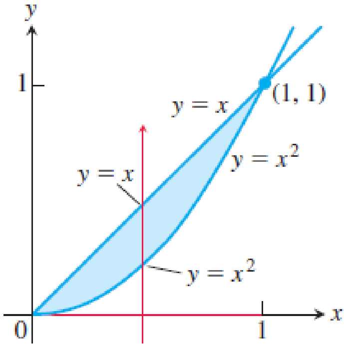
EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure), noting where the two curves intersect at the origin and (1, 1), and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

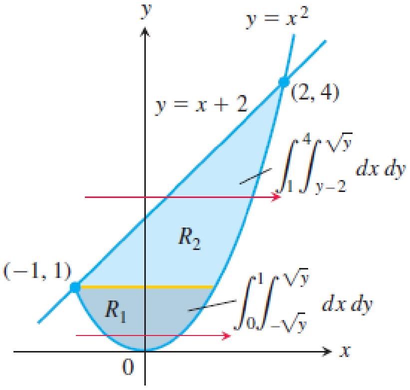
Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6. ■



EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$



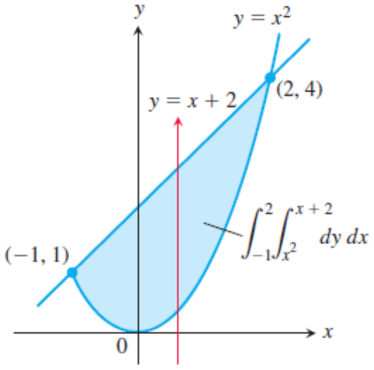
(a)

On the other hand, reversing the order of integration (Figure b) gives

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx.$$

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^{2}\Big[y\Big]_{x^2}^{x+2} dx = \int_{-1}^{2}(x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}. \qquad ■$$

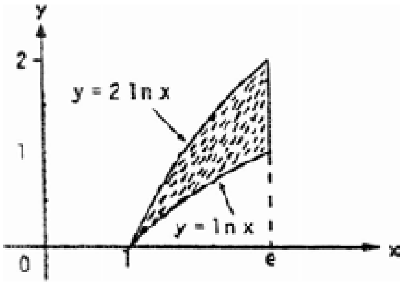


(b)

Examples: Sketch the region bounded by the given lines and curves. Then express the region’s area as an iterated double integral and evaluate the integral.

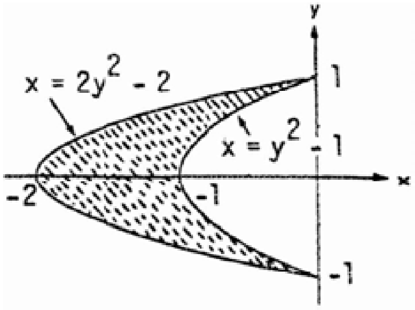
1. The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant

$$\int_1^e\int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx = \left[x\ln x - x\right]_1^e$$
$$= (e-e)-(0-1) = 1$$



2. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

$$\int_{-1}^{1}\int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^{1}\left(y^2 - 1 - 2y^2 + 2\right)dy$$
$$= \int_{-1}^{1}\left(1-y^2\right)dy = \left[y - \frac{y^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$



Double Integrals in Polar Form

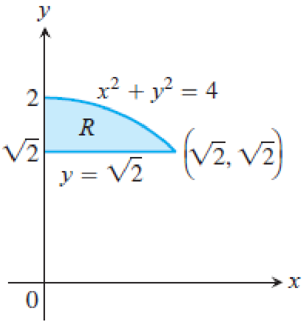
Double integrals are sometimes easier to evaluate if we change to polar coordinates.

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

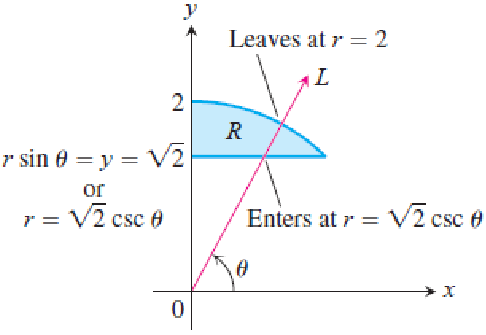
$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Finding Limits of Integration

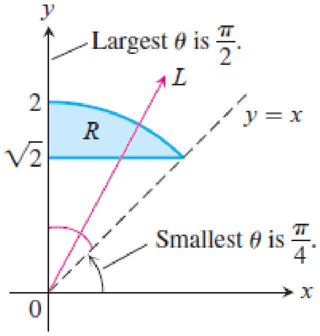
1. *Sketch*: Sketch the region and label the bounding curves.



2. *Find the r-limits of integration*: Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.



3. *Find the θ-limits of integration*: Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.



The integral is

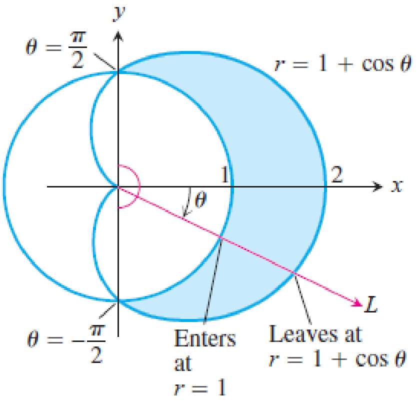
$$\iint_R f(r,\theta)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\,r\,dr\,d\theta.$$

EXAMPLE Finding Limits of Integration

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves .
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.



3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta.$$

If $f(r,\theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

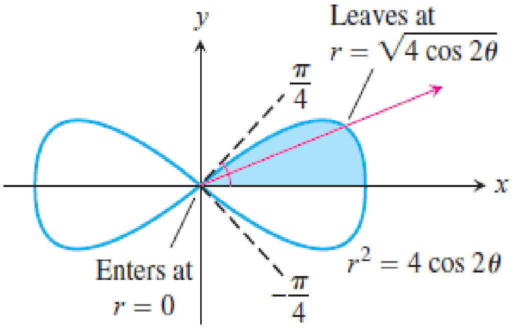
EXAMPLE Finding Area in Polar Coordinates

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

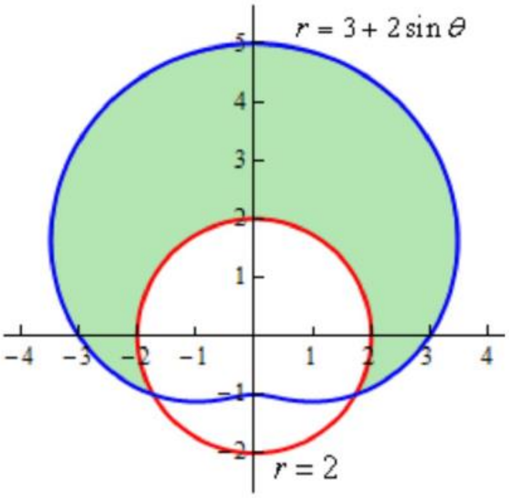
$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Hide Solution ▼

Here is a sketch of the region, D, that we want to determine the area of.



To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations equal and solving.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.

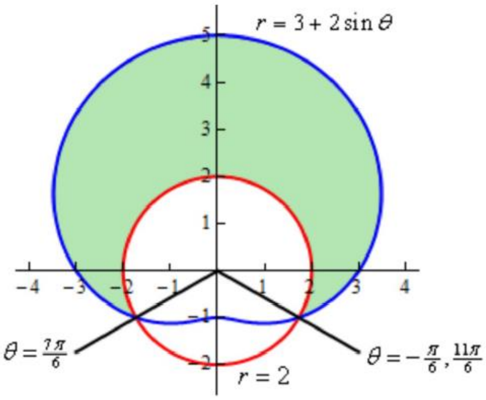
Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$ then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$
$$2 \leq r \leq 3 + 2\sin\theta$$

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region D is then,



$$\begin{aligned}
A &= \iint\limits_D dA \\
&= \int_{-\pi/6}^{7\pi/6} \int_{2}^{3+2\sin\theta} r\, dr d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \Big|_{2}^{3+2\sin\theta} d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\, d\theta \\
&= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta)\, d\theta \\
&= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187
\end{aligned}$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

where G denotes the region of integration in polar coordinates.
Notice that $dx\, dy$ is not replaced by $dr\, d\theta$ but by $r\, dr\, d\theta$.

EXAMPLE 3 Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

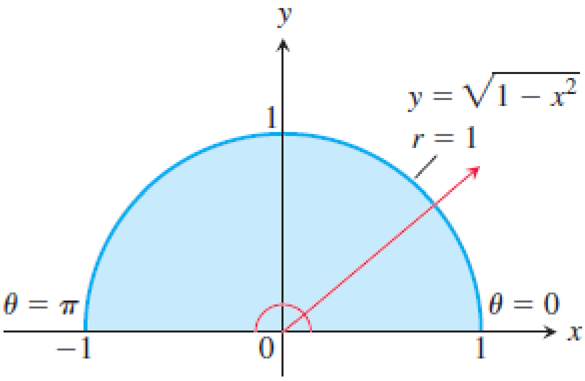
where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dy\, dx$ by $r\, dr\, d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^{1} e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$

The r in the $r\, dr\, d\theta$ was just what we needed to integrate e^{r^2}. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■



EXAMPLE Changing Cartesian Integrals to Polar Integrals

Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution We sketch the plate to determine the limits of integration . In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Integration with respect to y gives

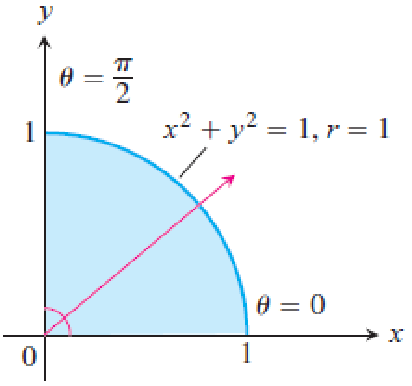
$$\int_0^1 \left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dx\, dy$ by $r\, dr\, d\theta$, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

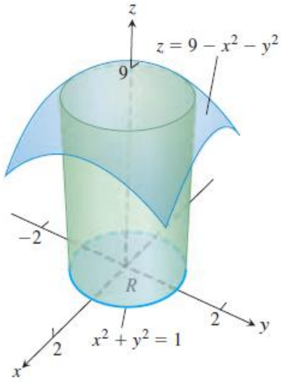
$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$



EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution The region of integration R is the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1$, $0 \le \theta \le 2\pi$. The solid region is shown in Figure The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$



$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

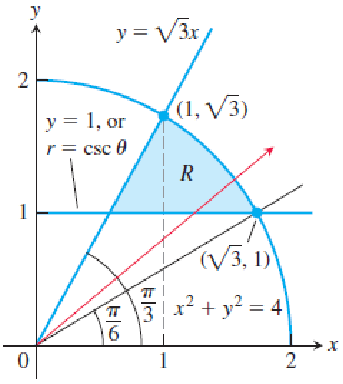
$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

■

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution A sketch of the region R is shown in Figure . First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.29.

Now, for the region R, as θ varies from $\pi/6$ to $\pi/3$, the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r\sin\theta$ for y in the equation for the horizontal line, we have $r\sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, r varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area then gives



$$\iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3}\frac{1}{2}[4 - \csc^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.$$

■

Example Evaluate the following integrals by converting them into polar coordinates.

$\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

Solution

$\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between the two circles, so we will have the following inequality for r.

$$2 \le r \le 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}} \int_2^5 2\,(r\cos\theta)\,(r\sin\theta)\,r\,dr\,d\theta$$

Don't forget to do the conversions and to add in the extra r. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2x\,y\,dA &= \int_0^{\frac{\pi}{2}} \int_2^5 r^3 \sin(2\theta)\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \left.\frac{1}{4} r^4 \sin(2\theta)\right|_2^5 d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4} \sin(2\theta)\,d\theta \\ &= -\left.\frac{609}{8}\cos(2\theta)\right|_0^{\frac{\pi}{2}} \\ &= \frac{609}{4}\end{aligned}$$

Example: Evaluating a Double Integral by Converting from Rectangular Coordinates

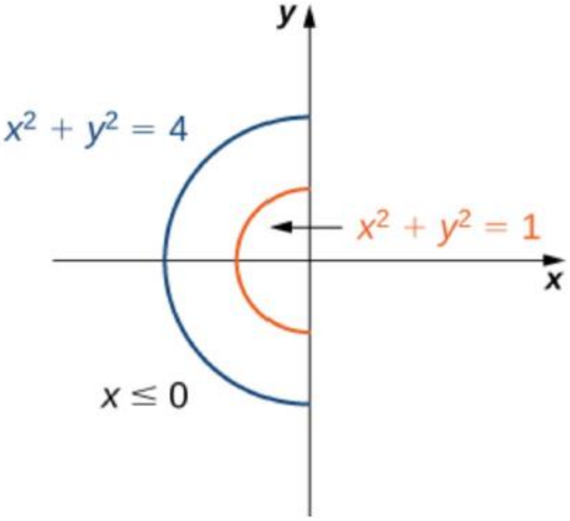
Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R = \{(x,y)\,|\,1 \le x^2+y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r,\theta)\,|\,1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).



Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

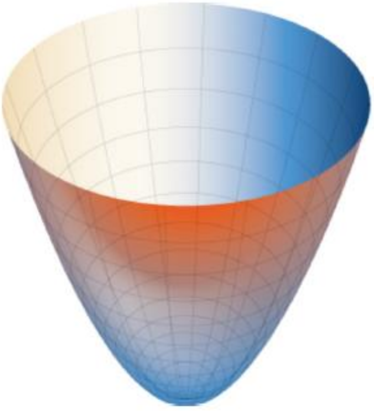
$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

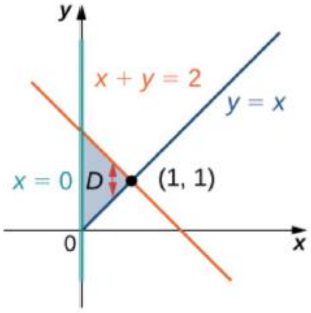
Example: Finding a Volume Using a Double Integral

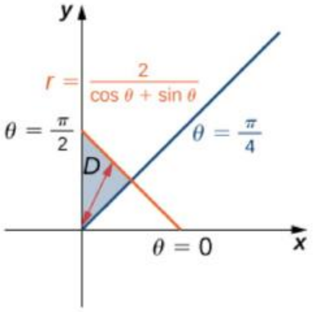
Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.







The region D is $\{(x, y) \,|\, 0 \leq x \leq 1,\, x \leq y \leq 2 - x\}$. Converting the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\,\theta + \sin\,\theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D = \{(r, \theta) \,|\, \pi/4 \leq \theta \leq \pi/2,\, 0 \leq r \leq 2/(\cos\,\theta + \sin\,\theta)\}$.

Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned}
V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\
&= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\,\theta+\sin\,\theta)} r^2 r\, dr d\theta \\
&= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\,\theta+\sin\,\theta)} d\theta \\
&= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\,\theta + \sin\,\theta}\right)^4 d\theta \\
&= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\,\theta + \sin\,\theta}\right)^4 d\theta \\
&= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\,\theta + \sin\,\theta}\right)^4 d\theta.
\end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

$$\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Bigg|_0^1 \\
&= \frac{4}{3}\ \text{units}^3.
\end{aligned}$$

Triple Integrals in Rectangular Coordinates

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

Finding Limits of Integration in the Order *dz dy dx*

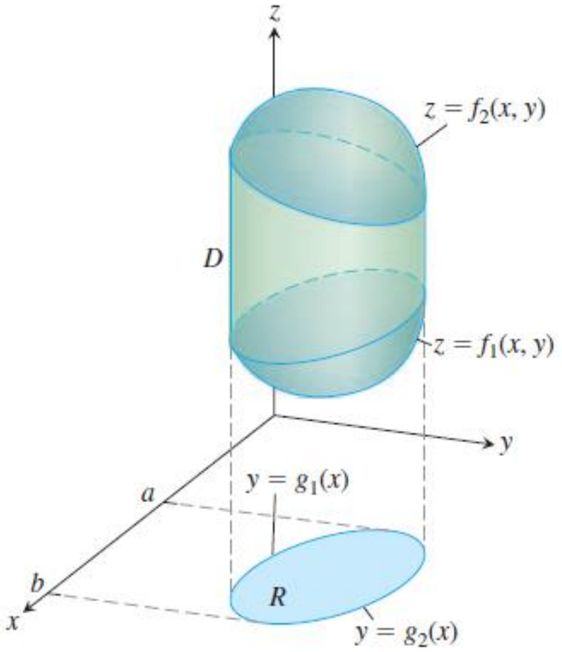
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.2) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these iterated integrals.

To evaluate

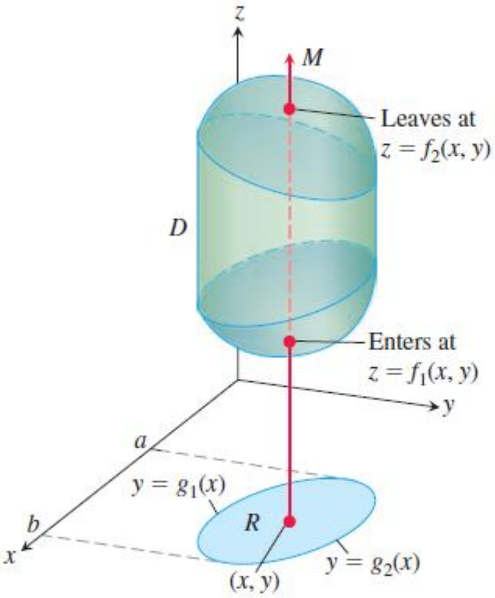
$$\iiint_D F(x, y, z)\, dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x. (You might choose a different order of integration, but the procedure is similar, as we illustrate in Example 2.)

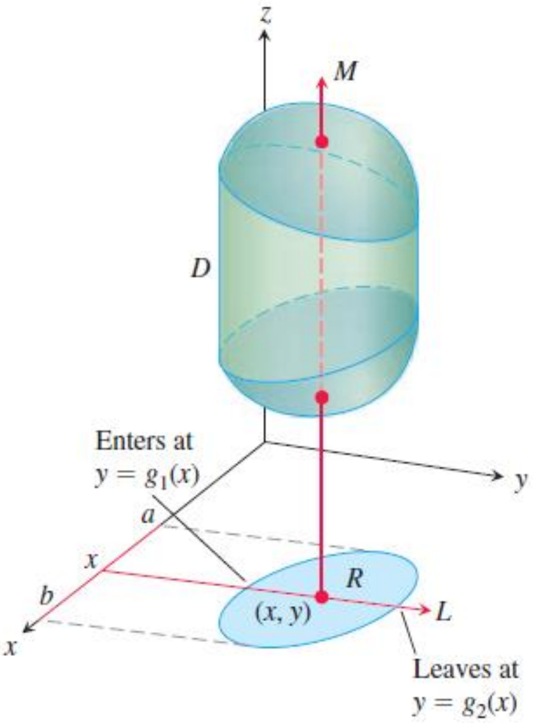
1. *Sketch.* Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x, y)}^{z=f_2(x, y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

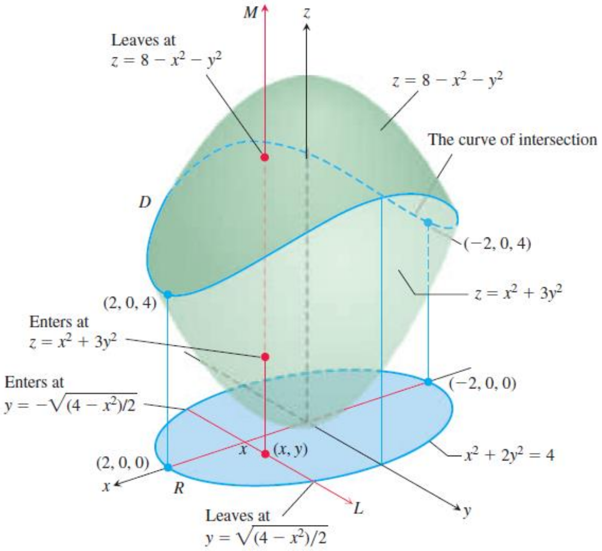
EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (**see** Figure) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.



Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$\begin{aligned} V &= \iiint_D dz\, dy\, dx \\ &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \\ &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \end{aligned}$$

$$= \int_{-2}^{2}\left[(8-2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right) dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\, dx$$

$= 8\pi\sqrt{2}.$ After integration with the substitution $x = 2\sin u$ ■

EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dy\, dz\, dx$.

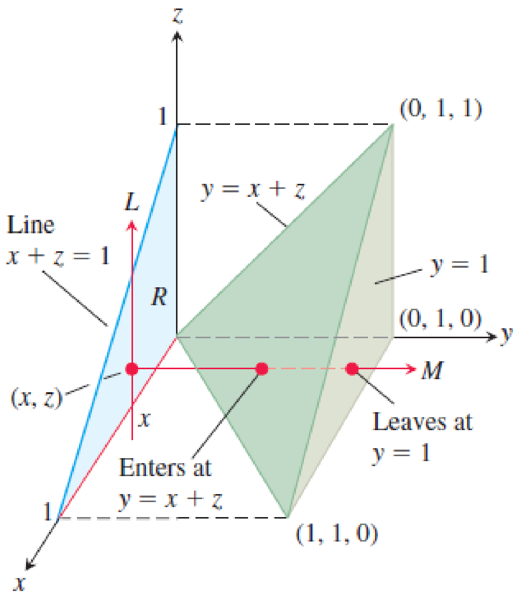
Solution We sketch D along with its "shadow" R in the xz-plane (**see** Figure). The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the y-limits of integration. The line through a typical point (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

Next we find the z-limits of integration. The line L through (x, z) parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. The integral is

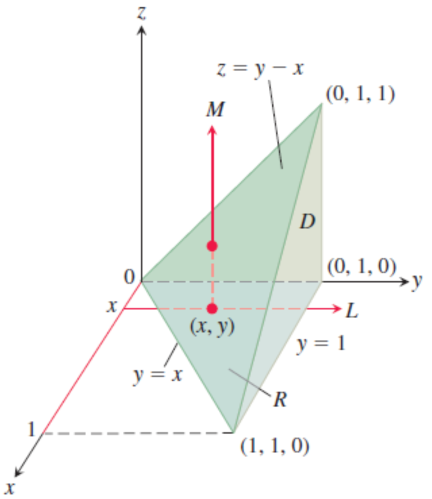
$$\int_{0}^{1}\int_{0}^{1-x}\int_{x+z}^{1} F(x, y, z)\, dy\, dz\, dx.$$ ■



EXAMPLE 3 Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 2 in the order $dz\,dy\,dx$, and then integrate in the order $dy\,dz\,dx$.

Solution First we find the z-limits of integration. A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (see Figure).

Next we find the y-limits of integration. On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exits at $y = 1$ (see Figure).



Finally we find the x-limits of integration. As the line L parallel to the y-axis in the previous step sweeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point (1, 1, 0) (see Figure 15.33). The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

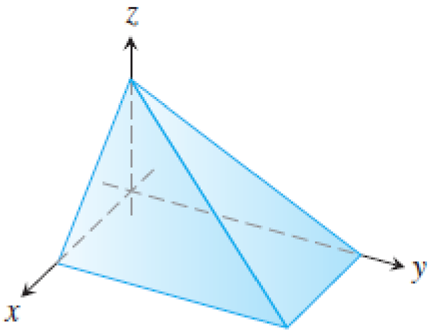
$$\begin{aligned} V &= \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx \\ &= \int_0^1 \int_x^1 (y - x)\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

We get the same result by integrating with the order *dy dz dx*. From Example 2,

$$\begin{aligned} V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx \\ &= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx \\ &= \int_0^1 \left[(1-x)z - \frac{1}{2} z^2 \right]_{z=0}^{z=1-x} dx \\ &= \int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2 \right] dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2\, dx \\ &= -\frac{1}{6}(1-x)^3 \Big]_0^1 = \frac{1}{6}. \end{aligned}$$

■

Example: The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



Solution:

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\ dz\ dx = \int_0^1 \int_0^{1-x} (2-2z)\ dz\ dx$$

$$= \int_0^1 \left[2z - z^2 \right]_0^{1-x} dx = \int_0^1 \left(1 - x^2 \right) dx$$

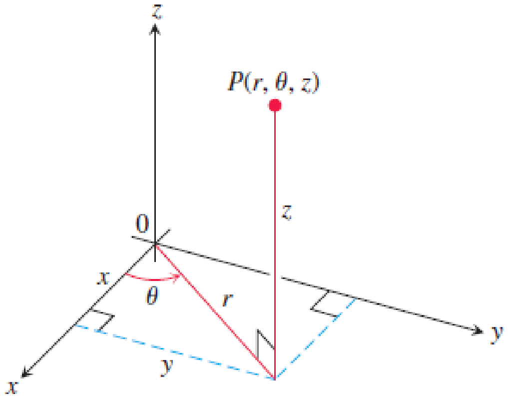
$$= \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

Triple Integrals in Cylindrical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.



Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

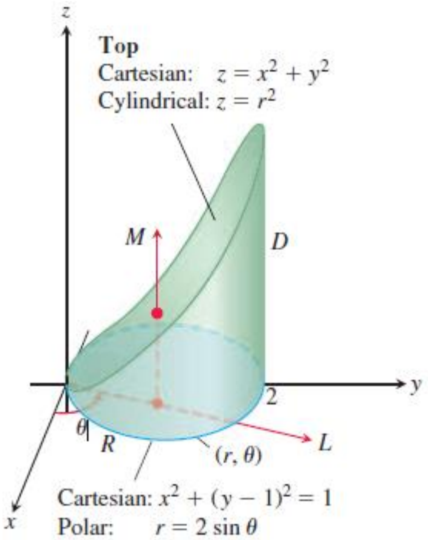
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

$$\lim_{n\to\infty} S_n = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y - 1)^2 = 1$$



$$\begin{aligned} x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

The region is sketched in Figure.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \quad ■$$

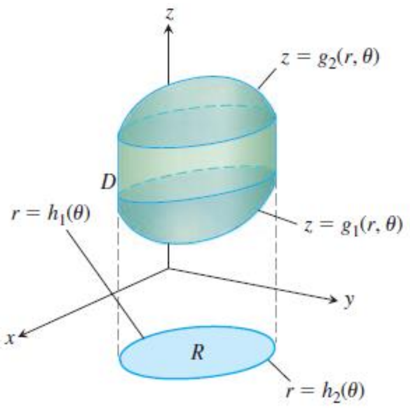
How to Integrate in Cylindrical Coordinates

To evaluate

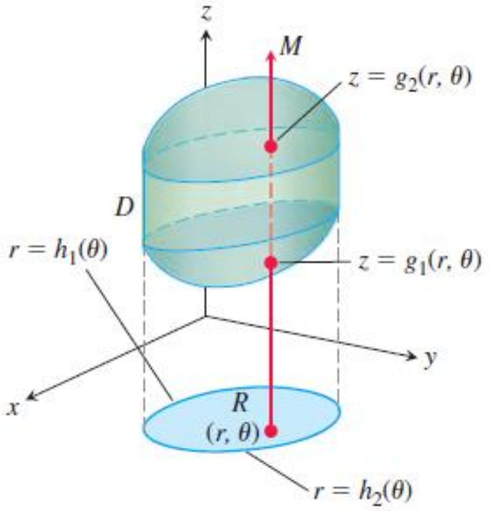
$$\iiint_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

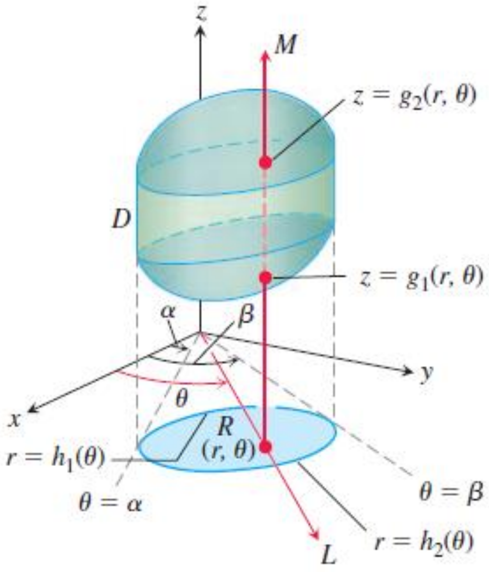
1. *Sketch.* Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration.* Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.

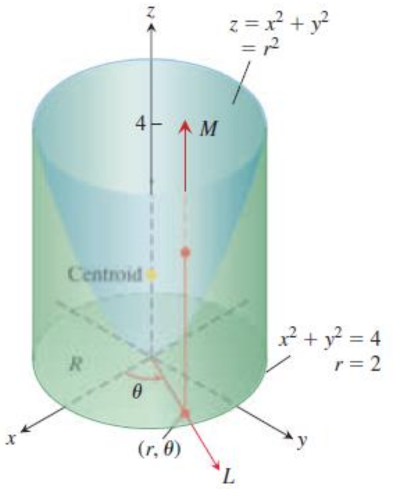


4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint\limits_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

EXAMPLE 2 Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (see Figure). Its base R is the disk $0 \leq r \leq 2$ in the xy-plane.



The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} \right]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^5}{2} dr\, d\theta = \int_0^{2\pi} \left[\frac{r^6}{12} \right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{2}\Big[z\Big]_0^{r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^3\, dr\, d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{2} d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi.$$

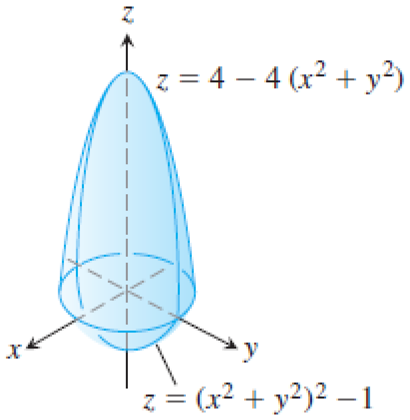
Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is (0, 0, 4 / 3). Notice that the centroid lies on the z-axis, outside the solid. ■

Examples: Find the volumes of the solids in Exercises

1.

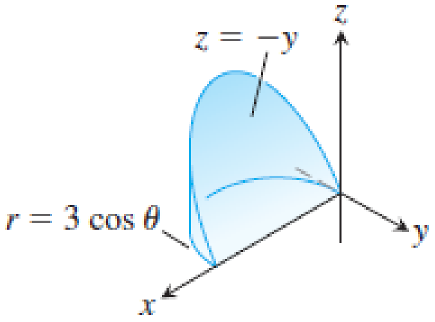


$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\ r\ dr\ d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(5r - 4r^3 - r^5\right) dr\ d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta = 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}$$

2.



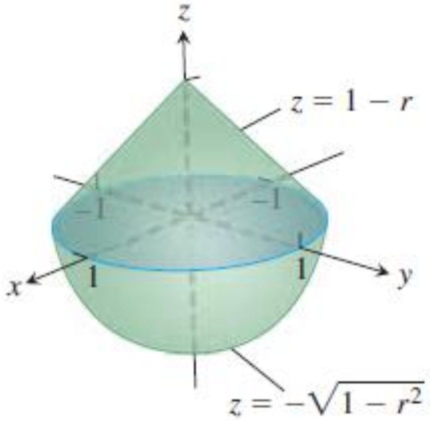
$$V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz\ r\ dr\ d\theta$$

$$= \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\left(-r^2\sin\theta\right) dr\ d\theta$$

$$= \int_{3\pi/2}^{2\pi}\left(-9\cos^3\theta\right)(\sin\theta)\ d\theta$$

$$= \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$$

3.



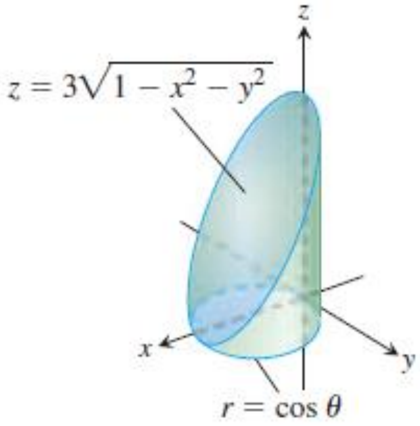
$$V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\ r\ dr\ d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(r - r^2 + r\sqrt{1-r^2}\right) dr\ d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

4.



$$V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\ r\ dr\ d\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}\ dr\ d\theta$$

$$= \int_0^{\pi/2}\left[-\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[-\left(1-\cos^2\theta\right)^{3/2} + 1\right] d\theta = \int_0^{\pi/2}\left(1 - \sin^3\theta\right) d\theta$$

$$= \left[\theta + \frac{\sin^2\theta\cos\theta}{3}\right]_0^{\pi/2} - \frac{2}{3}\int_0^{\pi/2}\sin\theta\ d\theta$$

$$= \frac{\pi}{2} + \frac{2}{3}\left[\cos\theta\right]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3} = \frac{3\pi-4}{6}$$