Functions and Graphs

Chapter Outline

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$. Since every nonnegative real number has a real-value square root,

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figure 1.2, Figure 1.3, and Figure 1.4.

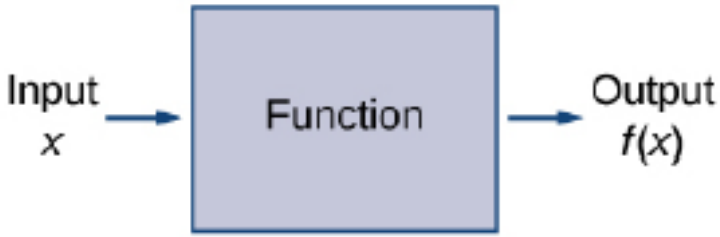


Figure 1.2 A function can be visualized as an input/output device.

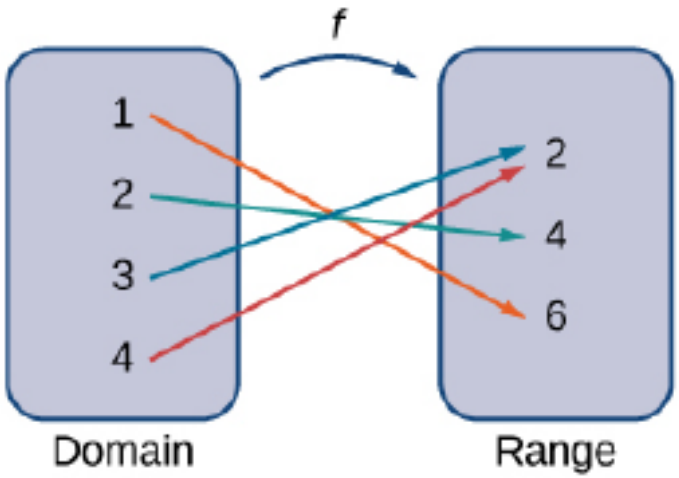


Figure 1.3 A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

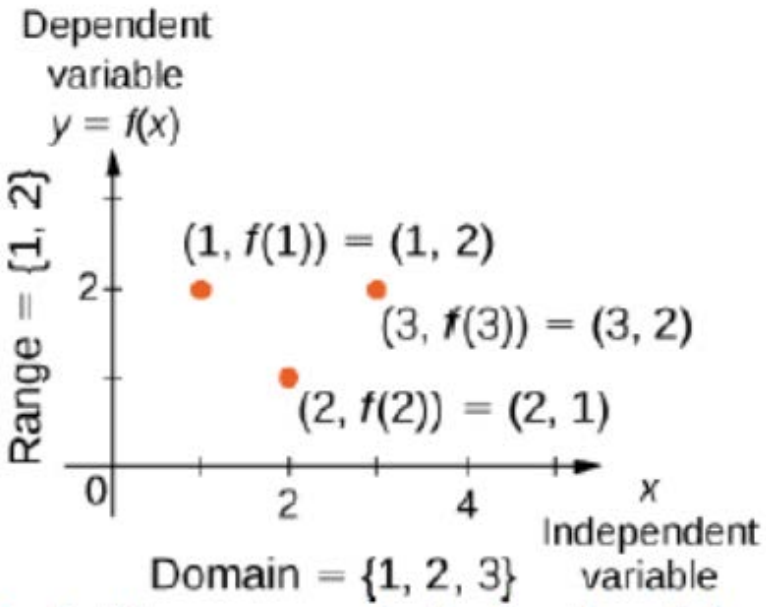


Figure 1.4 In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The **graph of a function** is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In **Figure 1.5**, we plot a graph of this function.

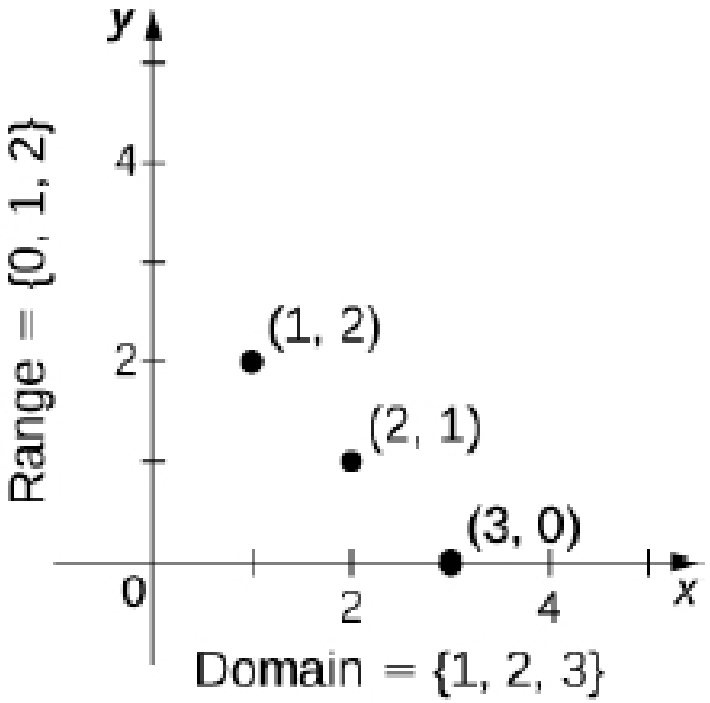


Figure 1.5 Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$,

the domain of $f(x) = x^2$ set of all real numbers.

the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers.

Example 1.1

Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

a. $f(-2)$

b. $f(\sqrt{2})$

c. $f(a+h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$

b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$

c. $$\begin{aligned} f(a+h) = 3(a+h)^2 + 2(a+h) - 1 &= 3\left(a^2 + 2ah + h^2\right) + 2a + 2h - 1 \\ &= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1 \end{aligned}$$

Example 1.2

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x-4)^2 + 5$

b. $f(x) = \sqrt{3x+2} - 1$

c. $f(x) = \dfrac{3}{x-2}$

a. Consider $f(x) = (x-4)^2 + 5$.

i. Since $f(x) = (x-4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.

ii. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$.

$$y \geq 5$$

b. Consider $f(x) = \sqrt{3x+2} - 1$.

i. To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x | x \geq -2/3\}$.

ii. To find the range of f,

$$\sqrt{3x+2} \geq 0,\ f(x) = \sqrt{3x+2} - 1 \geq -1.$$

the range of f is $\{y | y \geq -1\}$.

c. Consider $f(x) = 3/(x-2)$.

i. Since $3/(x-2)$ is defined when the denominator is nonzero, the domain is $\{x|x \neq 2\}$.

ii. To find the range of f,

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$.

Thus, the range is $\{y|y \neq 0\}$.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

Hours after Midnight	Temperature (°F)	Hours after Midnight	Temperature (°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

Table 1.1 Temperature as a Function of Time of Day

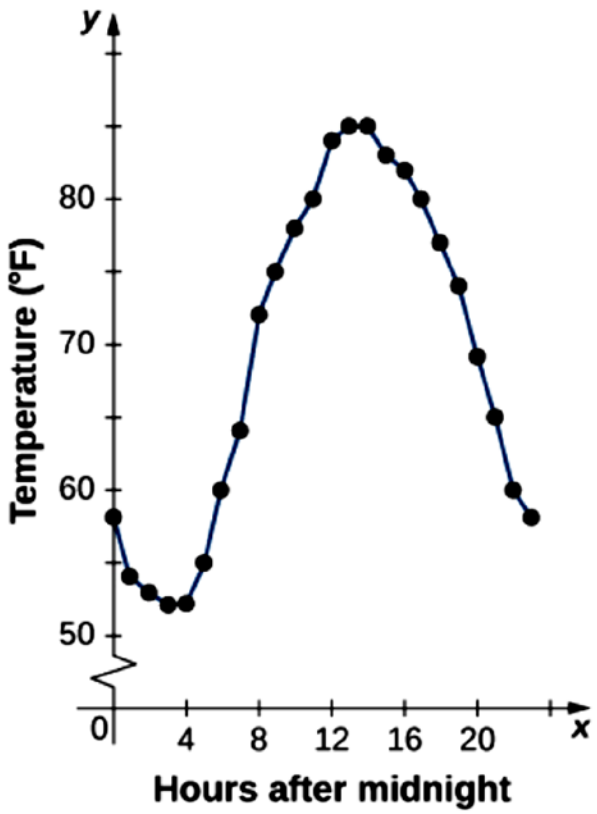


Figure 1.7 Connecting the dots in **Figure 1.6** shows the general pattern of the data.

the area of a circle of radius r is given by the formula $A(r) = \pi r^2$.

Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination, so they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

To see this, notice by inspecting similar triangles in Figure 1.15 that $m_1 = a/h$, and $m_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

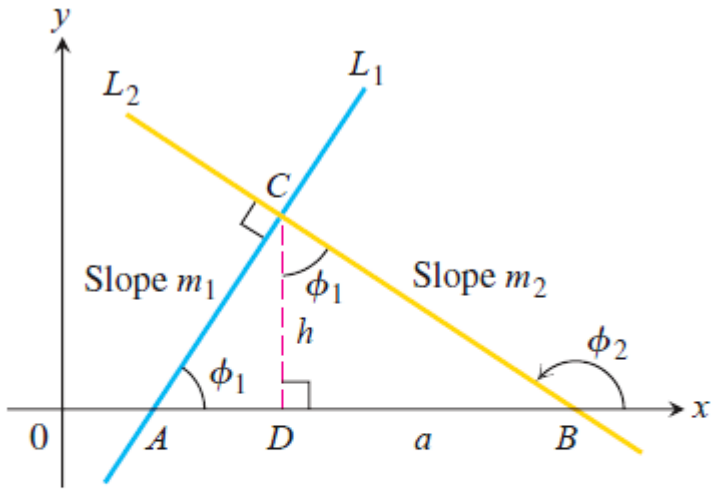


FIGURE 1.15 ΔADC is similar to ΔCDB. Hence ϕ_1 is also the upper angle in ΔCDB. From the sides of ΔCDB, we read $\tan \phi_1 = a/h$.

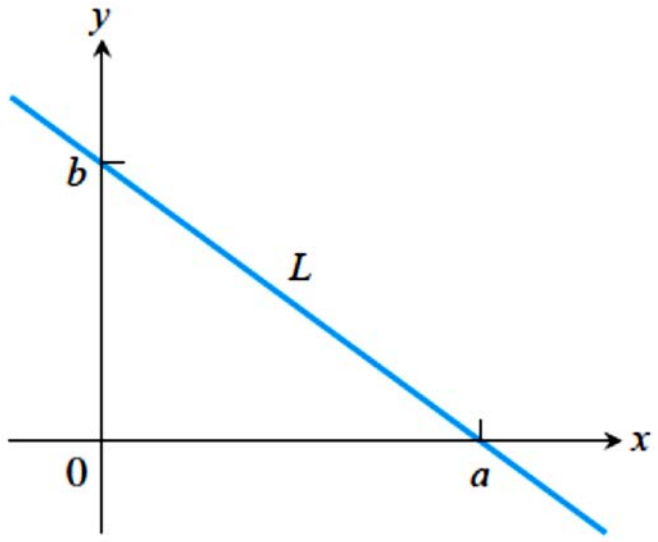


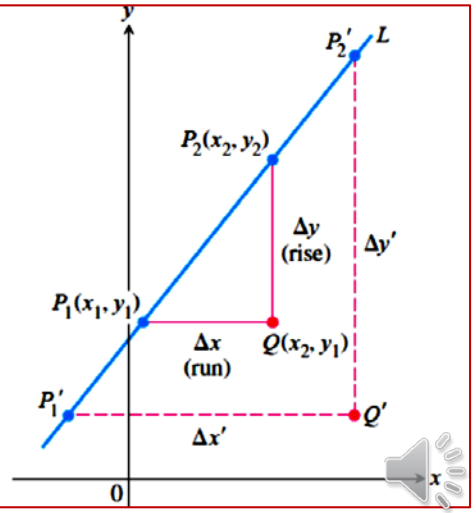
FIGURE 1.14 Line L has x-intercept a and y-intercept b.

DEFINITION Slope

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line $P_1 P_2$.



Distance and Circles in the Plane

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Figure 1.16).

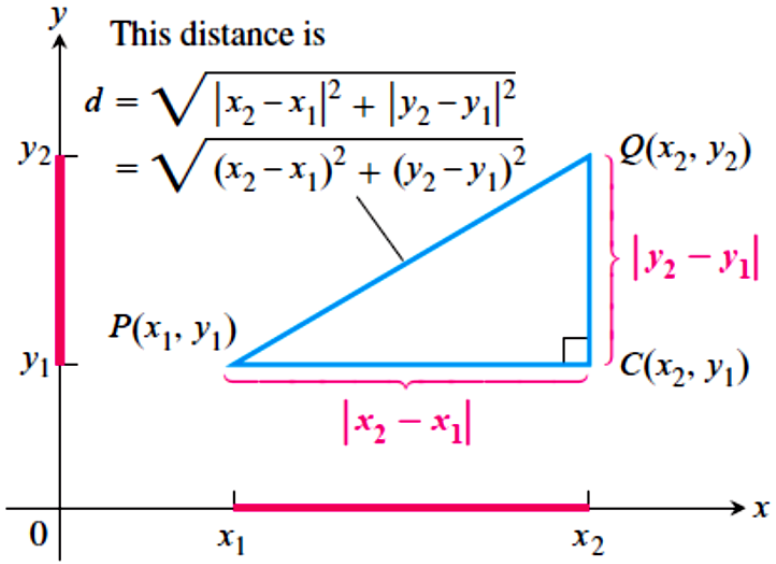


FIGURE 1.16 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle PCQ.

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 5 Calculating Distance

(a) The distance between $P(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3-(-1))^2+(4-2)^2} = \sqrt{(4)^2+(2)^2} = \sqrt{20} = \sqrt{4\cdot 5} = 2\sqrt{5}.$$

(b) The distance from the origin to $P(x, y)$ is

$$\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}.$$

By definition, a **circle** of radius a is the set of all points $P(x, y)$ whose distance from some center $C(h, k)$ equals a (Figure 1.17). From the distance formula, P lies on the circle if and only if

$$\sqrt{(x-h)^2+(y-k)^2} = a,$$

so

$$(x-h)^2+(y-k)^2 = a^2. \qquad (1)$$

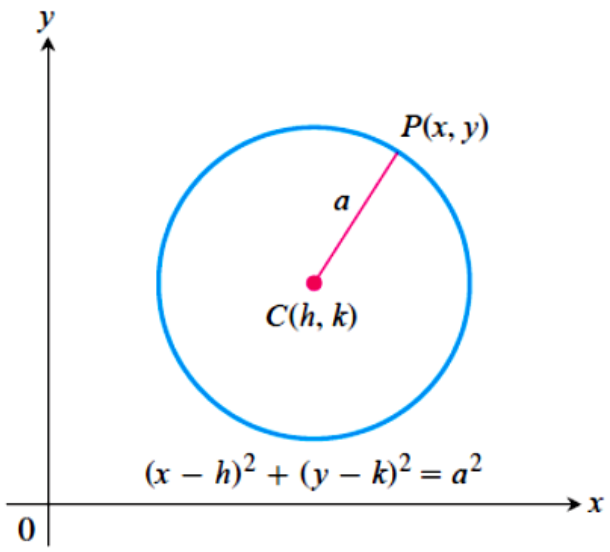


FIGURE 1.17 A circle of radius a in the xy-plane, with center at (h, k).

Equation (1) is the **standard equation** of a circle with center (h, k) and radius a. The circle of radius $a = 1$ and centered at the origin is the **unit circle** with equation

$$x^2 + y^2 = 1.$$

$$(x - h)^2 + (y - k)^2 = a^2. \qquad (1)$$

EXAMPLE 6

(a) The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x - 3)^2 + (y - 4)^2 = 2^2 = 4.$$

(b) The circle

$$(x - 1)^2 + (y + 5)^2 = 3$$

has $h = 1$, $k = -5$, and $a = \sqrt{3}$. The center is the point $(h, k) = (1, -5)$ and the radius is $a = \sqrt{3}$. ■

$$(x-h)^2+(y-k)^2=a^2.$$

EXAMPLE 7 Finding a Circle's Center and Radius

Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$

Solution We convert the equation to standard form by completing the squares in x and y:

$x^2+y^2+4x-6y-3=0$	Start with the given equation.
$(x^2+4x\quad)+(y^2-6y\quad)=3$	Gather terms. Move the constant to the right-hand side.
$\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)+\left(y^2-6y+\left(\frac{-6}{2}\right)^2\right)=3+\left(\frac{4}{2}\right)^2+\left(\frac{-6}{2}\right)^2$	Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.
$(x^2+4x+4)+(y^2-6y+9)=3+4+9$	
$(x+2)^2+(y-3)^2=16$	Write each quadratic as a squared linear expression.

The center is $(-2,3)$ and the radius is $a=4$. ■

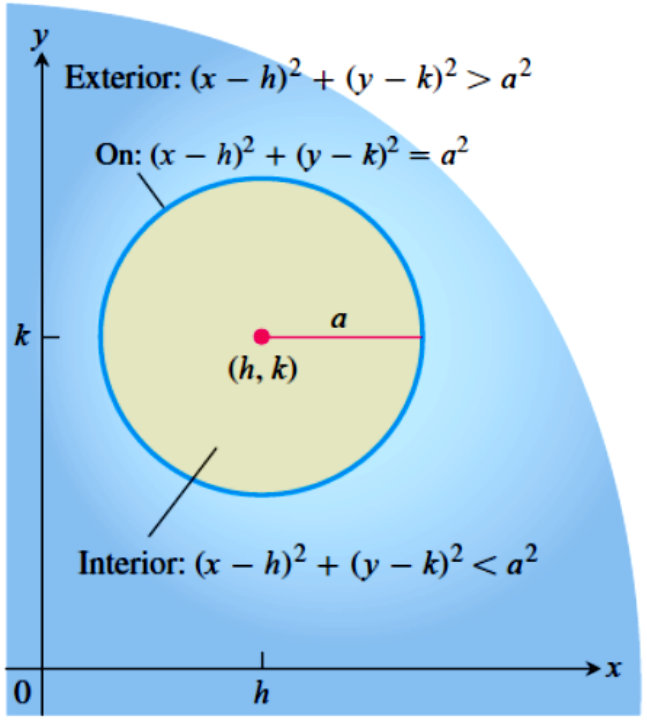


FIGURE 1.18 The interior and exterior of the circle $(x-h)^2+(y-k)^2=a^2$.

The points (x,y) satisfying the inequality

$$(x-h)^2+(y-k)^2<a^2$$

make up the **interior** region of the circle with center (h,k) and radius a (Figure 1.18). The circle's **exterior** consists of the points (x,y) satisfying

$$(x-h)^2+(y-k)^2>a^2.$$

Parabolas

The geometric definition and properties of general parabolas are reviewed in Section 10.1. Here we look at parabolas arising as the graphs of equations of the form $y = ax^2 + bx + c$.

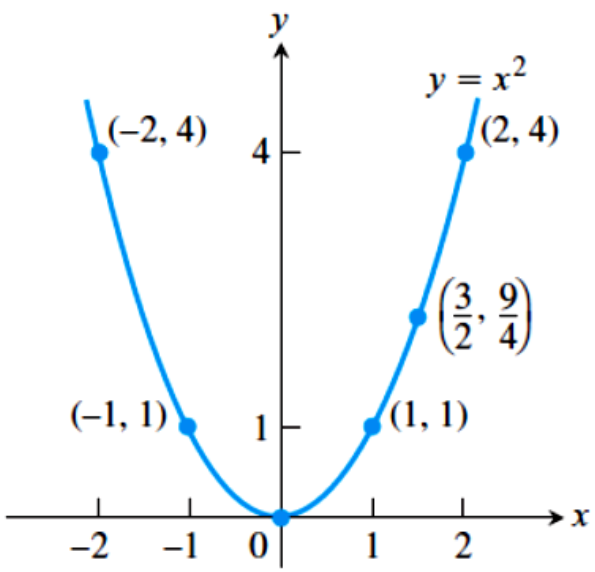


FIGURE 1.19 The parabola $y = x^2$ (Example 8).

EXAMPLE 8 The Parabola $y = x^2$

Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $\left(\frac{3}{2}, \frac{9}{4}\right)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Figure 1.19). ■

The graph of an equation of the form

$$y = ax^2$$

is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a > 0$ and downward if $a < 0$. The larger the value of $|a|$, the narrower the parabola (Figure 1.20).

Generally, the graph of $y = ax^2 + bx + c$ is a shifted and scaled version of the parabola $y = x^2$. We discuss shifting and scaling of graphs in more detail in Section 1.5.

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$

The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The **axis** is the line

$$x = -\frac{b}{2a}. \tag{2}$$

The **vertex** of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

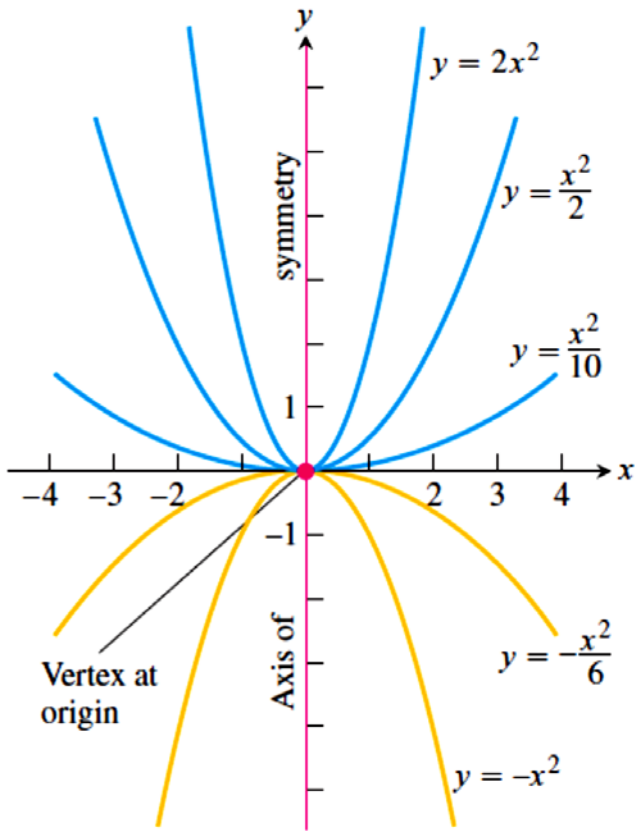


FIGURE 1.20 Besides determining the direction in which the parabola $y = ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

EXAMPLE 9 Graphing a Parabola

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.

Solution Comparing the equation with $y = ax^2 + bx + c$ we see that

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4.$$

Since $a < 0$, the parabola opens downward. From Equation (2) the axis is the vertical line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1.$$

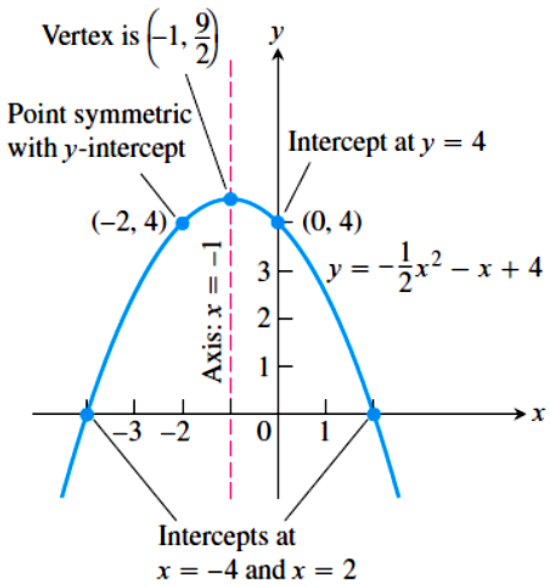


FIGURE 1.21 The parabola in Example 9.

When $x = -1$, we have

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

The x-intercepts are where $y = 0$:

$$-\frac{1}{2}x^2 - x + 4 = 0$$
$$x^2 + 2x - 8 = 0$$
$$(x - 2)(x + 4) = 0$$
$$x = 2, \qquad x = -4$$

We plot some points, sketch the axis, and use the direction of opening to complete the graph in Figure 1.21. ■

EXAMPLE 10 Sketching a Graph

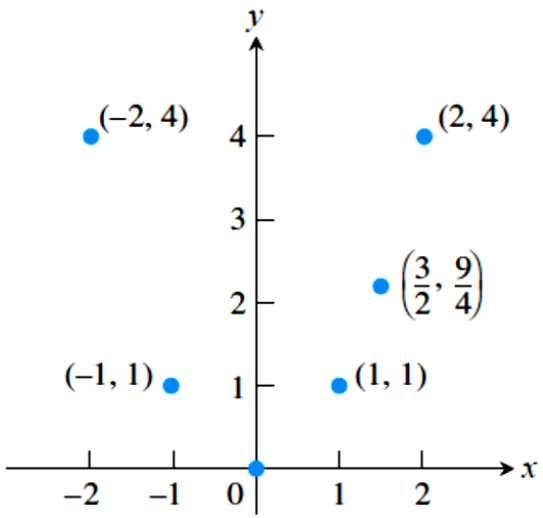
Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution

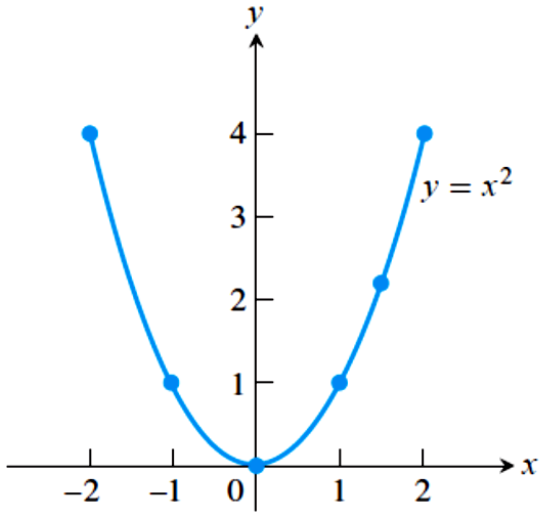
1. Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

2. Plot the points (x, y) whose coordinates appear in the table. Use fractions when they are convenient computationally.



3. Draw a smooth curve through the plotted points. Label the curve with its equation.



How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

Identifying Functions

There are a number of important types of functions frequently encountered in calculus. We identify and briefly summarize them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**. Figure 1.34 shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. Constant functions result when the slope $m = 0$ (Figure 1.35).

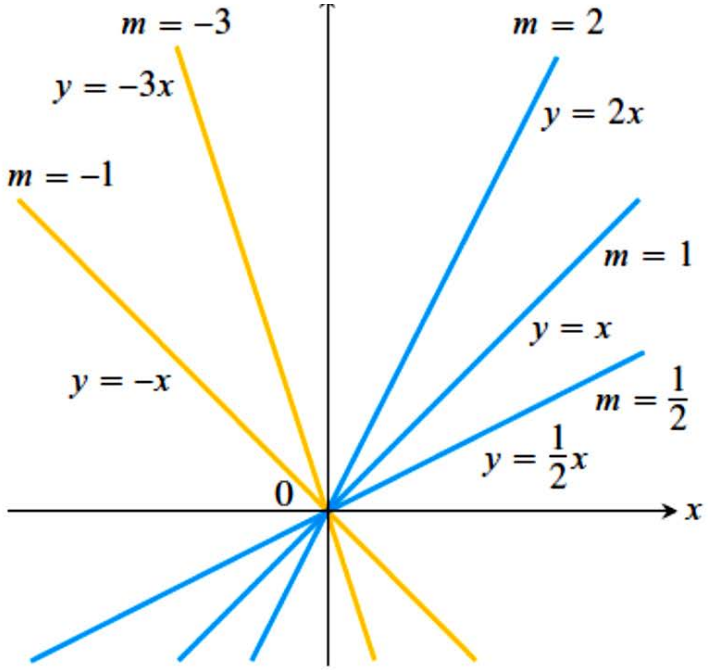


FIGURE 1.34 The collection of lines $y = mx$ has slope m and all lines pass through the origin.

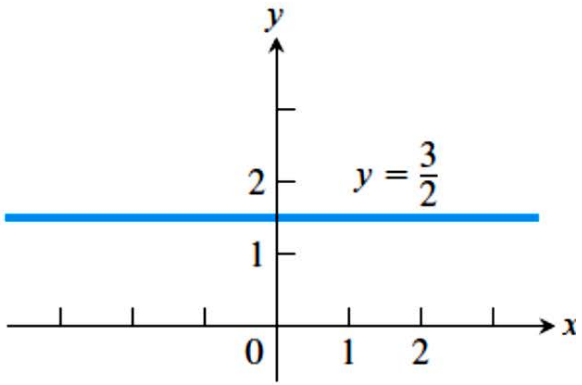


FIGURE 1.35 A constant function has slope $m = 0$.

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.36. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin.

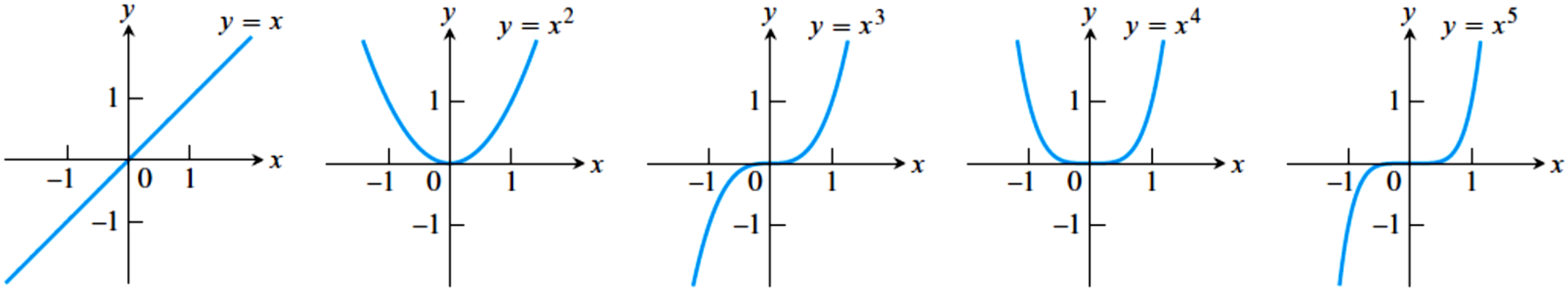


FIGURE 1.36 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

(b) $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.37. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$ which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes.

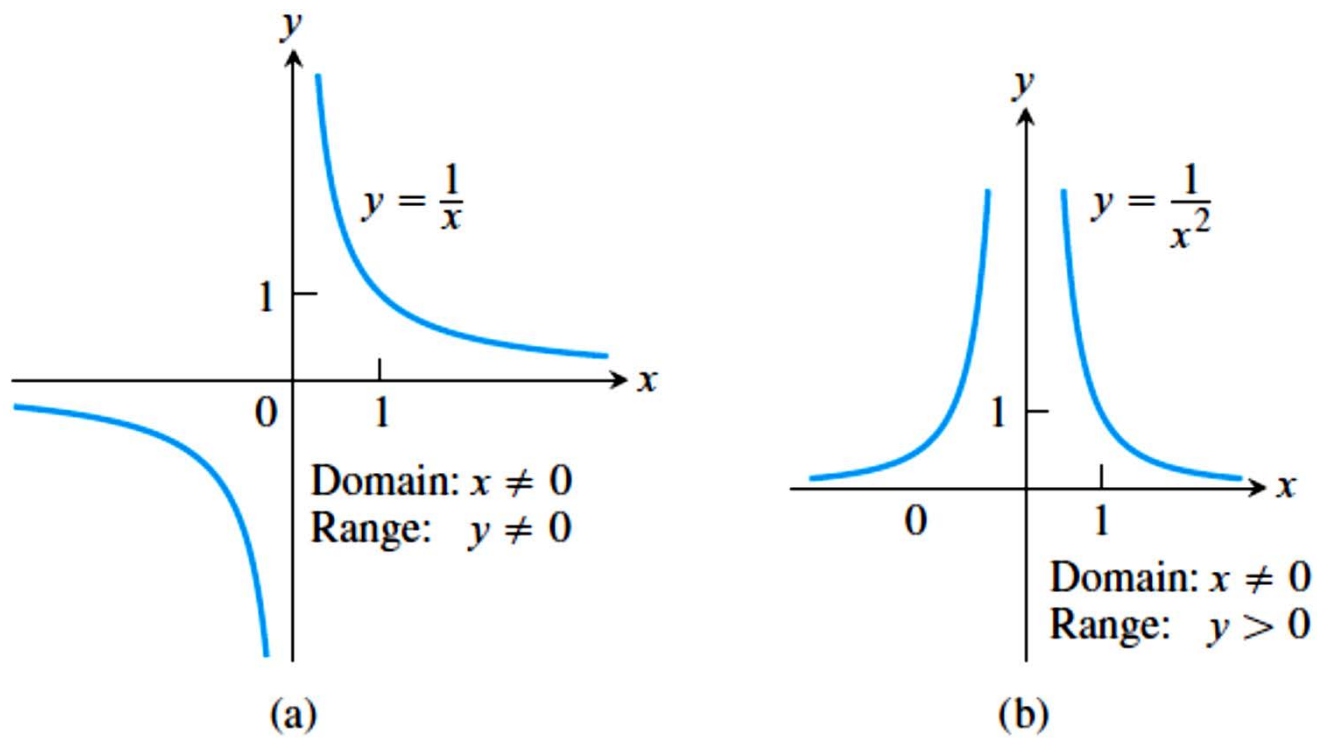


FIGURE 1.37 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.38 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

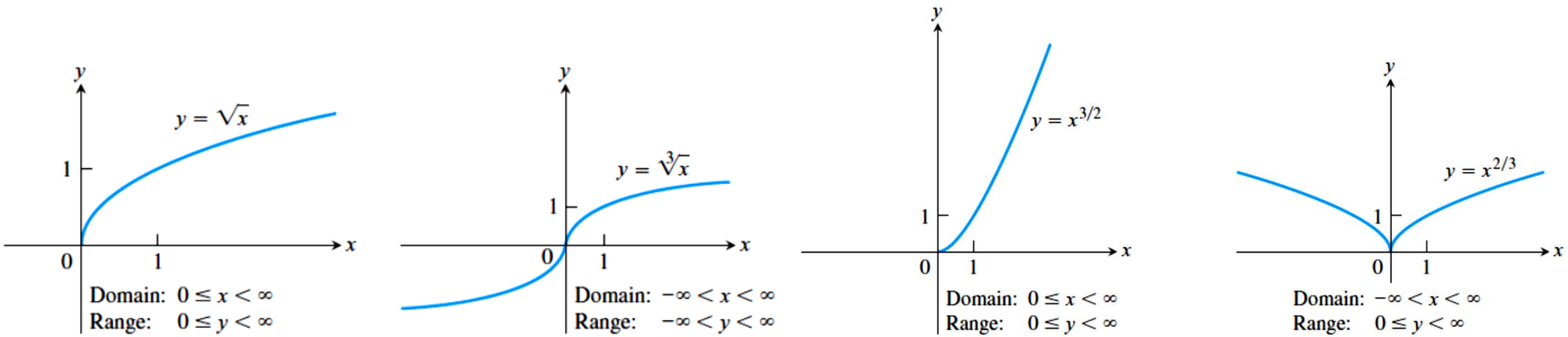


FIGURE 1.38 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.39 shows the graphs of three polynomials.

$$\boxed{f(x) = mx + b}$$

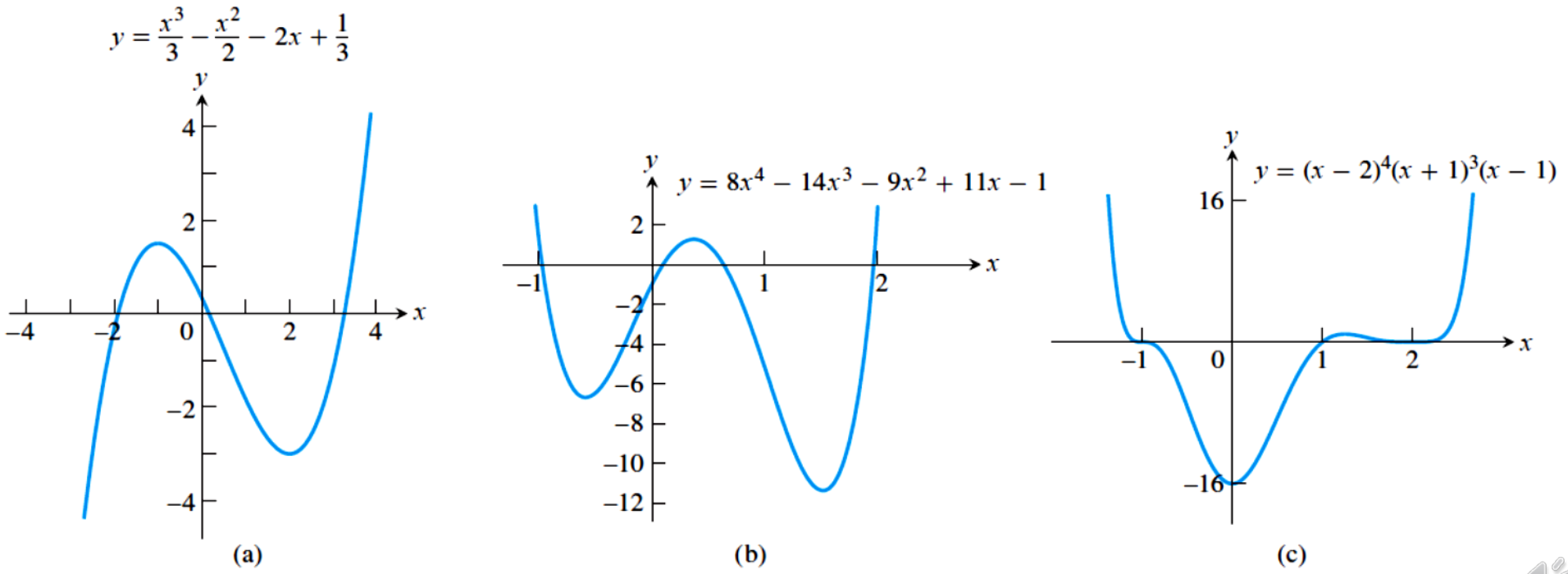


FIGURE 1.39 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function with domain $\{x \mid x \neq -4/7\}$. Its graph is shown in Figure 1.40a with the graphs of two other rational functions in Figures 1.40b and 1.40c.

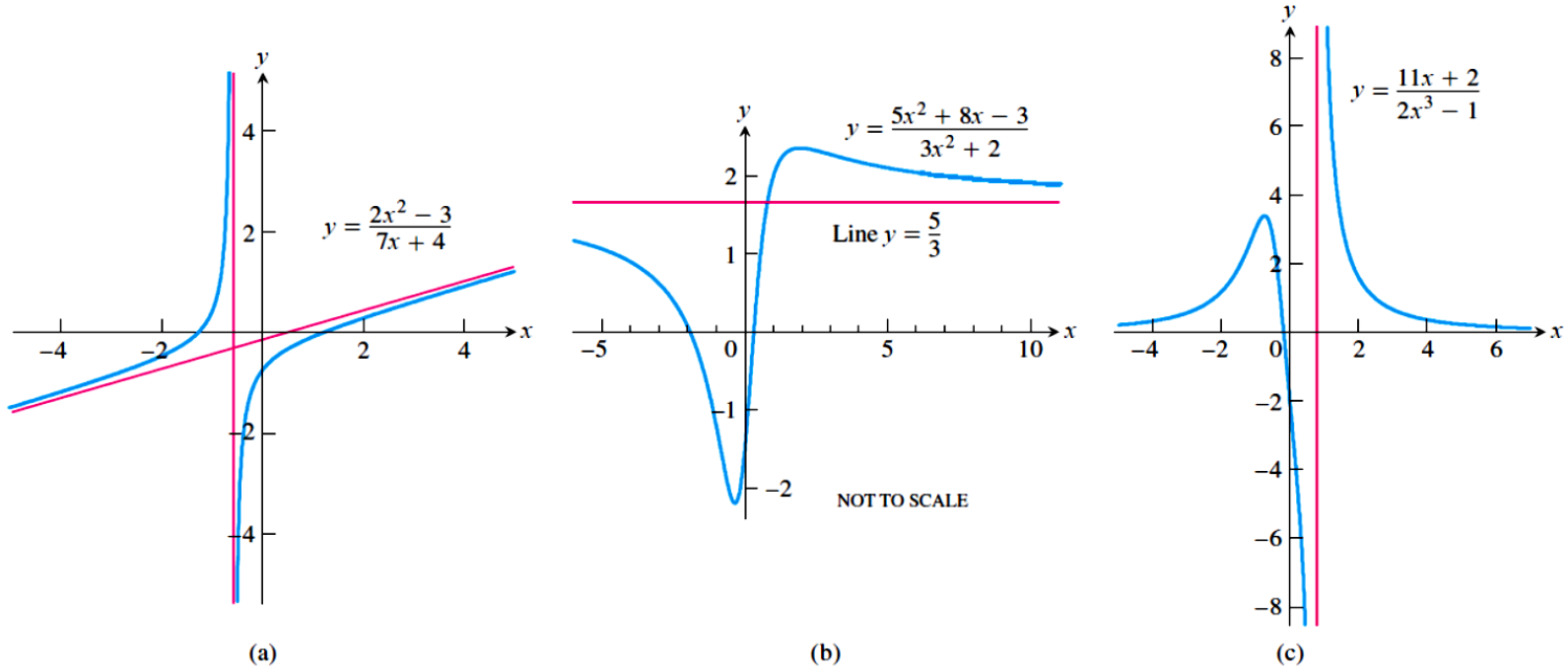


FIGURE 1.40 Graphs of three rational functions.

Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions. Figure 1.41 displays the graphs of three algebraic functions.

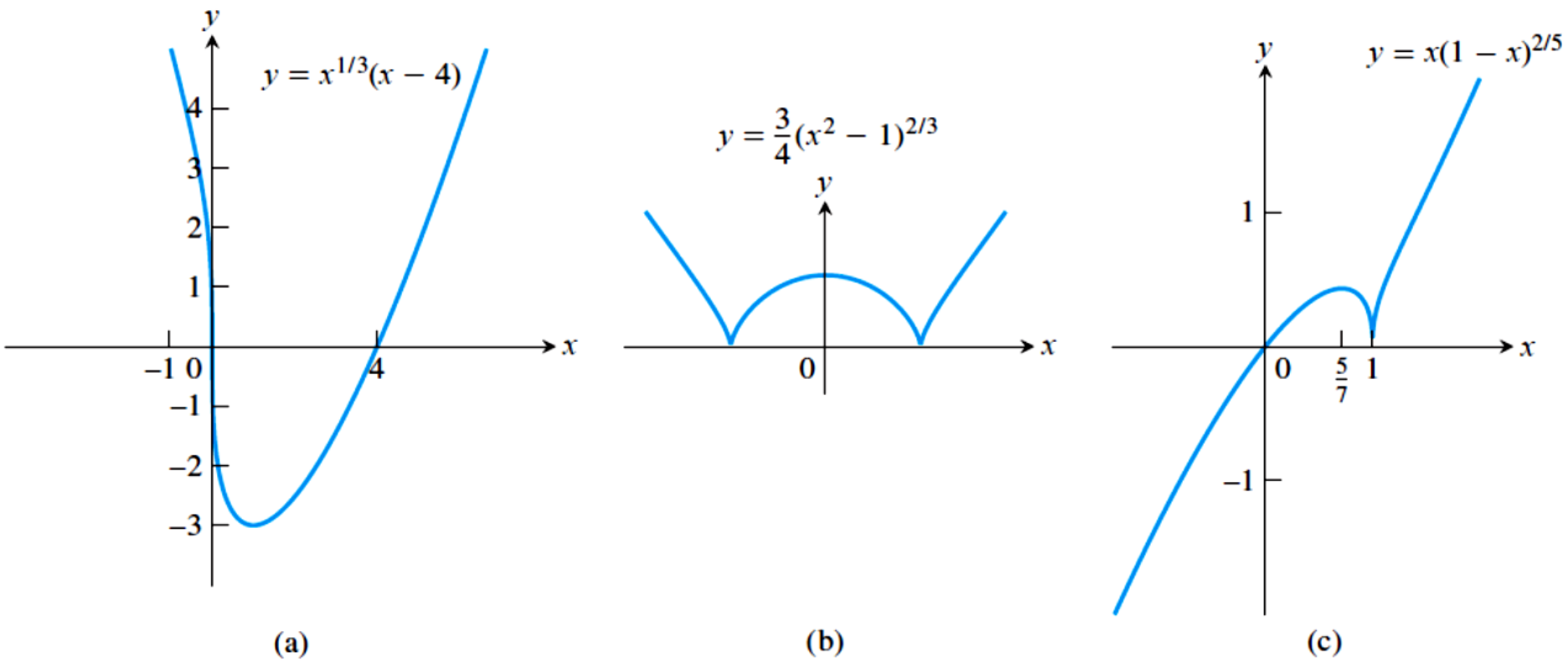
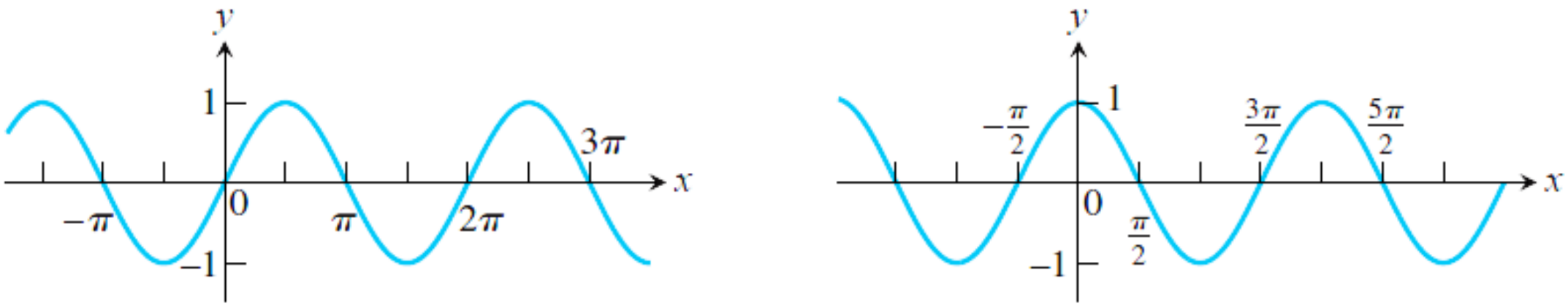


FIGURE 1.41 Graphs of three algebraic functions.

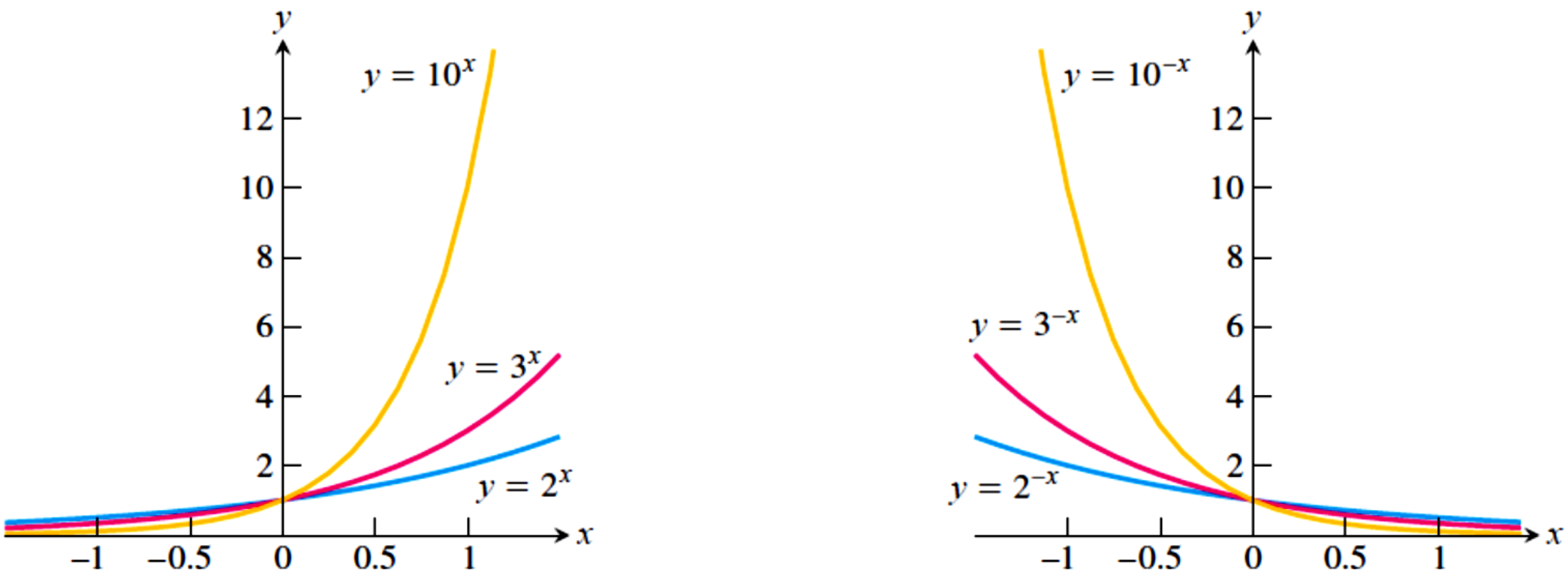
Trigonometric Functions We review trigonometric functions in Section 1.6. The graphs of the sine and cosine functions are shown in Figure 1.42.



(a) $f(x) = \sin x$ (b) $f(x) = \cos x$

FIGURE 1.42 Graphs of the sine and cosine functions.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$. So an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1.43.



(a) $y = 2^x, y = 3^x, y = 10^x$ (b) $y = 2^{-x}, y = 3^{-x}, y = 10^{-x}$

FIGURE 1.43 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and the calculus of these functions is studied in Chapter 7. Figure 1.44 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

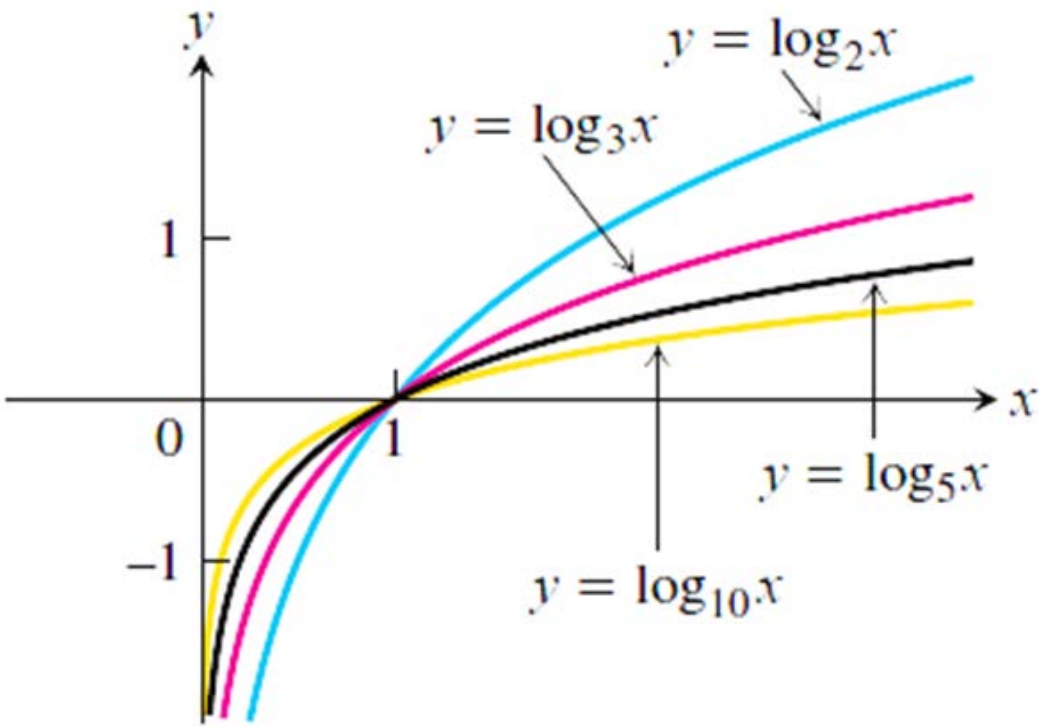
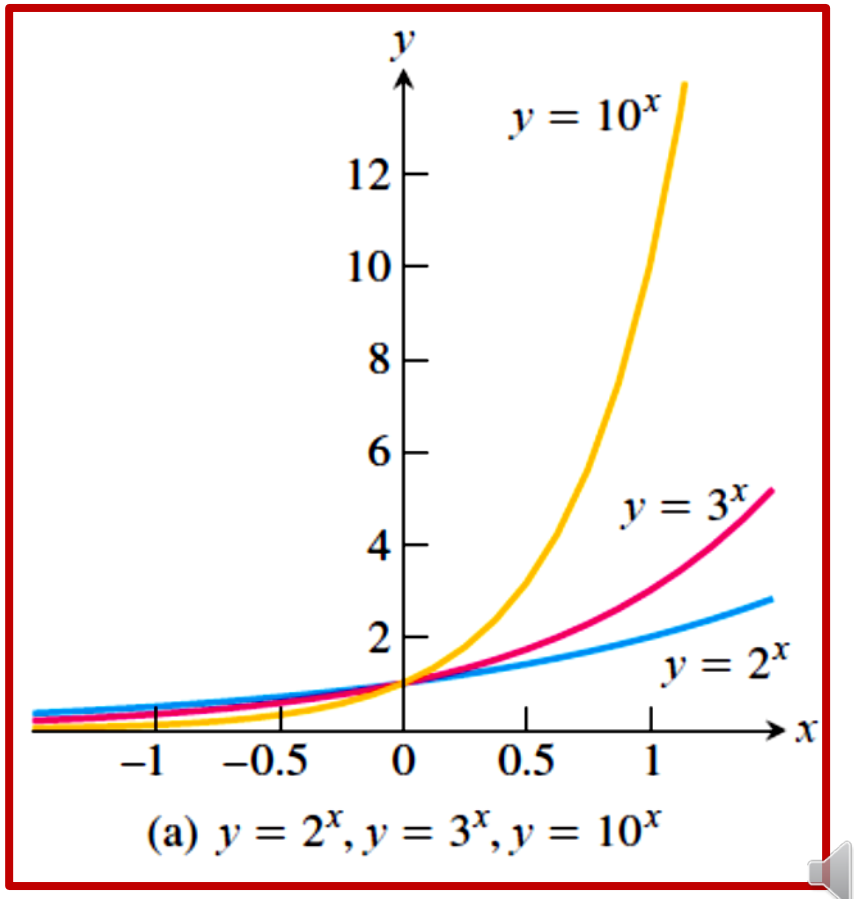


FIGURE 1.44 Graphs of four logarithmic functions.



(a) $y = 2^x, y = 3^x, y = 10^x$

EXAMPLE 1 Recognizing Functions

Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category. For example, $f(x) = x^2$ is both a power function and a polynomial of second degree.

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ **(b)** $g(x) = 7^x$ **(c)** $h(z) = z^7$

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$

Solution

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ is a polynomial of degree 5.

(b) $g(x) = 7^x$ is an exponential function with base 7. Notice that the variable x is the exponent.

(c) $h(z) = z^7$ is a power function. (The variable z is the base.)

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$ is a trigonometric function. ■

Practice example :

Recognizing Functions

In Exercises 1–4, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

1. a. $f(x) = 7 - 3x$ **b.** $g(x) = \sqrt[5]{x}$

c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ **d.** $r(x) = 8^x$

2. a. $F(t) = t^4 - t$ **b.** $G(t) = 5^t$

c. $H(z) = \sqrt{z^3 + 1}$ **d.** $R(z) = \sqrt[3]{z^7}$

3. a. $y = \dfrac{3 + 2x}{x - 1}$ **b.** $y = x^{5/2} - 2x + 1$

c. $y = \tan \pi x$ **d.** $y = \log_7 x$

4. a. $y = \log_5 \left(\dfrac{1}{t}\right)$ **b.** $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$

c. $g(x) = 2^{1/x}$ **d.** $w = 5 \cos \left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$

Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$(f + g)(x) = f(x) + g(x).$$
$$(f - g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 Combining Functions Algebraically

The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x},$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	**Formula**	**Domain**
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

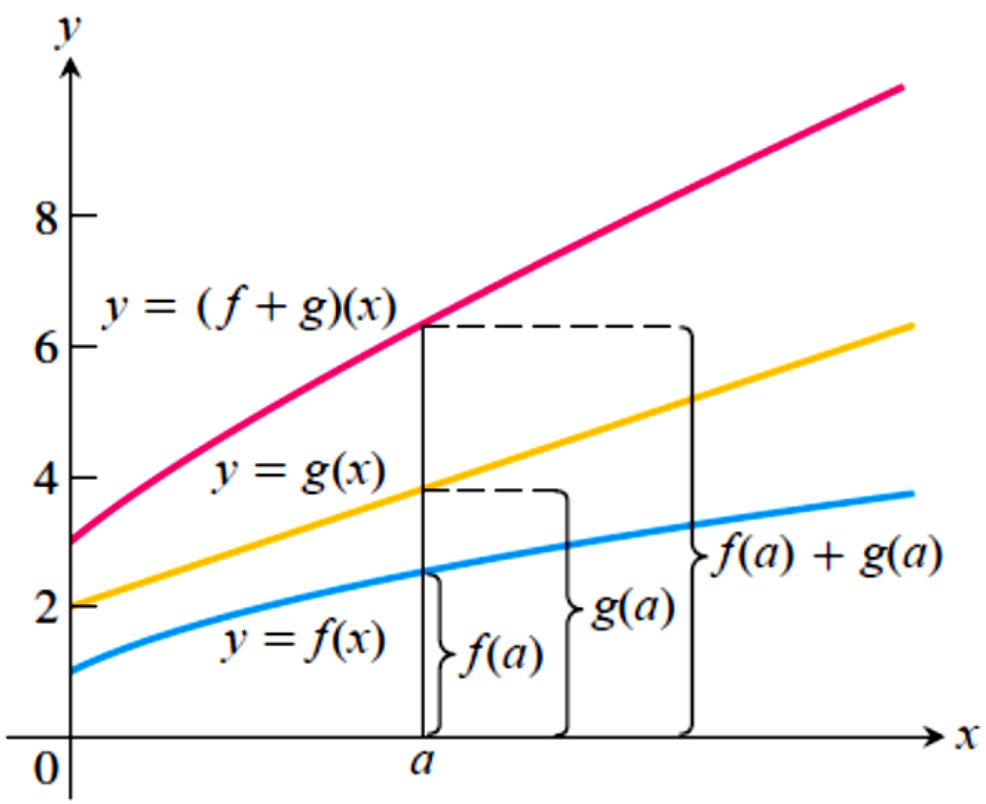


FIGURE 1.50 Graphical addition of two functions.

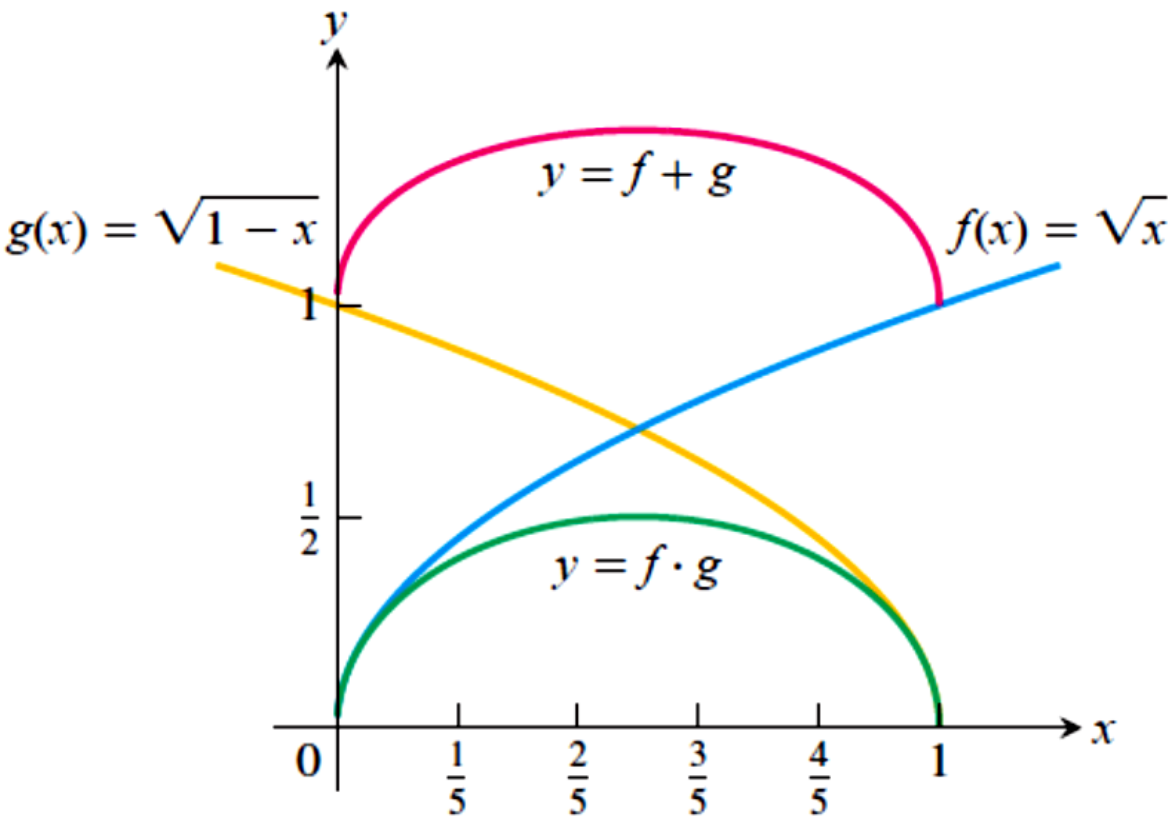


FIGURE 1.51 The domain of the function $f + g$ is the intersection of the domains of f and g, the interval $[0, 1]$ on the x-axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite Functions

Composition is another method for combining functions.

DEFINITION Composition of Functions

If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

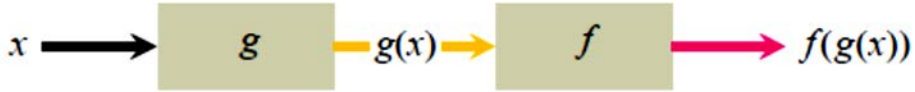


FIGURE 1.52 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

The definition says that $f \circ g$ can be formed when the range of g lies in the domain of f. To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.52 pictures $f \circ g$ as a machine diagram and Figure 1.53 shows the composite as an arrow diagram.

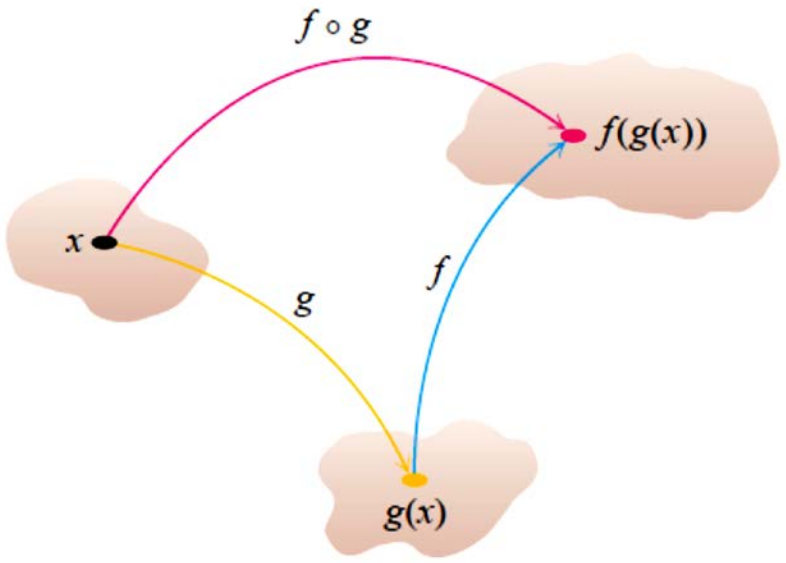


FIGURE 1.53 Arrow diagram for $f \circ g$.

EXAMPLE 2 Viewing a Function as a Composite

The function $y = \sqrt{1 - x^2}$ can be thought of as first calculating $1 - x^2$ and then taking the square root of the result. The function y is the composite of the function $g(x) = 1 - x^2$ and the function $f(x) = \sqrt{x}$. Notice that $1 - x^2$ cannot be negative. The domain of the composite is $[-1, 1]$. ■

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 3 Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = \left(\sqrt{x}\right)^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$.

Shifting a Graph of a Function

To shift the graph of a function $y = f(x)$ straight up, add a positive constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of a function $y = f(x)$ straight down, add a negative constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of $y = f(x)$ to the left, add a positive constant to x. To shift the graph of $y = f(x)$ to the right, add a negative constant to x.

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\lvert h \rvert$ units if $h < 0$

EXAMPLE 4 Shifting a Graph

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.54).

(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.54).

(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.55).

(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.56).

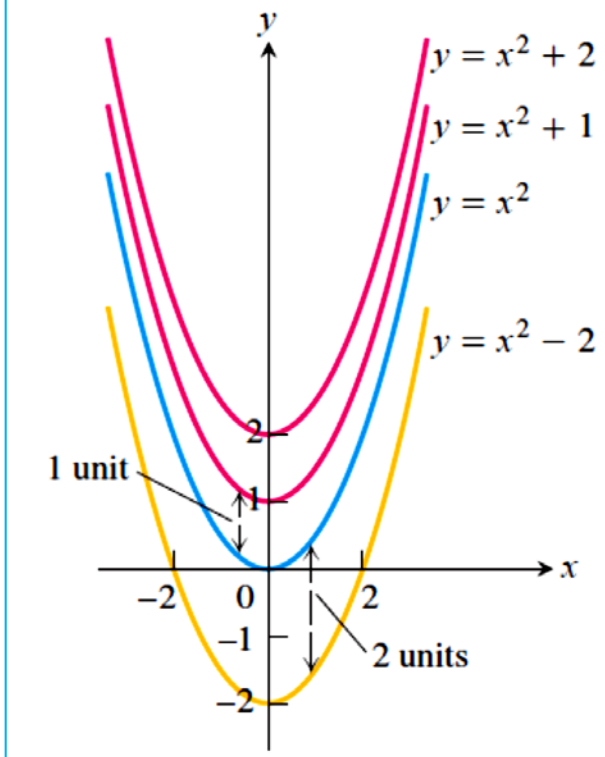


FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

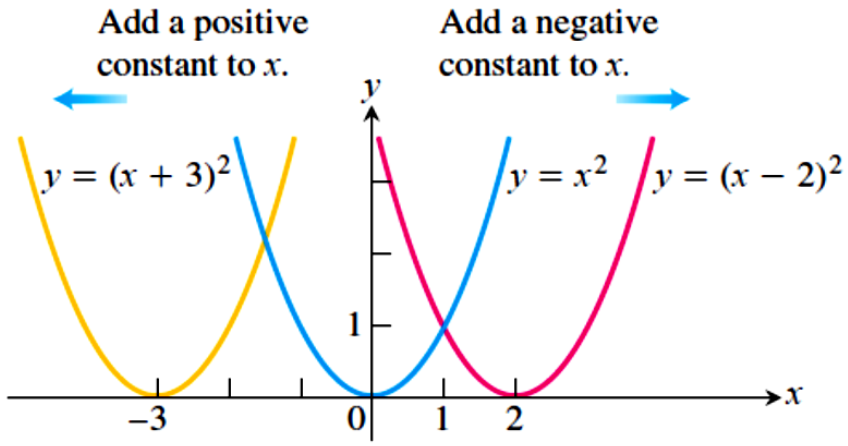


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x (Example 4c).

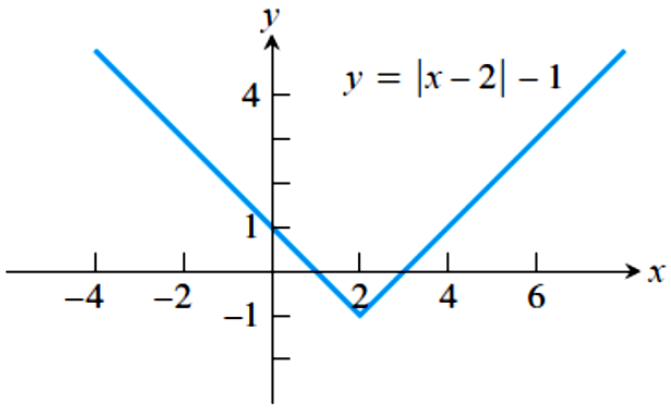


FIGURE 1.56 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLE 5 Scaling and Reflecting a Graph

(a) Vertical: Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph by a factor of 3 (Figure 1.57).

(b) Horizontal: The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.58). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) Reflection: The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.59).

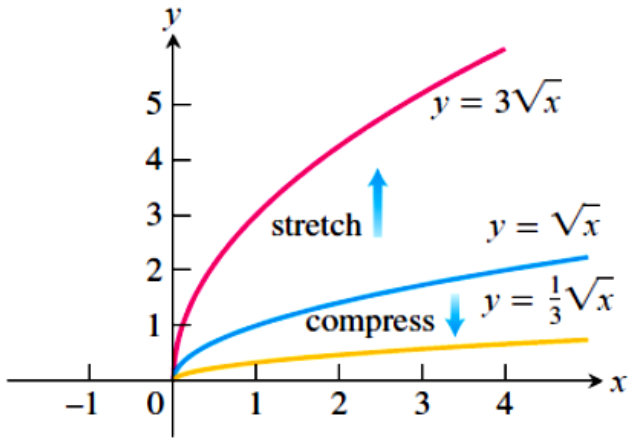


FIGURE 1.57 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5a).

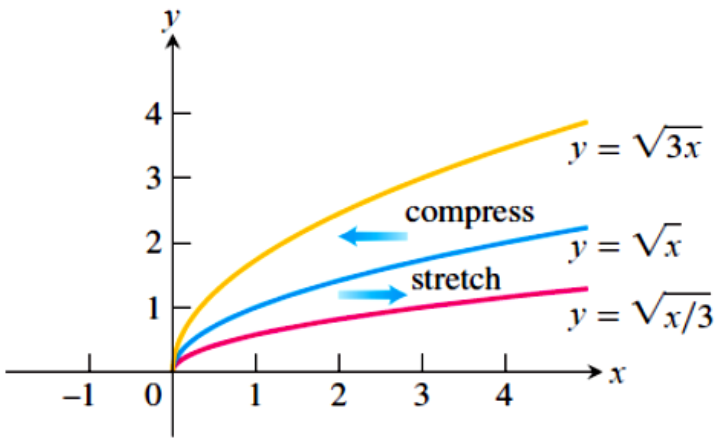


FIGURE 1.58 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 5b).

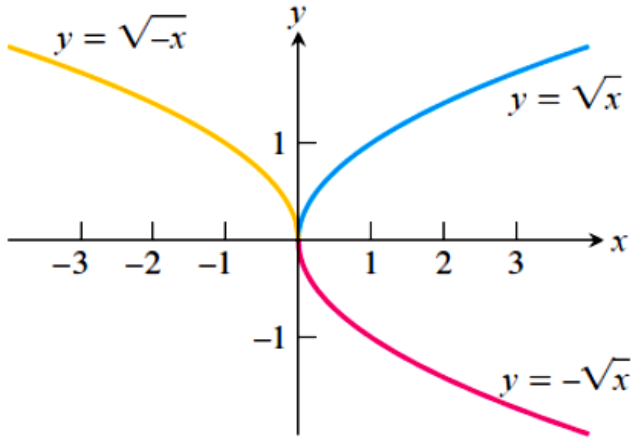


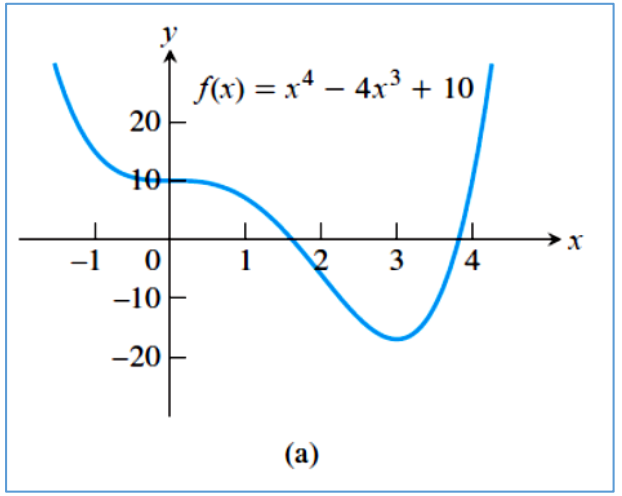
FIGURE 1.59 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 5c).

EXAMPLE 6 Combining Scalings and Reflections

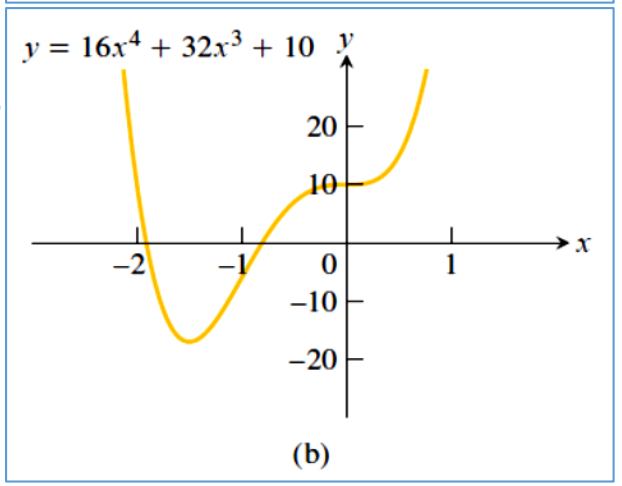
Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.60a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.60b).

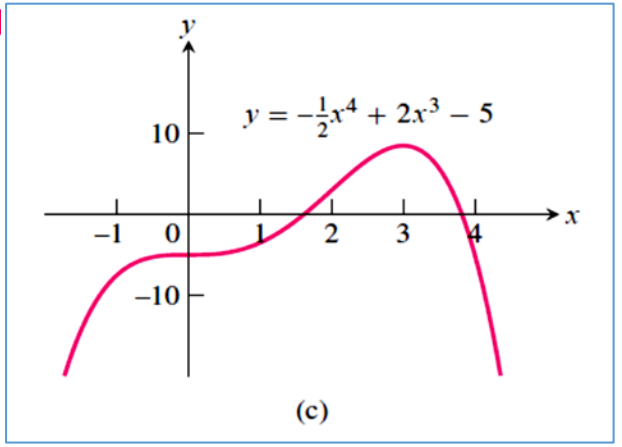
(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.60c).



(a)



(b)



(c)

Solution

(a) The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$

Trigonometric Functions

This section reviews the basic trigonometric functions. The trigonometric functions are important because they are periodic, or repeating.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called *radians* because of the way they simplify later calculations.

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.63) equals the length of the arc that ACB cuts from the unit circle. Figure 1.63 shows that $s = r\theta$ is the **length of arc** cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians.

Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by

$$\pi \text{ radians} = 180°.$$

For example, $45°$ in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°.$$

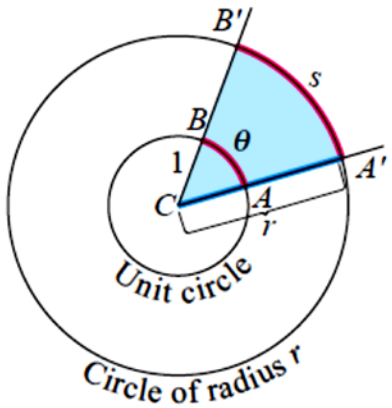


FIGURE 1.63 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C. The value of θ can be found from any other circle, however, as the ratio s/r. Thus $s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.

Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$

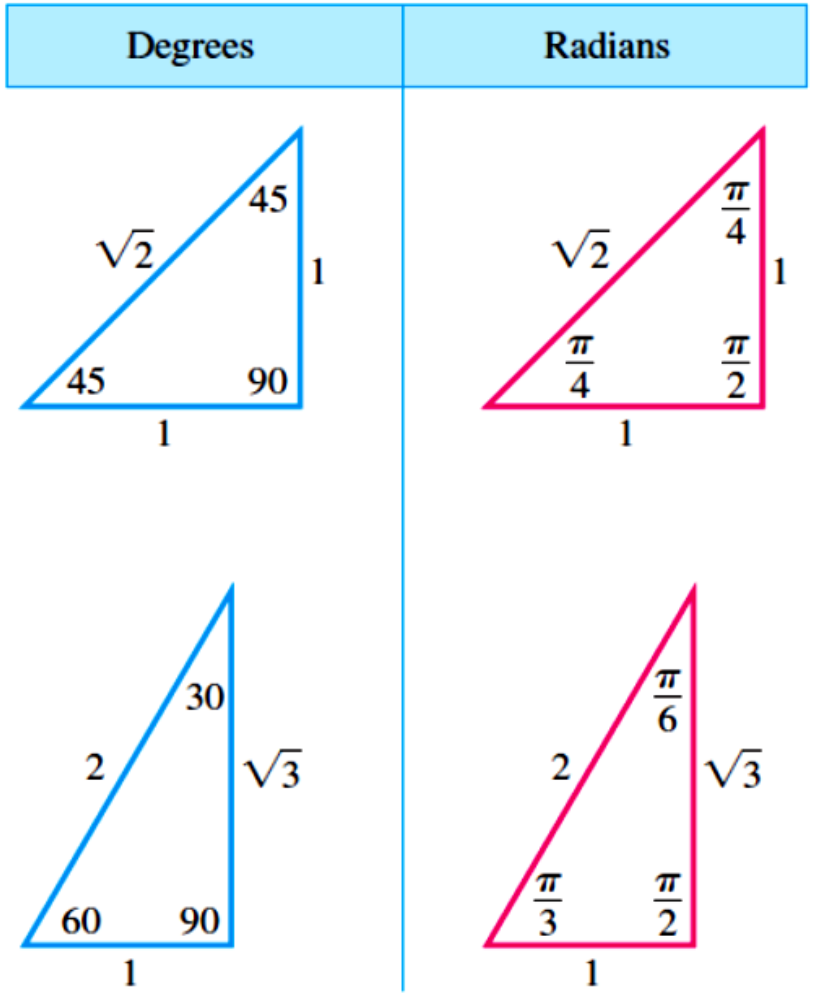


FIGURE 1.64 The angles of two common triangles, in degrees and radians.

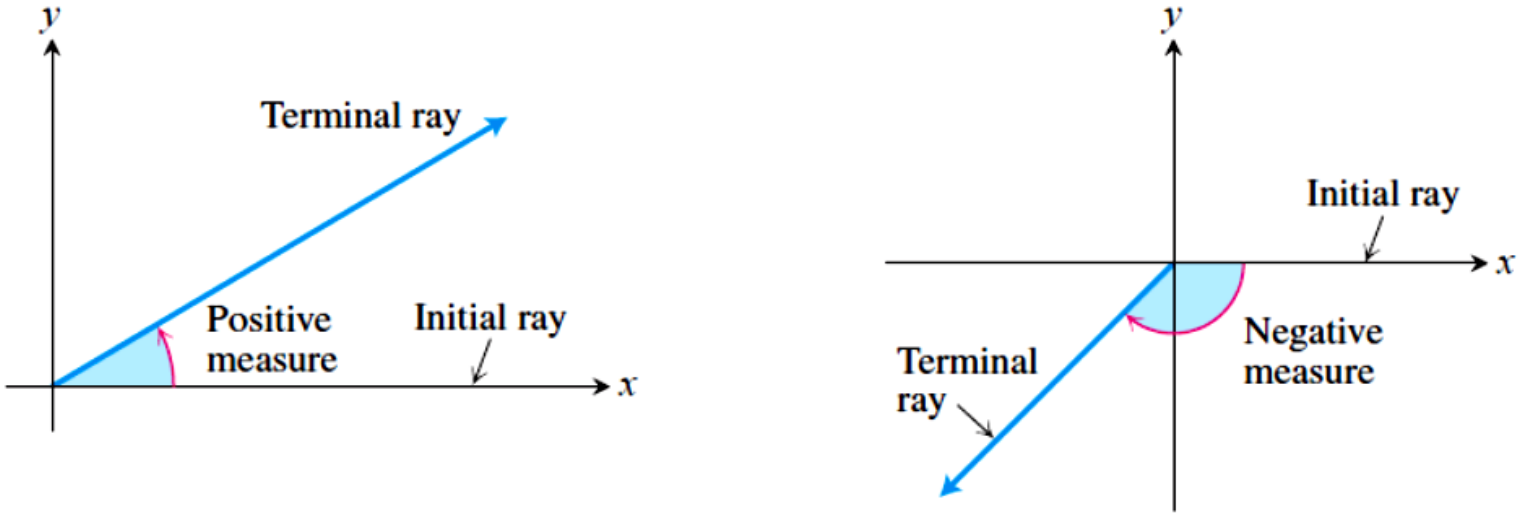


FIGURE 1.65 Angles in standard position in the xy-plane.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.67). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.68).

$$\textbf{sine:} \quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:} \quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:} \quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:} \quad \sec\theta = \frac{r}{x}$$

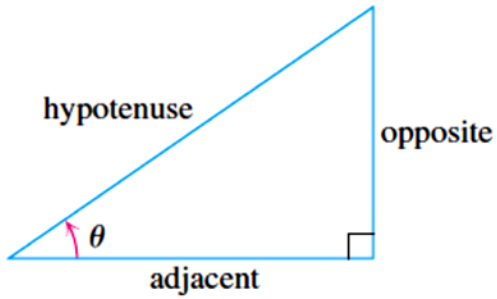
$$\textbf{tangent:} \quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:} \quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is acute (Figure 1.69).

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

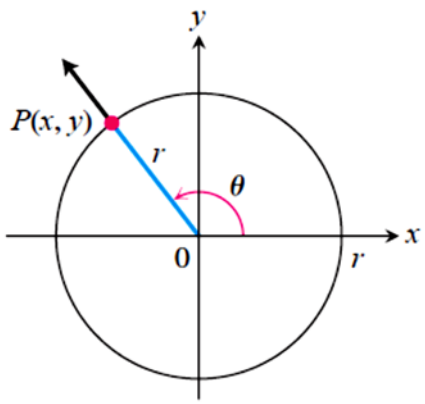


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

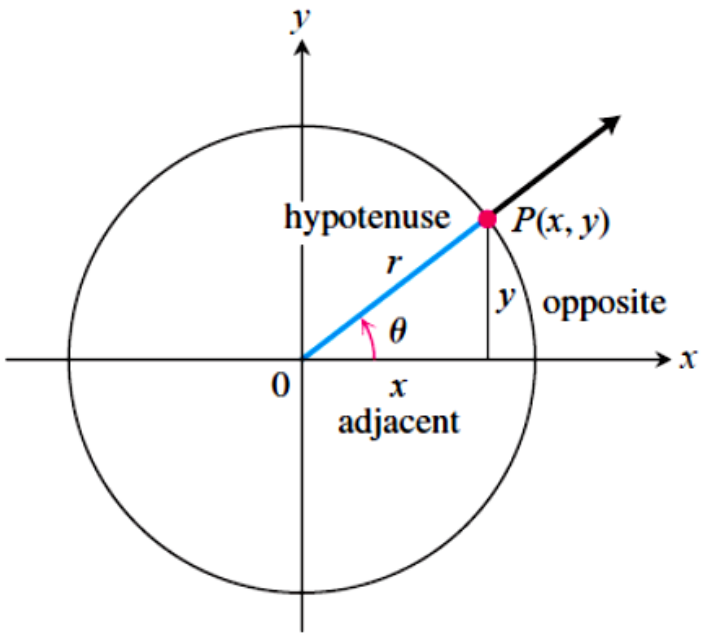


FIGURE 1.69 The new and old definitions agree for acute angles.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The CAST rule (Figure 1.70) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.71, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

Using a similar method we determined the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in Table 1.4.

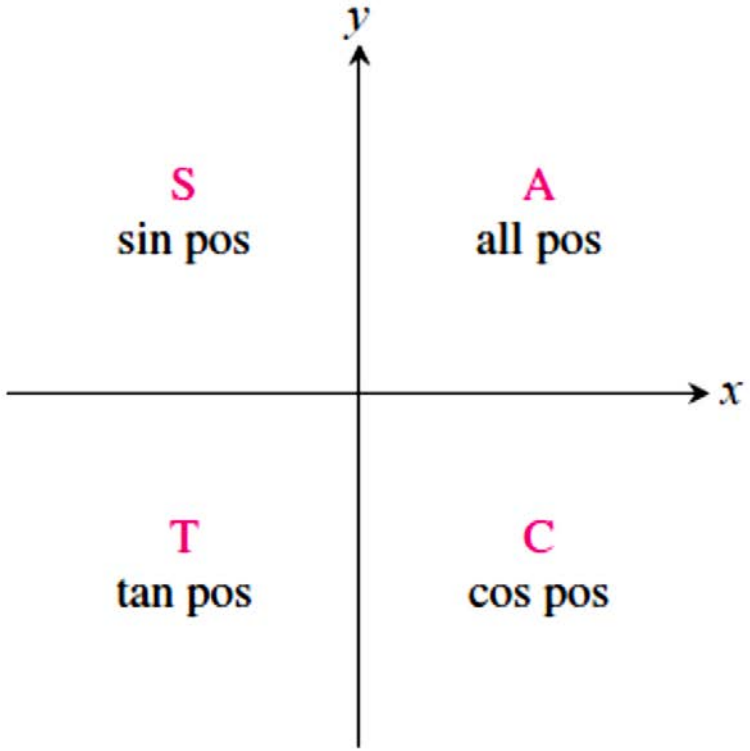


FIGURE 1.70 The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.

$$\left(\cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

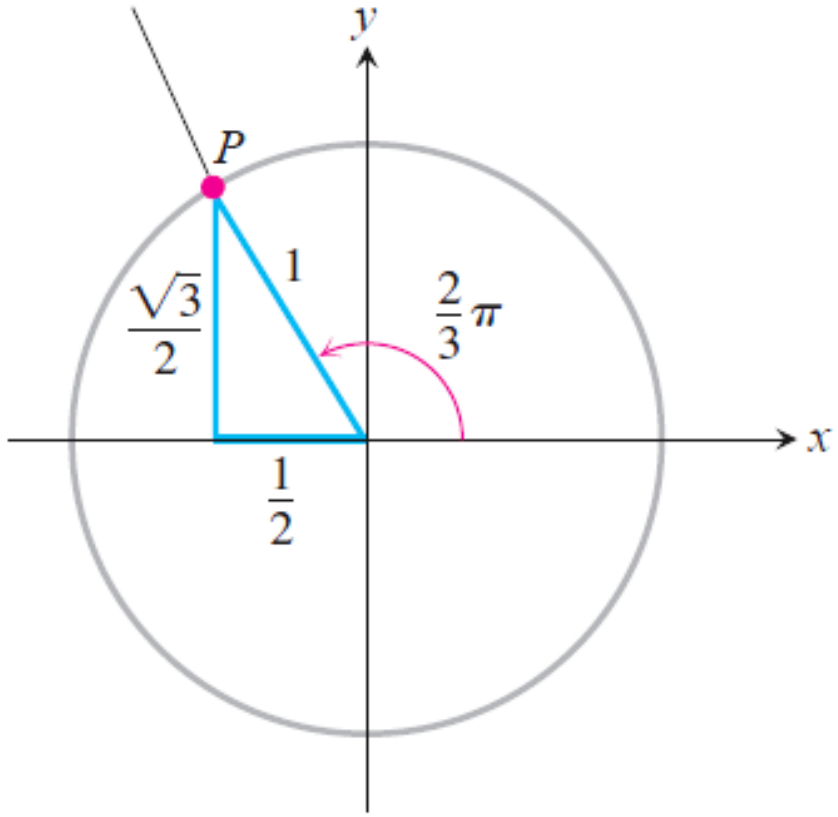


FIGURE 1.71 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0		0

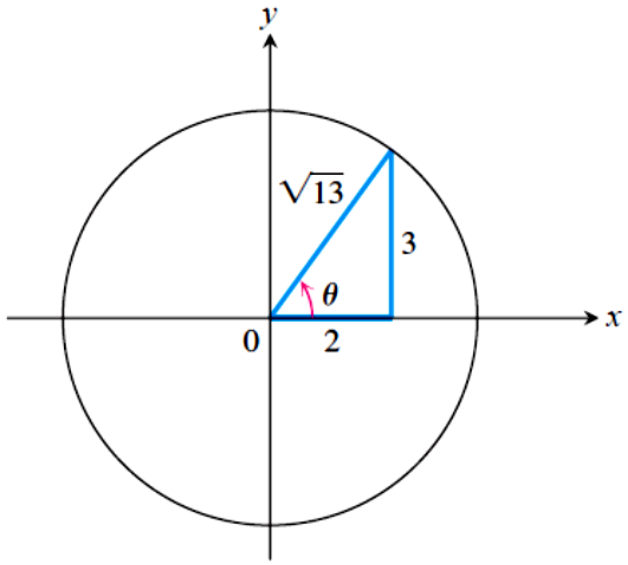


FIGURE 1.72 The triangle for calculating the trigonometric functions in Example 1.

EXAMPLE 1 Finding Trigonometric Function Values

If $\tan\theta = 3/2$ and $0 < \theta < \pi/2$, find the five other trigonometric functions of θ.

Solution From $\tan\theta = 3/2$, we construct the right triangle of height 3 (opposite) and base 2 (adjacent) in Figure 1.72. The Pythagorean theorem gives the length of the hypotenuse, $\sqrt{4+9} = \sqrt{13}$. From the triangle we write the values of the other five trigonometric functions:

$$\cos\theta = \frac{2}{\sqrt{13}}, \qquad \sin\theta = \frac{3}{\sqrt{13}}, \qquad \sec\theta = \frac{\sqrt{13}}{2}, \qquad \csc\theta = \frac{\sqrt{13}}{3}, \qquad \cot\theta = \frac{2}{3}$$

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

7. $\cos x = -\frac{4}{5}$, $\tan x = -\frac{3}{4}$

8. $\sin x = \frac{2}{\sqrt{5}}$, $\cos x = \frac{1}{\sqrt{5}}$

9. $\sin x = -\frac{\sqrt{8}}{3}$, $\tan x = -\sqrt{8}$

10. $\sin x = \frac{12}{13}$, $\tan x = -\frac{12}{5}$

Periodicity and Graphs of the Trigonometric Functions

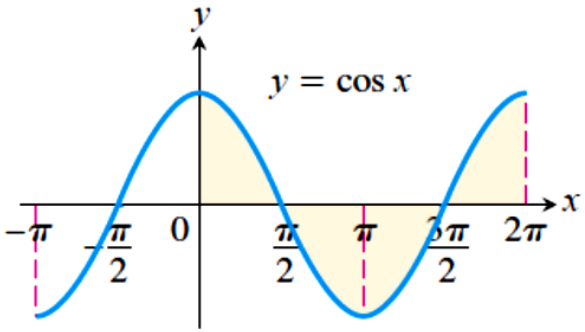
When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \qquad \cot(\theta + 2\pi) = \cot\theta$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

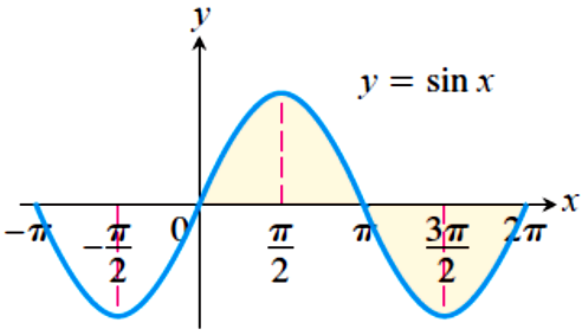
DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.



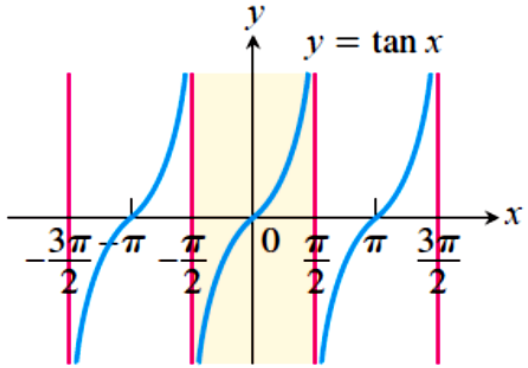
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



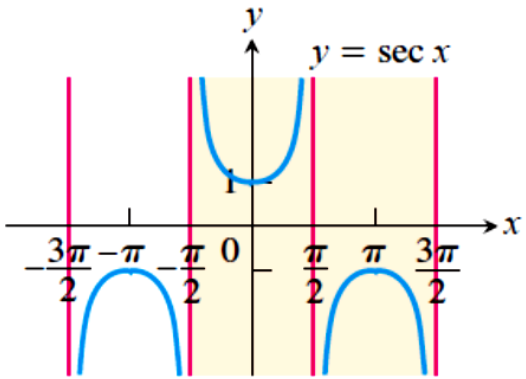
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



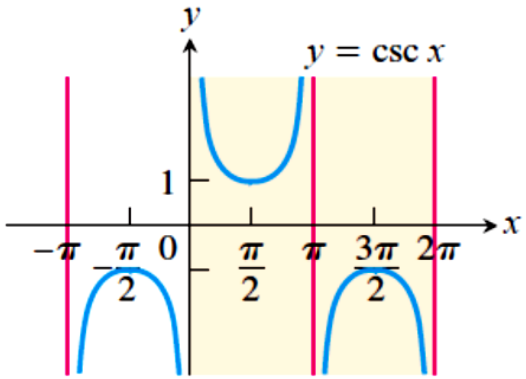
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



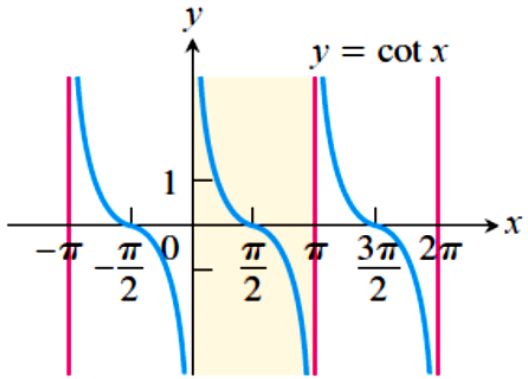
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

FIGURE 1.73 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Periods of Trigonometric Functions

Period π:

$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:

$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

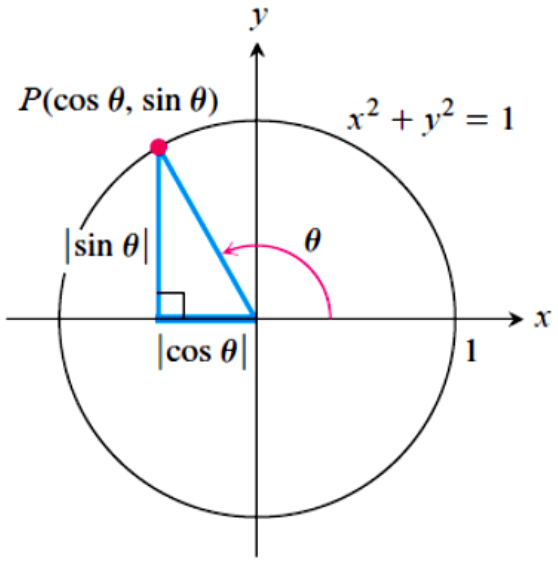


FIGURE 1.74 The reference triangle for a general angle θ.

Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Figure 1.69). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.74 and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \tag{2}$$

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \tag{3}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{4}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{5}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{6}$$

This equation is called the **law of cosines**.

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Figure 1.75. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

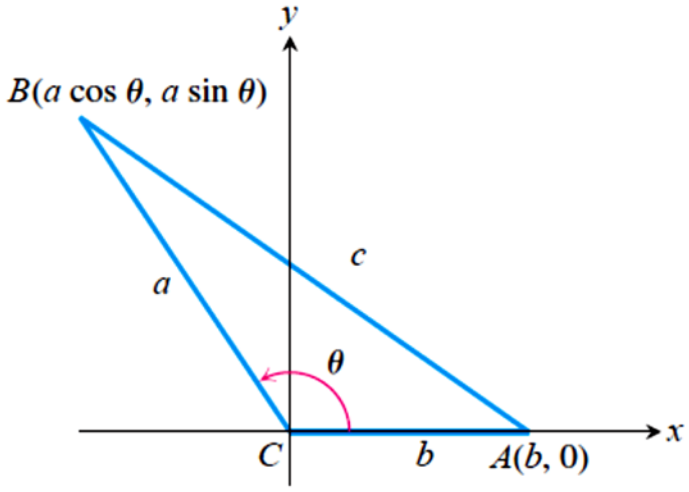


FIGURE 1.75 The square of the distance between A and B gives the law of cosines.

Use the addition formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Addition Formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (2)$$

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

31. $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\left(-\frac{\pi}{2}\right) - \sin x \sin\left(-\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(-1) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cos\left(\frac{\pi}{2}\right) - \sin x \sin\left(\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(1) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\left(\frac{\pi}{2}\right) + \cos x \sin\left(\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(1) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\left(-\frac{\pi}{2}\right) + \cos x \sin\left(-\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(-1) = -\cos x$

35. $\cos(A - B) = \cos(A + (-B)) = \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B - \sin A(-\sin B)$
$= \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B + \cos A(-\sin B)$
$= \sin A \cos B - \cos A \sin B$

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$ **40.** $\sin(2\pi - x)$

39. $\cos(\pi + x) = \cos\pi\cos x - \sin\pi\sin x = (-1)(\cos x) - (0)(\sin x) = -\cos x$

40. $\sin(2\pi - x) = \sin 2\pi\cos(-x) + \cos(2\pi)\sin(-x) = (0)(\cos(-x)) + (1)(\sin(-x)) = -\sin x$

Find the function values in Exercises 47–50.

47. $\cos^2\frac{\pi}{8}$ **48.** $\cos^2\frac{\pi}{12}$

49. $\sin^2\frac{\pi}{12}$ **50.** $\sin^2\frac{\pi}{8}$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \quad (4)$$

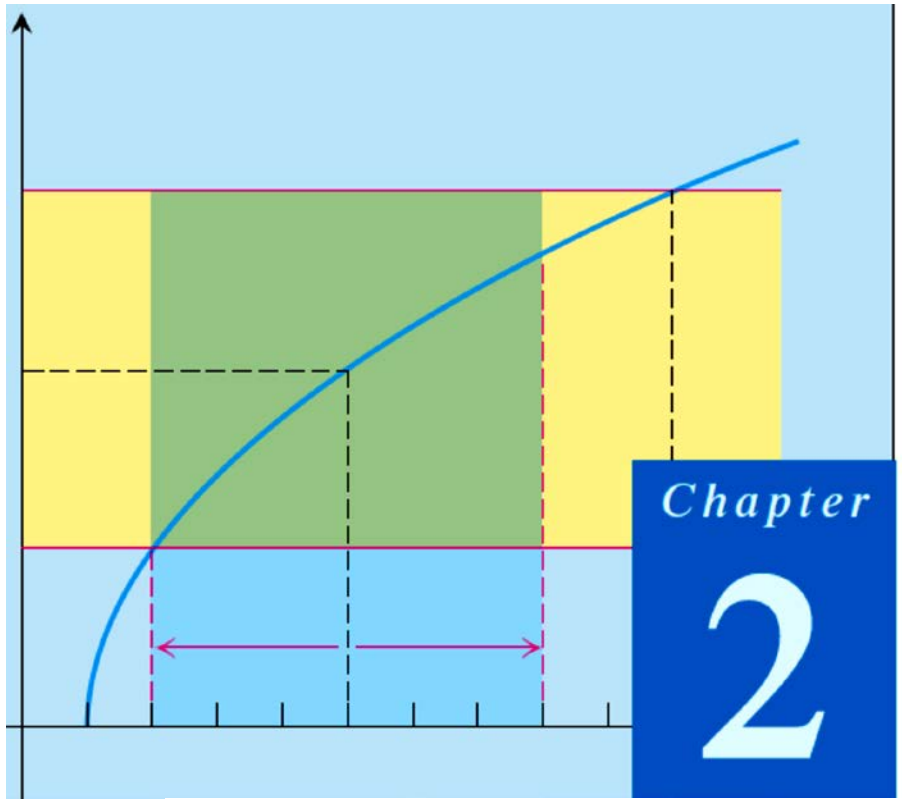
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \quad (5)$$

47. $\cos^2\frac{\pi}{8} = \frac{1+\cos\left(\frac{2\pi}{8}\right)}{2} = \frac{1+\frac{\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}$

48. $\cos^2\frac{\pi}{12} = \frac{1+\cos\left(\frac{2\pi}{12}\right)}{2} = \frac{1+\frac{\sqrt{3}}{2}}{2} = \frac{2+\sqrt{3}}{4}$

49. $\sin^2\frac{\pi}{12} = \frac{1-\cos\left(\frac{2\pi}{12}\right)}{2} = \frac{1-\frac{\sqrt{3}}{2}}{2} = \frac{2-\sqrt{3}}{4}$

50. $\sin^2\frac{\pi}{8} = \frac{1-\cos\left(\frac{2\pi}{8}\right)}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$



LIMITS AND CONTINUITY

OVERVIEW The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is fundamental to finding the tangent to a curve or the velocity of an object.

Rates of Change and Limits

In this section, we introduce average and instantaneous rates of change. These lead to the main idea of the section, the idea of limit.

Average and Instantaneous Speed

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 Finding an Average Speed

A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

the only force acting on the falling body. We call this type of motion **free fall**.) If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the constant of proportionality.

The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt.

(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$

The next example examines what happens when we look at the average speed of a falling object over shorter and shorter time intervals.

EXAMPLE 3 The Average Growth Rate of a Laboratory Population

Figure 2.2 shows how a population of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure 2.2). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}.$$

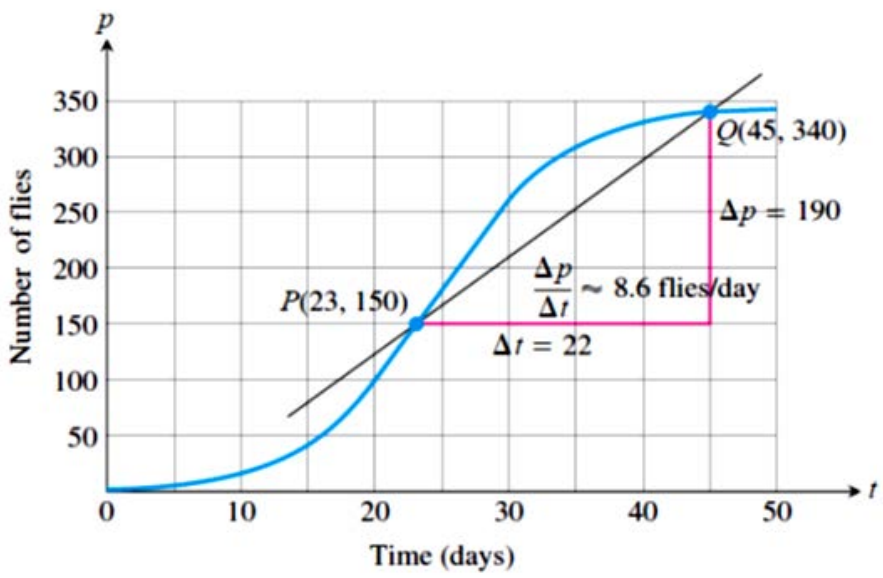


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

This average is the slope of the secant through the points P and Q on the graph in Figure 2.2. ■

The average rate of change from day 23 to day 45 calculated in Example 3 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 4 The Growth Rate on Day 23

How fast was the number of flies in the population of Example 3 growing on day 23?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.3).

$$\boxed{\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day (approximately).}}$$

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$

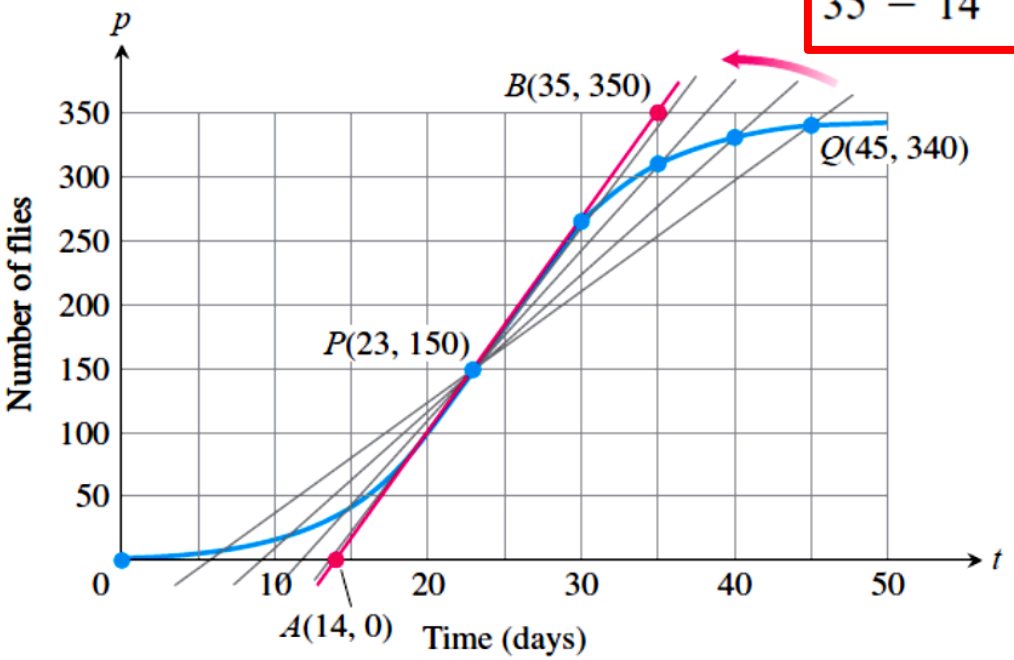


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

we determine limiting values, or *limits*, as we will soon call them.

Limits of Function Values

Let $f(x)$ be defined on an open interval about x_0, *except possibly at* x_0 *itself*. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L".

EXAMPLE 5 Behavior of a Function Near a Point

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

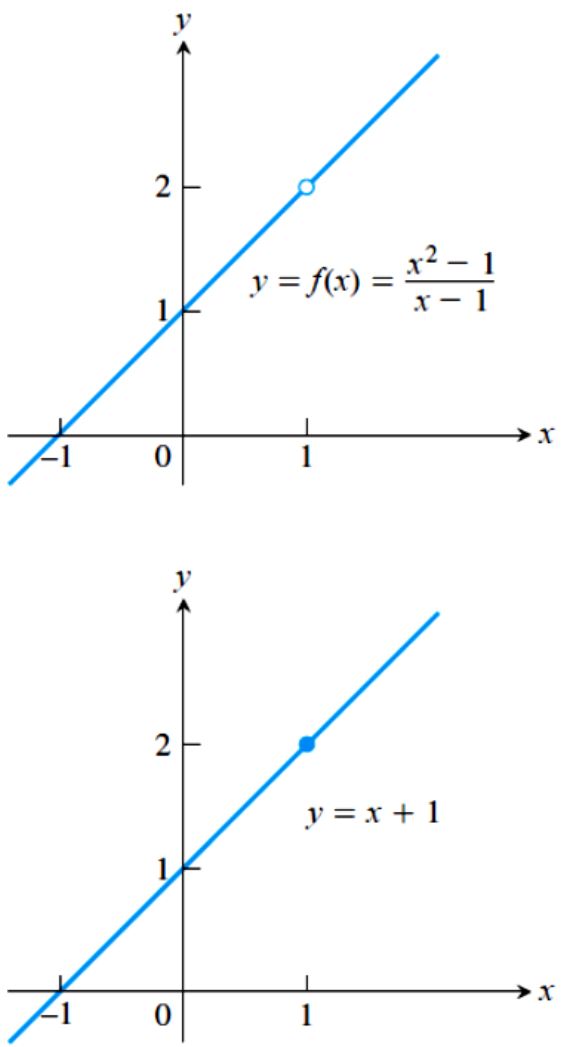


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

The graph of f is thus the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a "hole" in Figure 2.4. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 (Table 2.2).

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

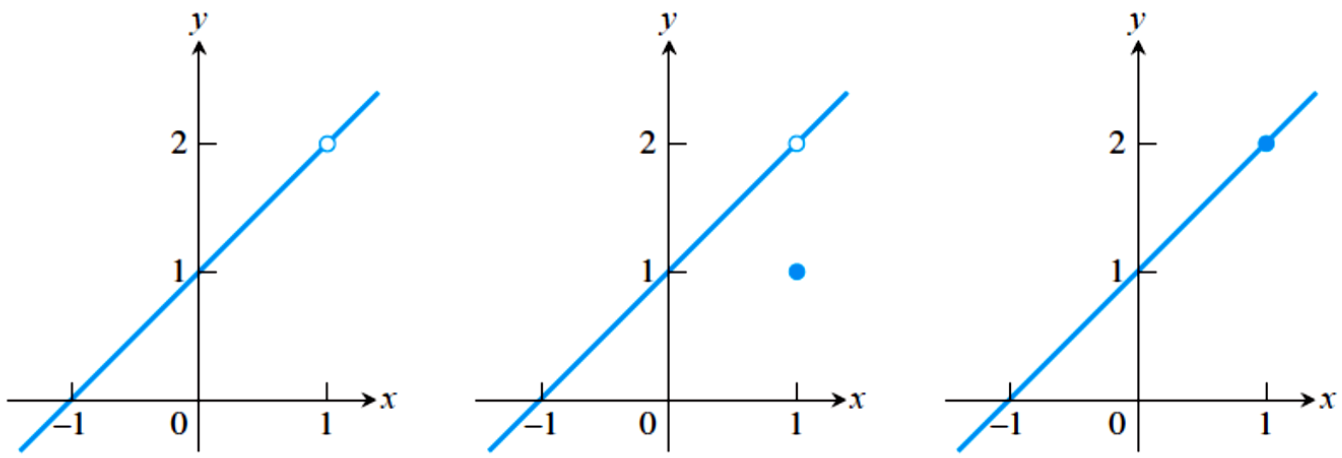
Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

EXAMPLE 6 The Limit Value Does Not Depend on How the Function Is Defined at x_0

The function f in Figure 2.5 has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value is special, and we return to it in Section 2.6. ■

Sometimes $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined. (We will say more about this in Sections 2.2 and 2.6.)

EXAMPLE 8 The Identity and Constant Functions Have Limits at Every Point

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.6a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.6b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instance,

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

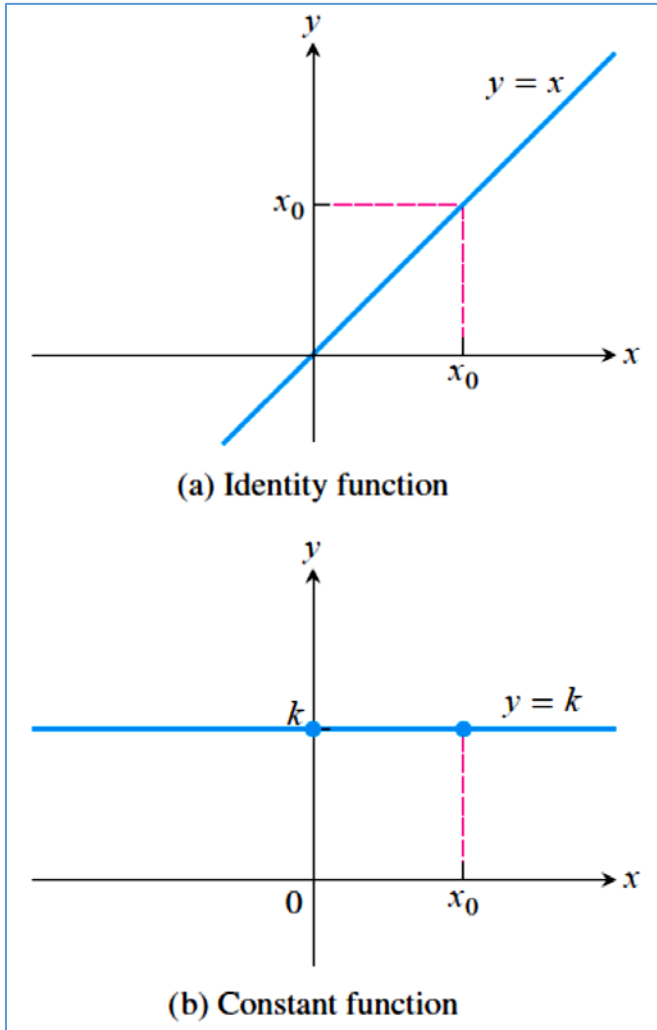


FIGURE 2.6 The functions in Example 8.

Calculating Limits Using the Limit Laws

The Limit Laws

The next theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x \to c} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

EXAMPLE 1 Using the Limit Laws

properties of limits to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

Solution

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Product and Multiple Rules

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

(c)

$$\begin{aligned}
\lim_{x \to -2} \sqrt{4x^2 - 3} &= \sqrt{\lim_{x \to -2} (4x^2 - 3)} && \text{Power Rule with } r/s = 1/2 \\
&= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} && \text{Difference Rule} \\
&= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules} \\
&= \sqrt{16 - 3} \\
&= \sqrt{13}
\end{aligned}$$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step.

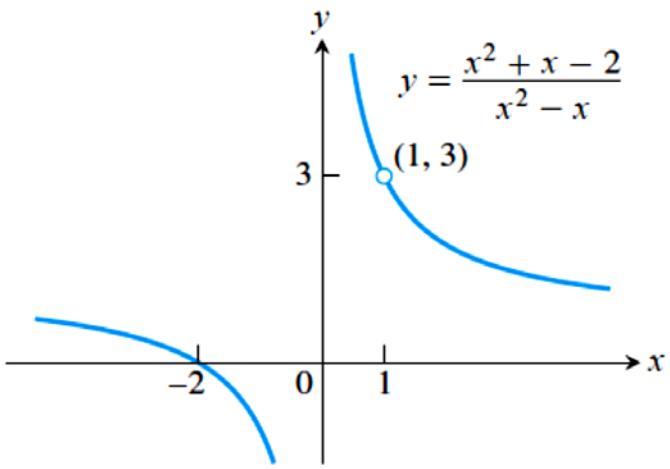
Eliminating Zero Denominators Algebraically

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

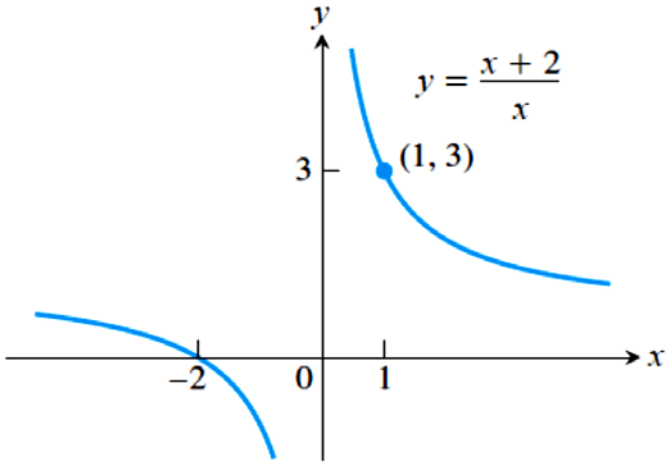
EXAMPLE 3 Canceling a Common Factor

Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$



(a)



(b)

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.8. ■

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

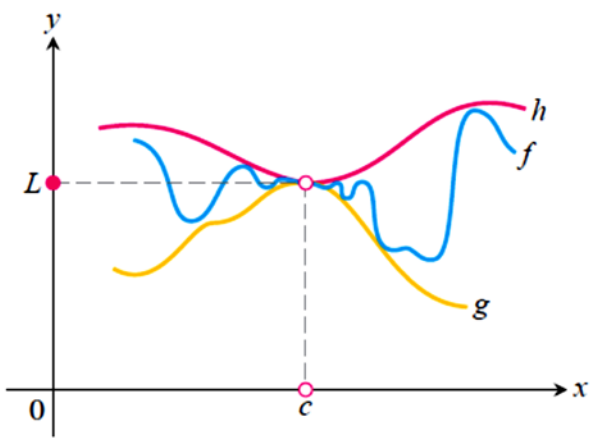


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

The Sandwich Theorem is sometimes called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 5 Applying the Sandwich Theorem

Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.10).

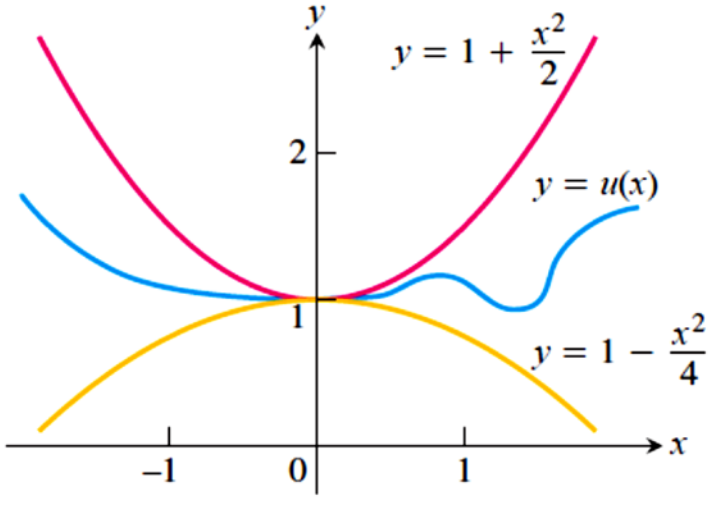


FIGURE 2.10 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 5).

EXAMPLE 6 More Applications of the Sandwich Theorem

(a) (Figure 2.11a). It follows from the definition of $\sin\theta$ that $-|\theta| \le \sin\theta \le |\theta|$ for all θ, and since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, we have

$$\lim_{\theta\to 0} \sin\theta = 0.$$

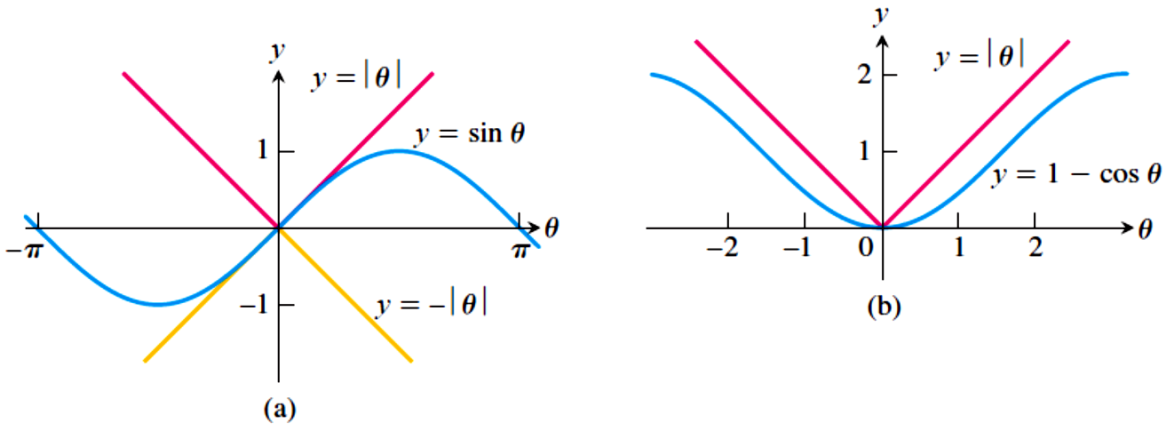


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta\to 0}\sin\theta = 0$ and (b) $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ (Example 6).

(b) (Figure 2.11b). From the definition of $\cos\theta$, $0 \le 1 - \cos\theta \le |\theta|$ for all θ, and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ or

$$\lim_{\theta\to 0} \cos\theta = 1.$$

(c) For any function $f(x)$, if $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$. The argument: $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$. ■

Another important property of limits is given by the next theorem. A proof is given in the next section.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

The assertion resulting from replacing the less than or equal to $\leq$ inequality by the strict $<$ inequality in Theorem 5 is false. Figure 2.11a shows that for $\theta \neq 0$, $-|\theta| < \sin\theta < |\theta|$, but in the limit as $\theta \to 0$, equality holds.

2.3 The Precise Definition of a Limit

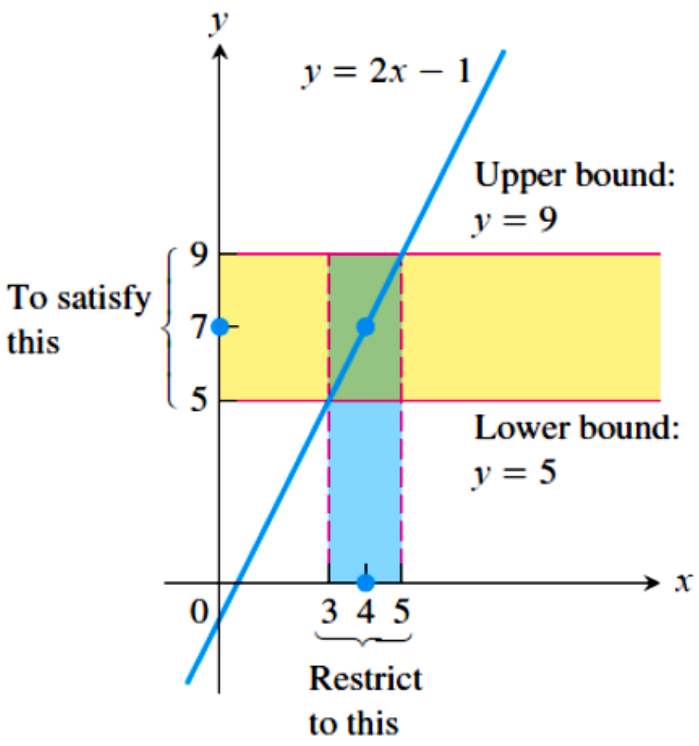


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

EXAMPLE 1 A Linear Function

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it is clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 \\ -1 < x - 4 &< 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Figure 2.12). ■

DEFINITION **Limit of a Function**

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

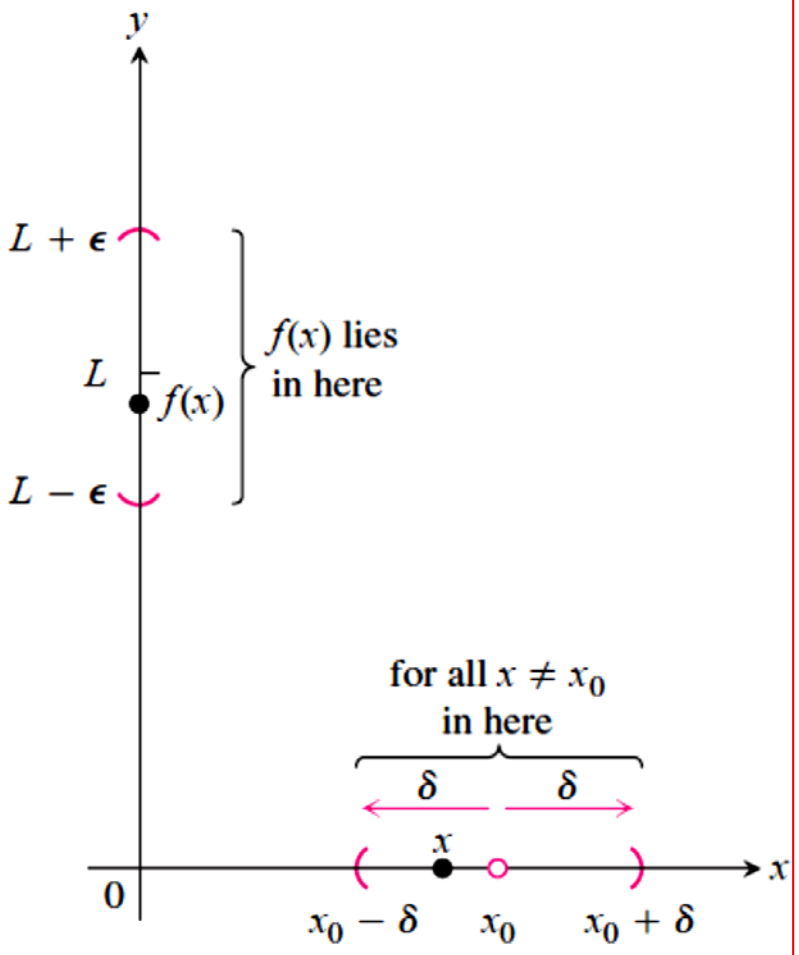


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

Examples: Testing the Definition

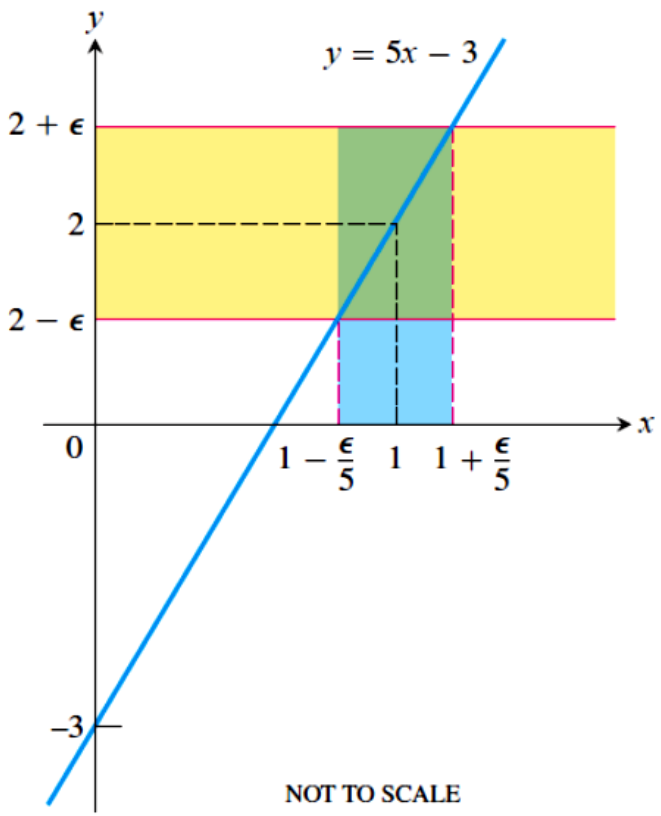


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

EXAMPLE 2 Testing the Definition

Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.15). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x \to 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ■

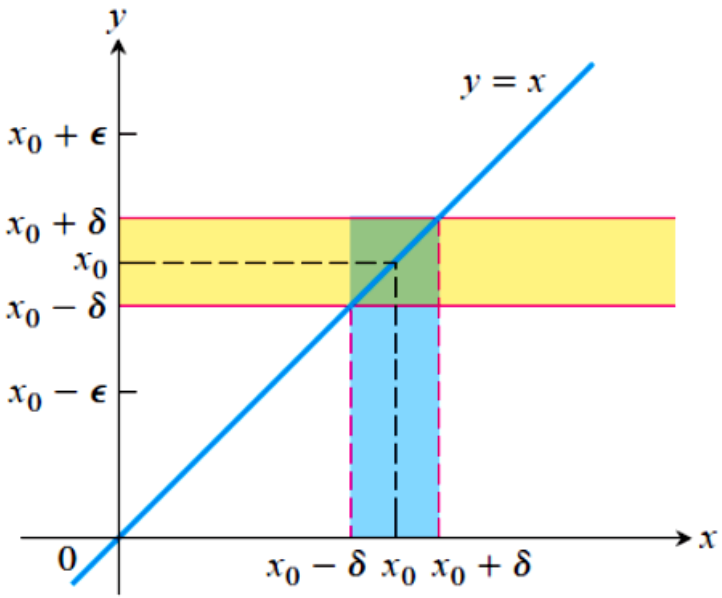


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \le \epsilon$ (Example 3a).

EXAMPLE 3 Limits of the Identity and Constant Functions

Prove:

(a) $\lim_{x \to x_0} x = x_0$ **(b)** $\lim_{x \to x_0} k = k$ (k constant).

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Figure 2.16). This proves that $\lim_{x \to x_0} x = x_0$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.17). This proves that $\lim_{x \to x_0} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

EXAMPLE 4 Finding Delta Algebraically

For the limit $\lim_{x\to 5}\sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x-5| < \delta \quad \Rightarrow \quad |\sqrt{x-1} - 2| < 1.$$

Solution We organize the search into two steps, as discussed below.

1. *Solve the inequality* $|\sqrt{x-1} - 2| < 1$ *to find an interval containing* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$\begin{aligned} |\sqrt{x-1} - 2| &< 1 \\ -1 < \sqrt{x-1} - 2 &< 1 \\ 1 < \sqrt{x-1} &< 3 \\ 1 < x - 1 &< 9 \\ 2 < x &< 10 \end{aligned}$$

The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well (see Figure 2.19).

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.18). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x-5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1} - 2| < 1$ (Figure 2.19)

$$0 < |x-5| < 3 \quad \Rightarrow \quad |\sqrt{x-1} - 2| < 1.$$

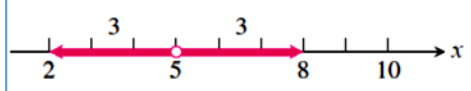


FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

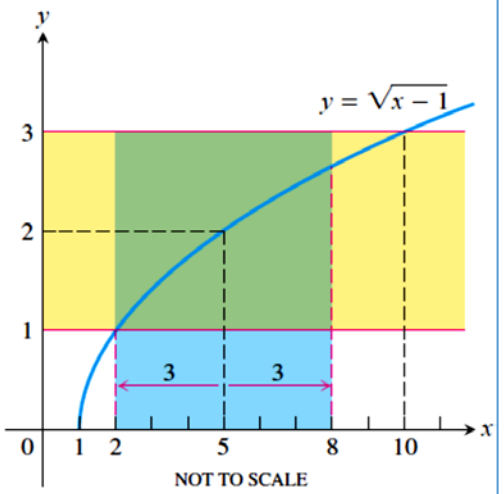


FIGURE 2.19 The function and intervals in Example 4.

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.

2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

EXAMPLE 5 Finding Delta Algebraically

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

1. *Solve the inequality $|f(x) - 4| < \epsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \neq x_0$.*

For $x \neq x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$\begin{aligned} |x^2 - 4| &< \epsilon \\ -\epsilon < x^2 - 4 &< \epsilon \\ 4 - \epsilon < x^2 &< 4 + \epsilon \\ \sqrt{4 - \epsilon} < |x| &< \sqrt{4 + \epsilon} \\ \sqrt{4 - \epsilon} < x &< \sqrt{4 + \epsilon}. \end{aligned}$$

Assumes $\epsilon < 4$; see below.

An open interval about $x_0 = 2$ that solves the inequality

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$ (Figure 2.20).

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$ (Figure 2.20).

2. *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$. In other words, take $\delta = \min\left\{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\right\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \qquad \Rightarrow \qquad |f(x) - 4| < \epsilon.$$

This completes the proof.

2.4 One-Sided Limits and Limits at Infinity

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

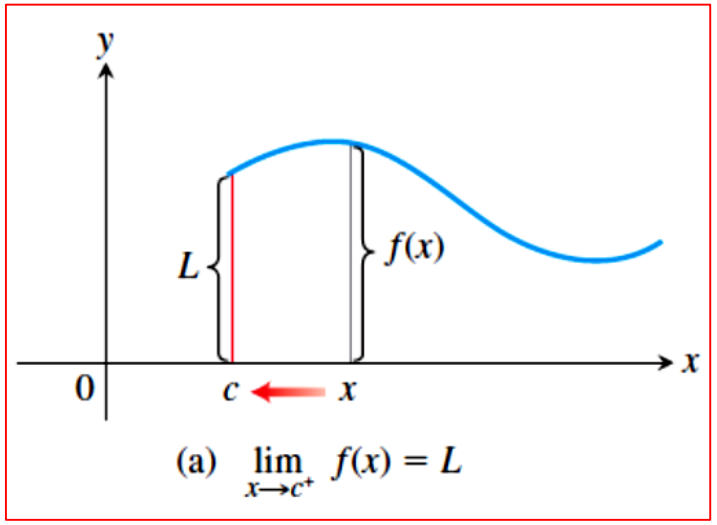
Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$



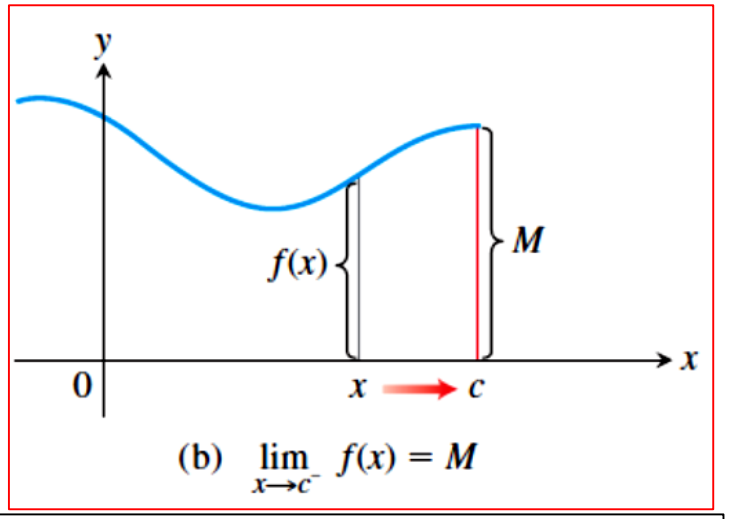


FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x

EXAMPLE 1 One-Sided Limits for a Semicircle

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.23. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

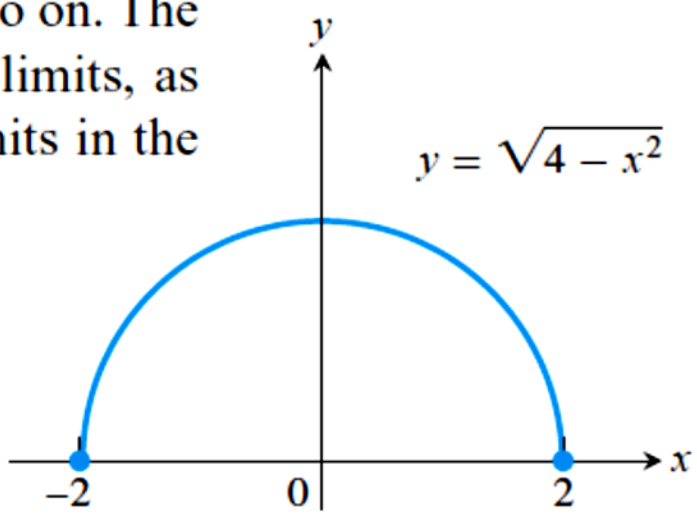


FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

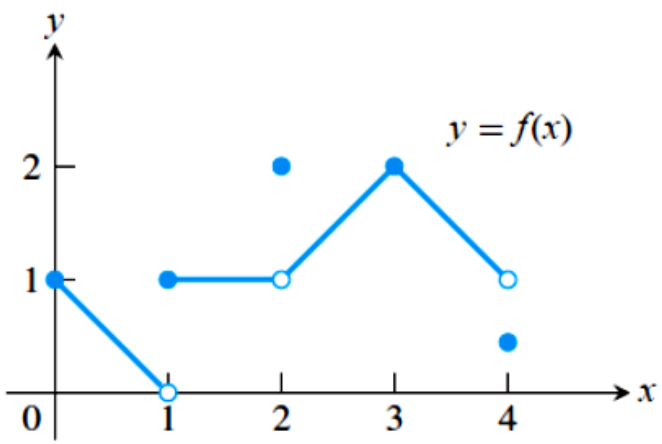


FIGURE 2.24 Graph of the function in Example 2.

EXAMPLE 2 Limits of the Function Graphed in Figure 2.24

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

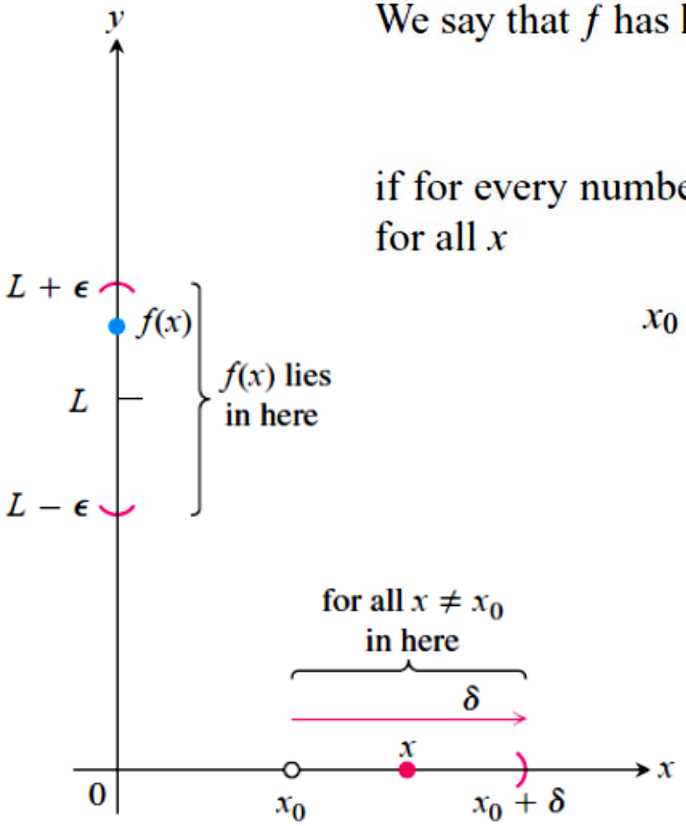


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

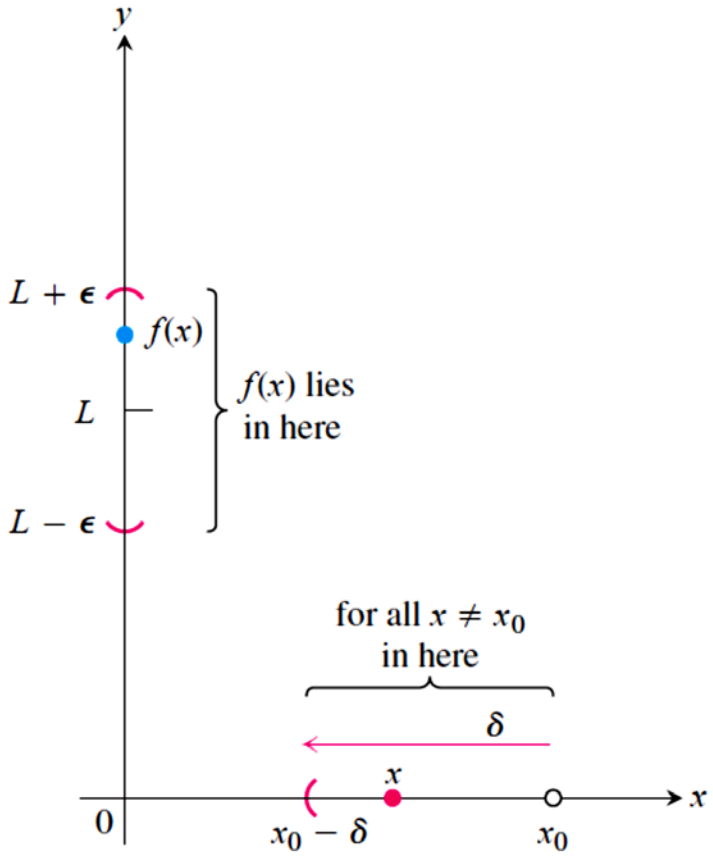


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

EXAMPLE 3 Applying the Definition to Find Delta

Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon.$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.27). ■

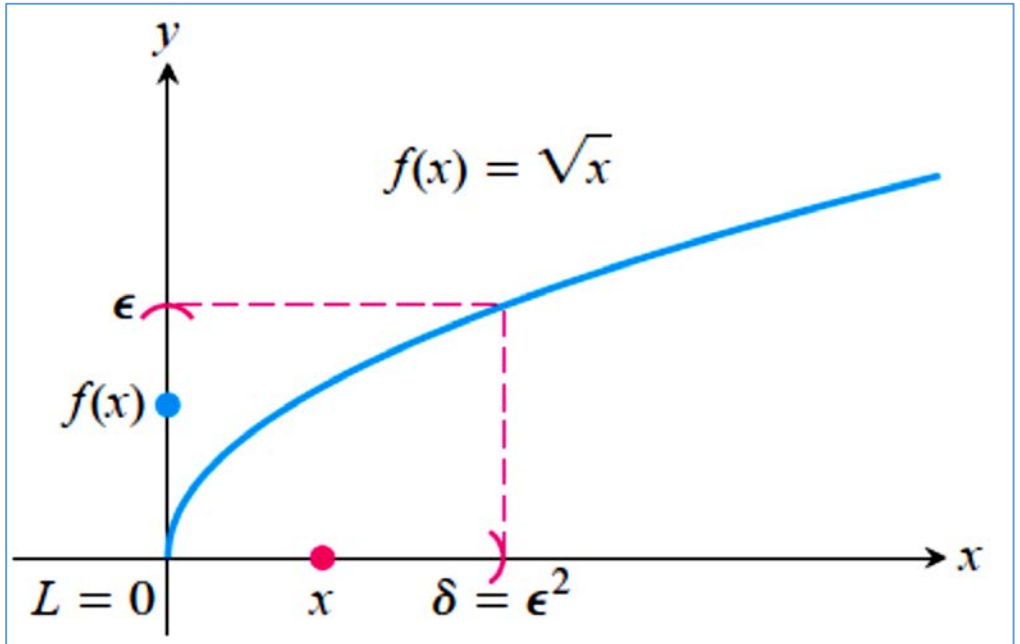


FIGURE 2.27 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.29 and confirm it algebraically using the Sandwich Theorem.

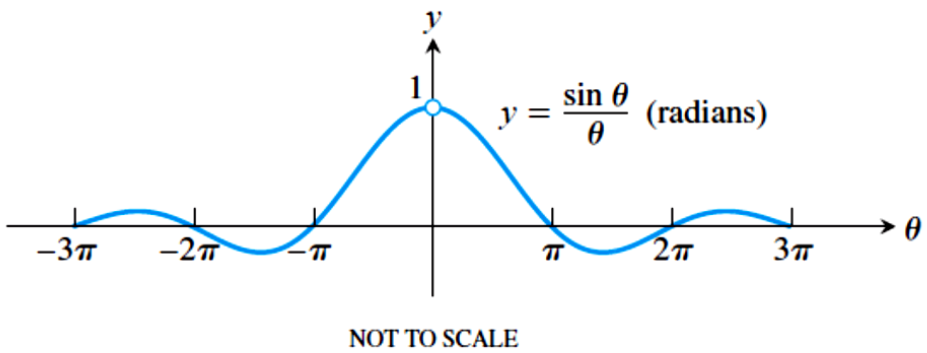


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

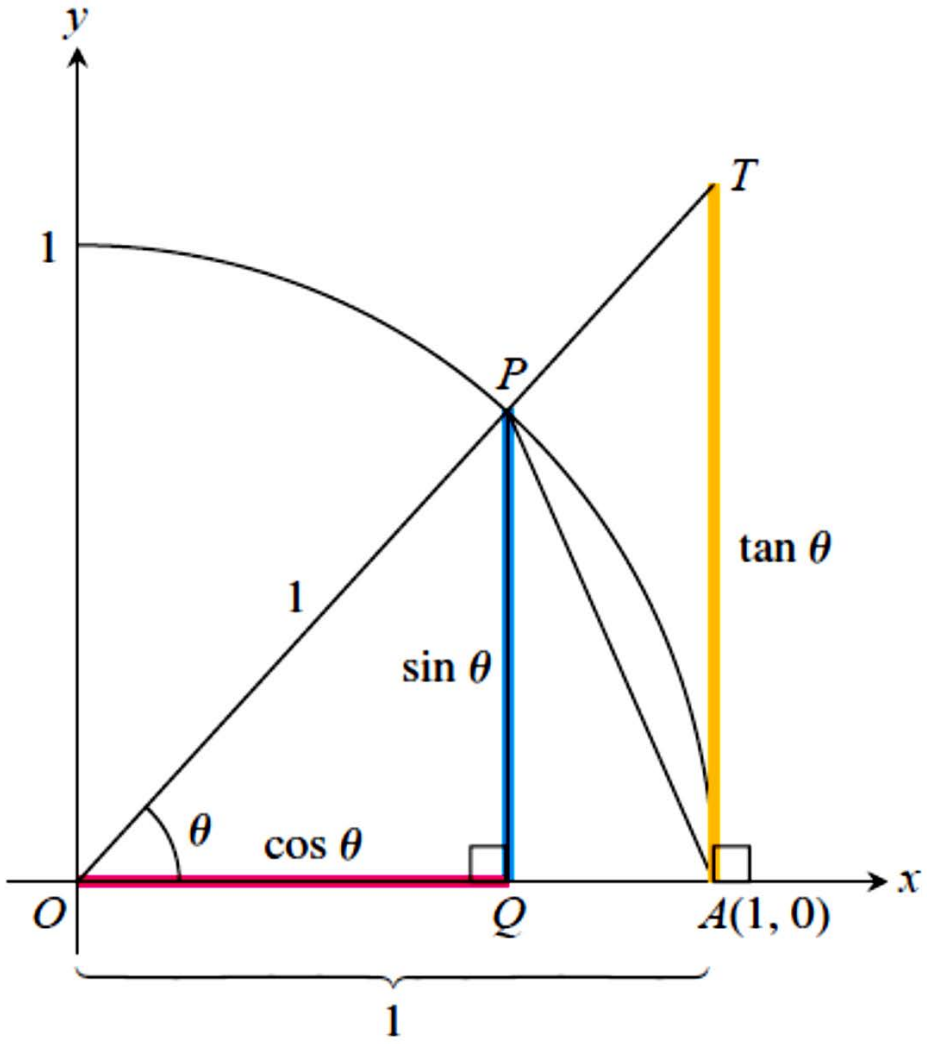


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

EXAMPLE 5 Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Show that **(a)** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \tag{4}$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \tag{5}$$

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned} \lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} \\ &= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta \qquad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0. \end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 2x}{5x} &= \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\ &= \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} \qquad \text{Now, Eq. (1) applies with } \theta = 2x. \\ &= \frac{2}{5}(1) = \frac{2}{5} \end{aligned}$$

■

DEFINITIONS Limit as x approaches ∞ or $-\infty$

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 7 Using Theorem 8

(a) $$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x}$$ Sum Rule

$$= 5 + 0 = 5$$ Known limits

(b) $$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$

$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}$$ Product rule

$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$$ Known limits

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

EXAMPLE 8 Numerator and Denominator of Same Degree

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$

Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

See Fig. 2.33. ■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$

Divide numerator and denominator by x^3.

$$= \frac{0 + 0}{2 - 0} = 0$$

See Fig. 2.34. ■

We give an example of the case when the degree of the numerator is greater than the degree of the denominator in the next section (Example 8, Section 2.5).

PRELIMINARIES

OVERVIEW This chapter reviews the basic ideas you need to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, and functions.

Real Numbers

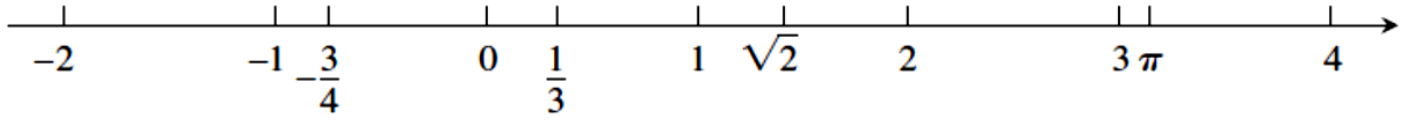
Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol $\mathbb{R}$ denotes either the real number system or, equivalently, the real line.

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

(a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

(b) eventually repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}$$

The bar indicates the block of repeating digits.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$. Since every decimal expansion represents a real number, it should be clear that there are infinitely many irrational numbers. Both rational and irrational numbers are found arbitrarily close to any point on the real line.

Set notation is very useful for specifying a particular subset of real numbers. A **set** is a collection of objects, and these objects are the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. If S and T are sets, then $S \cup T$ is their **union** and consists of all elements belonging either to S or T (or to both S and T). The **intersection** $S \cap T$ consists of all elements belonging to both S and T. The **empty set** $\emptyset$ is the set that contains no elements. For example, the intersection of the rational numbers and the irrational numbers is the empty set.

Some sets can be described by *listing* their elements in braces. For instance, the set A consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\}.$$

The entire set of integers is written as

$$\{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x \mid x \text{ is an integer and } 0 < x < 6\}$$

is the set of positive integers less than 6.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \leq x \leq 5$.

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x-1} \geq 5$

Solution

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \quad \text{Add 1 to both sides.}$$

$$x < 4 \quad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \quad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \quad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \quad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \quad \text{Divide by 7.}$$

The solution set is the open interval $(-3/7, \infty)$ (Figure 1.1b).

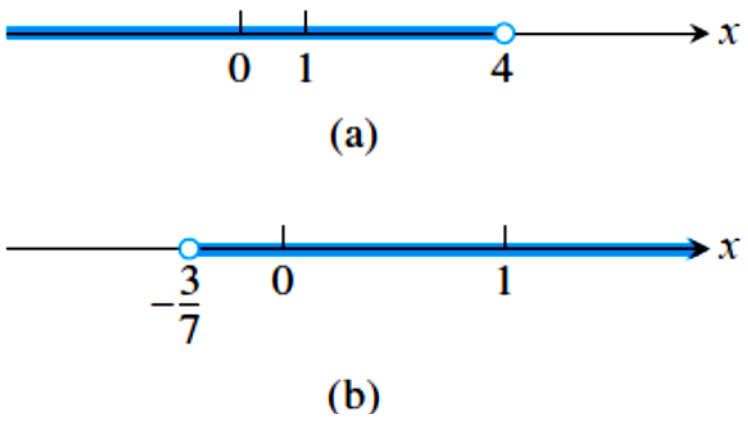


FIGURE 1.1 Solution sets for the inequalities in Example 1.

(c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval $(1, 11/5]$ (Figure 1.1c).

(c)

FIGURE 1.1 Solution sets for the inequalities in Example 1.

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

a. $0 < x < 4$

b. $0 < x - 2 < 4$

c. $1 < \frac{x}{2} < 3$

d. $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$

e. $1 < \frac{6}{x} < 3$

f. $|x - 4| < 2$

g. $-6 < -x < 2$

h. $-6 < -x < -2$

NT = necessarily true, NNT = Not necessarily true. Given: $2 < x < 6$.

a) NNT. 5 is a counter example.

b) NT. $2 < x < 6 \Rightarrow 2 - 2 < x - 2 < 6 - 2 \Rightarrow 0 < x - 2 < 4$

c) NT. $2 < x < 6 \Rightarrow 2/2 < x/2 < 6/2 \Rightarrow 1 < x < 3$.

d) NT. $2 < x < 6 \Rightarrow 1/2 > 1/x > 1/6 \Rightarrow 1/6 < 1/x < 1/2$.

e) NT. $2 < x < 6 \Rightarrow 1/2 > 1/x > 1/6 \Rightarrow 1/6 < 1/x < 1/2 \Rightarrow 6(1/6) < 6(1/x) < 6(1/2) \Rightarrow 1 < 6/x < 3$.

f) NT. $2 < x < 6 \Rightarrow x < 6 \Rightarrow (x - 4) < 2$ and $2 < x < 6 \Rightarrow x > 2 \Rightarrow -x < -2 \Rightarrow -x + 4 < 2 \Rightarrow -(x - 4) < 2$.
The pair of inequalities $(x - 4) < 2$ and $-(x - 4) < 2 \Rightarrow |x - 4| < 2$.

g) NT. $2 < x < 6 \Rightarrow -2 > -x > -6 \Rightarrow -6 < -x < -2$. But $-2 < 2$. So $-6 < -x < -2 < 2$ or $-6 < -x < 2$.

h) NT. $2 < x < 6 \Rightarrow -1(2) > -1(x) < -1(6) \Rightarrow -6 < -x < -2$

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
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Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

In Exercises 5– 8 solve the inequalities and show the solution sets on the real line.

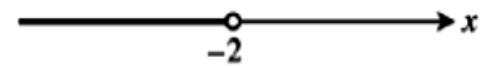
5. $-2x > 4$

6. $8 - 3x \geq 5$

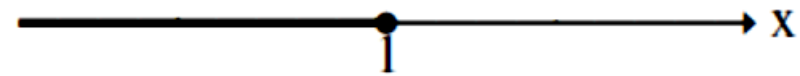
7. $5x - 3 \leq 7 - 3x$

8. $3(2 - x) > 2(3 + x)$

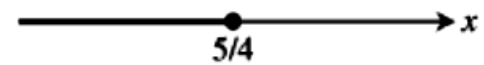
5. $-2x > 4 \Rightarrow x < -2$



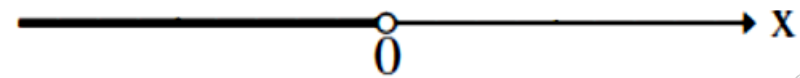
6. $8 - 3x \geq 5 \Rightarrow -3x \geq -3 \Rightarrow x \leq 1$



7. $5x - 3 \leq 7 - 3x \Rightarrow 8x \leq 10 \Rightarrow x \leq \frac{5}{4}$



8. $3(2 - x) > 2(3 + x) \Rightarrow 6 - 3x > 6 + 2x$
$\Rightarrow 0 > 5x \Rightarrow 0 > x$



Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

13. $y = 3$ or $y = -3$

14. $y - 3 = 7$ or $y - 3 = -7 \Rightarrow y = 10$ or $y = -4$

15. $2t + 5 = 4$ or $2t + 5 = -4 \Rightarrow 2t = -1$ or $2t = -9 \Rightarrow t = -\frac{1}{2}$ or $t = -\frac{9}{2}$

16. $1 - t = 1$ or $1 - t = -1 \Rightarrow -t = 0$ or $-t = -2 \Rightarrow t = 0$ or $t = 2$

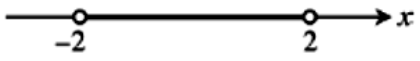
17. $8 - 3s = \frac{9}{2}$ or $8 - 3s = -\frac{9}{2} \Rightarrow -3s = -\frac{7}{2}$ or $-3s = -\frac{25}{2} \Rightarrow s = \frac{7}{6}$ or $s = \frac{25}{6}$

18. $\frac{s}{2} - 1 = 1$ or $\frac{s}{2} - 1 = -1 \Rightarrow \frac{s}{2} = 2$ or $\frac{s}{2} = 0 \Rightarrow s = 4$ or $s = 0$

Solve the inequalities in Exercises 19-24 expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

19. $\lvert x \rvert < 2$	**20.** $\lvert x \rvert \le 2$	**21.** $\lvert t - 1 \rvert \le 3$
22. $\lvert t + 2 \rvert < 1$	**23.** $\lvert 3y - 7 \rvert < 4$	**24.** $\lvert 2y + 5 \rvert < 1$

19. $-2 < x < 2$; solution interval $(-2, 2)$

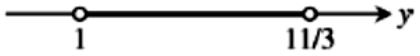


20. $-2 \le x \le 2$; solution interval $[-2, 2]$

21. $-3 \le t - 1 \le 3 \Rightarrow -2 \le t \le 4$; solution interval $[-2, 4]$

22. $-1 < t + 2 < 1 \Rightarrow -3 < t < -1$;
solution interval $(-3, -1)$

23. $-4 < 3y - 7 < 4 \Rightarrow 3 < 3y < 11 \Rightarrow 1 < y < \frac{11}{3}$;
solution interval $\left(1, \frac{11}{3}\right)$



24. $-1 < 2y + 5 < 1 \Rightarrow -6 < 2y < -4 \Rightarrow -3 < y < -2$;
solution interval $(-3, -2)$

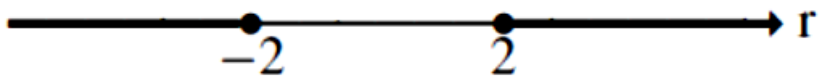
Quadratic Inequalities

Solve the inequalities in Exercises 35-37 Express the solution sets as intervals or unions of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$ as appropriate.

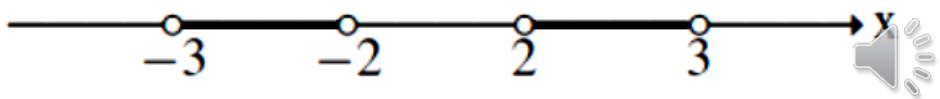
35. $x^2 < 2$ **36.** $4 \le x^2$ **37.** $4 < x^2 < 9$

35. $x^2 < 2 \Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$;
solution interval $\left(-\sqrt{2}, \sqrt{2}\right)$

36. $4 \le x^2 \Rightarrow 2 \le |x| \Rightarrow x \ge 2$ or $x \le -2$;
solution interval $(-\infty, -2] \cup [2, \infty)$



37. $4 < x^2 < 9 \Rightarrow 2 < |x| < 3 \Rightarrow 2 < x < 3$ or $2 < -x < 3$
$\Rightarrow 2 < x < 3$ or $-3 < x < -2$;
solution intervals $(-3, -2) \cup (2, 3)$



Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

a. $0 < x < 4$

b. $0 < x - 2 < 4$

c. $1 < \frac{x}{2} < 3$

d. $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$

e. $1 < \frac{6}{x} < 3$

f. $|x - 4| < 2$

g. $-6 < -x < 2$

h. $-6 < -x < -2$

NT = necessarily true, NNT = Not necessarily true. Given: $2 < x < 6$.

a) NNT. 5 is a counter example.

b) NT. $2 < x < 6 \Rightarrow 2 - 2 < x - 2 < 6 - 2 \Rightarrow 0 < x - 2 < 4$

c) NT. $2 < x < 6 \Rightarrow 2/2 < x/2 < 6/2 \Rightarrow 1 < x < 3$.

d) NT. $2 < x < 6 \Rightarrow 1/2 > 1/x > 1/6 \Rightarrow 1/6 < 1/x < 1/2$.

e) NT. $2 < x < 6 \Rightarrow 1/2 > 1/x > 1/6 \Rightarrow 1/6 < 1/x < 1/2 \Rightarrow 6(1/6) < 6(1/x) < 6(1/2) \Rightarrow 1 < 6/x < 3$.

f) NT. $2 < x < 6 \Rightarrow x < 6 \Rightarrow (x - 4) < 2$ and $2 < x < 6 \Rightarrow x > 2 \Rightarrow -x < -2 \Rightarrow -x + 4 < 2 \Rightarrow -(x - 4) < 2$.
The pair of inequalities $(x - 4) < 2$ and $-(x - 4) < 2 \Rightarrow |x - 4| < 2$.

g) NT. $2 < x < 6 \Rightarrow -2 > -x > -6 \Rightarrow -6 < -x < -2$. But $-2 < 2$. So $-6 < -x < -2 < 2$ or $-6 < -x < 2$.

h) NT. $2 < x < 6 \Rightarrow -1(2) > -1(x) < -1(6) \Rightarrow -6 < -x < -2$

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

In Exercises 5– 8 solve the inequalities and show the solution sets on the real line.

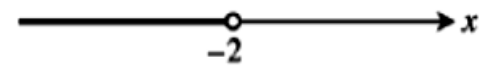
5. $-2x > 4$

6. $8 - 3x \geq 5$

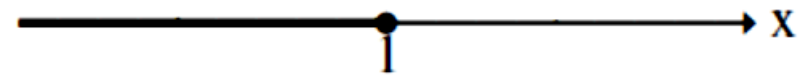
7. $5x - 3 \leq 7 - 3x$

8. $3(2 - x) > 2(3 + x)$

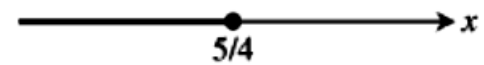
5. $-2x > 4 \Rightarrow x < -2$



6. $8 - 3x \geq 5 \Rightarrow -3x \geq -3 \Rightarrow x \leq 1$



7. $5x - 3 \leq 7 - 3x \Rightarrow 8x \leq 10 \Rightarrow x \leq \frac{5}{4}$



8. $3(2 - x) > 2(3 + x) \Rightarrow 6 - 3x > 6 + 2x$
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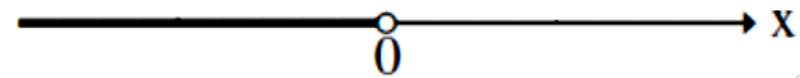

Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

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14. $y - 3 = 7$ or $y - 3 = -7 \Rightarrow y = 10$ or $y = -4$

15. $2t + 5 = 4$ or $2t + 5 = -4 \Rightarrow 2t = -1$ or $2t = -9 \Rightarrow t = -\frac{1}{2}$ or $t = -\frac{9}{2}$

16. $1 - t = 1$ or $1 - t = -1 \Rightarrow -t = 0$ or $-t = -2 \Rightarrow t = 0$ or $t = 2$

17. $8 - 3s = \frac{9}{2}$ or $8 - 3s = -\frac{9}{2} \Rightarrow -3s = -\frac{7}{2}$ or $-3s = -\frac{25}{2} \Rightarrow s = \frac{7}{6}$ or $s = \frac{25}{6}$

18. $\frac{s}{2} - 1 = 1$ or $\frac{s}{2} - 1 = -1 \Rightarrow \frac{s}{2} = 2$ or $\frac{s}{2} = 0 \Rightarrow s = 4$ or $s = 0$

Solve the inequalities in Exercises 19-24 expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

19. $\lvert x \rvert < 2$	**20.** $\lvert x \rvert \le 2$	**21.** $\lvert t - 1 \rvert \le 3$
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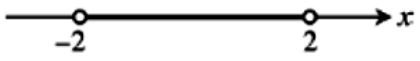

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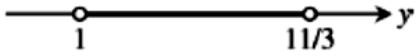
21. $-3 \le t - 1 \le 3 \Rightarrow -2 \le t \le 4$; solution interval $[-2, 4]$

22. $-1 < t + 2 < 1 \Rightarrow -3 < t < -1$;
solution interval $(-3, -1)$

23. $-4 < 3y - 7 < 4 \Rightarrow 3 < 3y < 11 \Rightarrow 1 < y < \frac{11}{3}$;
solution interval $\left(1, \frac{11}{3}\right)$



24. $-1 < 2y + 5 < 1 \Rightarrow -6 < 2y < -4 \Rightarrow -3 < y < -2$;
solution interval $(-3, -2)$

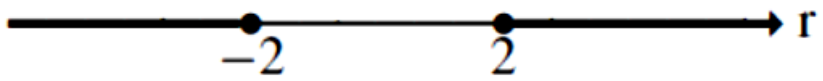
Quadratic Inequalities

Solve the inequalities in Exercises 35-37 Express the solution sets as intervals or unions of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$ as appropriate.

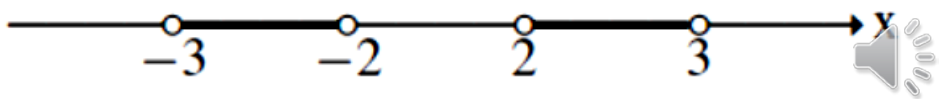
35. $x^2 < 2$ **36.** $4 \le x^2$ **37.** $4 < x^2 < 9$

35. $x^2 < 2 \Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$;
solution interval $\left(-\sqrt{2}, \sqrt{2}\right)$

36. $4 \le x^2 \Rightarrow 2 \le |x| \Rightarrow x \ge 2$ or $x \le -2$;
solution interval $(-\infty, -2] \cup [2, \infty)$



37. $4 < x^2 < 9 \Rightarrow 2 < |x| < 3 \Rightarrow 2 < x < 3$ or $2 < -x < 3$
$\Rightarrow 2 < x < 3$ or $-3 < x < -2$;
solution intervals $(-3, -2) \cup (2, 3)$



Increments and Distance

In Exercises 1–4, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B.

1. $A(-3, 2)$, $B(-1, -2)$

2. $A(-1, -2)$, $B(-3, 2)$

3. $A(-3.2, -2)$, $B(-8.1, -2)$

4. $A(\sqrt{2}, 4)$, $B(0, 1.5)$

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

1. $\Delta x = -1 - (-3) = 2$, $\Delta y = -2 - 2 = -4$; $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{4 + 16} = 2\sqrt{5}$

2. $\Delta x = -3 - (-1) = -2$, $\Delta y = 2 - (-2) = 4$; $d = \sqrt{(-2)^2 + 4^2} = 2\sqrt{5}$

3. $\Delta x = -8.1 - (-3.2) = -4.9$, $\Delta y = -2 - (-2) = 0$; $d = \sqrt{(-4.9)^2 + 0^2} = 4.9$

4. $\Delta x = 0 - \sqrt{2} = -\sqrt{2}$, $\Delta y = 1.5 - 4 = -2.5$; $d = \sqrt{\left(-\sqrt{2}\right)^2 + (-2.5)^2} = \sqrt{8.25}$

Slopes, Lines, and Intercepts

Plot the points in Exercises 9–12 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB.

9. $A(-1, 2)$, $B(-2, -1)$ **10.** $A(-2, 1)$, $B(2, -2)$

DEFINITION Slope

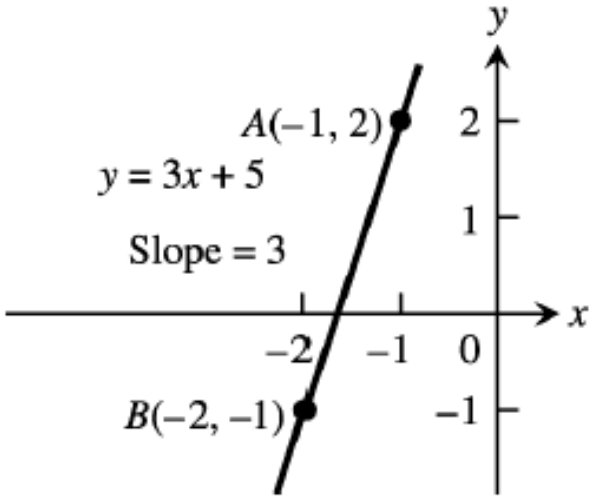
The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2.

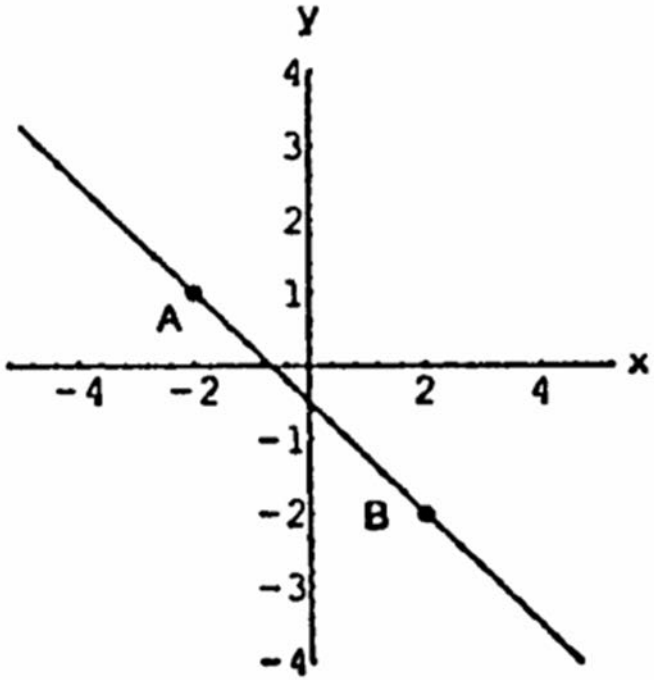
9. $m = \frac{\Delta y}{\Delta x} = \frac{-1-2}{-2-(-1)} = 3$

perpendicular slope $= -\frac{1}{3}$



10. $m = \frac{\Delta y}{\Delta x} = \frac{-2-1}{2-(-2)} = -\frac{3}{4}$

perpendicular slope $= \frac{4}{3}$

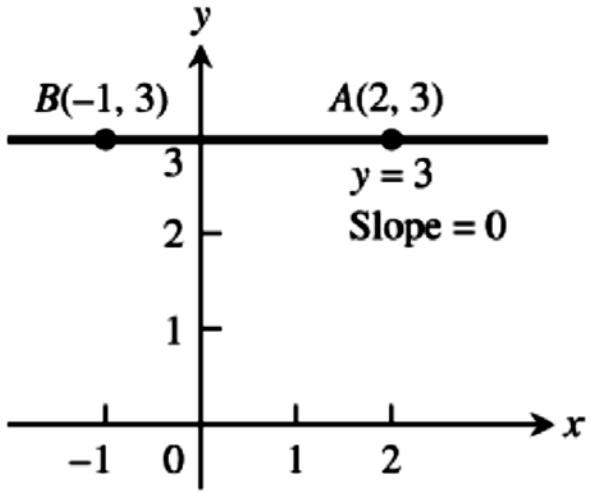


11. $A(2, 3)$, $B(-1, 3)$

12. $A(-2, 0)$, $B(-2, -2)$

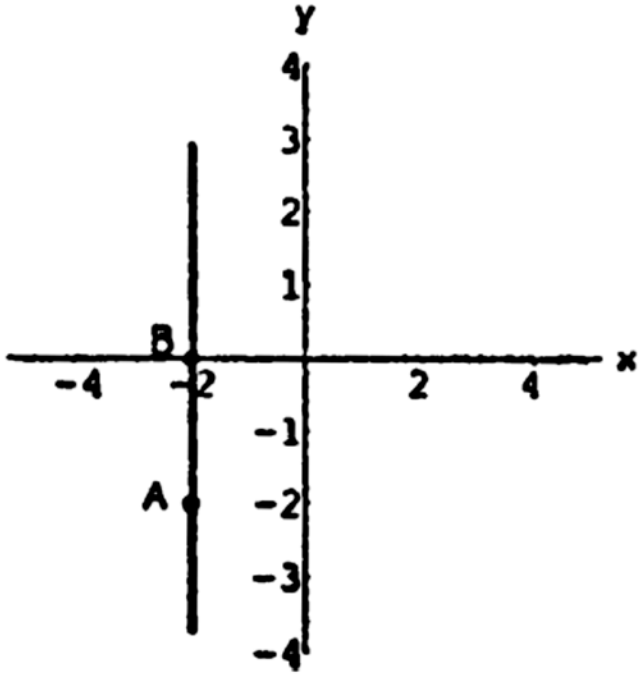
11. $m = \frac{\Delta y}{\Delta x} = \frac{3-3}{-1-2} = 0$

perpendicular slope does not exist



12. $m = \frac{\Delta y}{\Delta x} = \frac{-2-0}{-2-(-2)}$; no slope

perpendicular slope = 0



In Exercises **17-20** write an equation for each line described.

17. Passes through $(-1, 1)$ with slope -1

18. Passes through $(2, -3)$ with slope $1/2$

19. Passes through $(3, 4)$ and $(-2, 5)$

20. Passes through $(-8, 0)$ and $(-1, 3)$

DEFINITION Slope

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line $P_1 P_2$.

17. $P(-1, 1)$, $m = -1 \Rightarrow y - 1 = -1(x - (-1)) \Rightarrow y = -x$

18. $P(2, -3)$, $m = \frac{1}{2} \Rightarrow y - (-3) = \frac{1}{2}(x - 2) \Rightarrow y = \frac{1}{2}x - 4$

19. $P(3, 4)$, $Q(-2, 5) \Rightarrow m = \frac{\Delta y}{\Delta x} = \frac{5-4}{-2-3} = -\frac{1}{5} \Rightarrow y - 4 = -\frac{1}{5}(x - 3) \Rightarrow y = -\frac{1}{5}x + \frac{23}{5}$

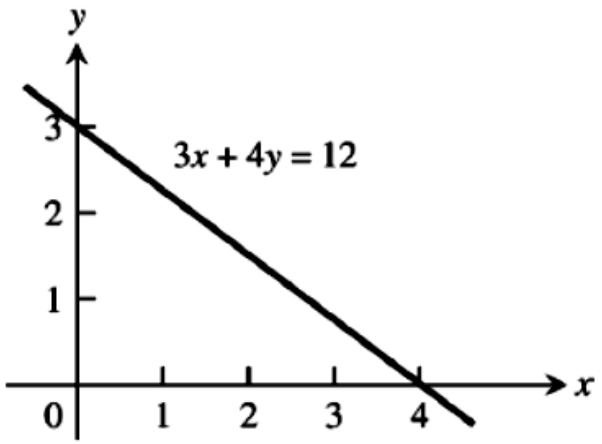
20. $P(-8, 0)$, $Q(-1, 3) \Rightarrow m = \frac{\Delta y}{\Delta x} = \frac{3-0}{-1-(-8)} = \frac{3}{7} \Rightarrow y - 0 = \frac{3}{7}(x - (-8)) \Rightarrow y = \frac{3}{7}x + \frac{24}{7}$

In Exercises **31-32**, find the line's x- and y-intercepts and use this information to graph the line.

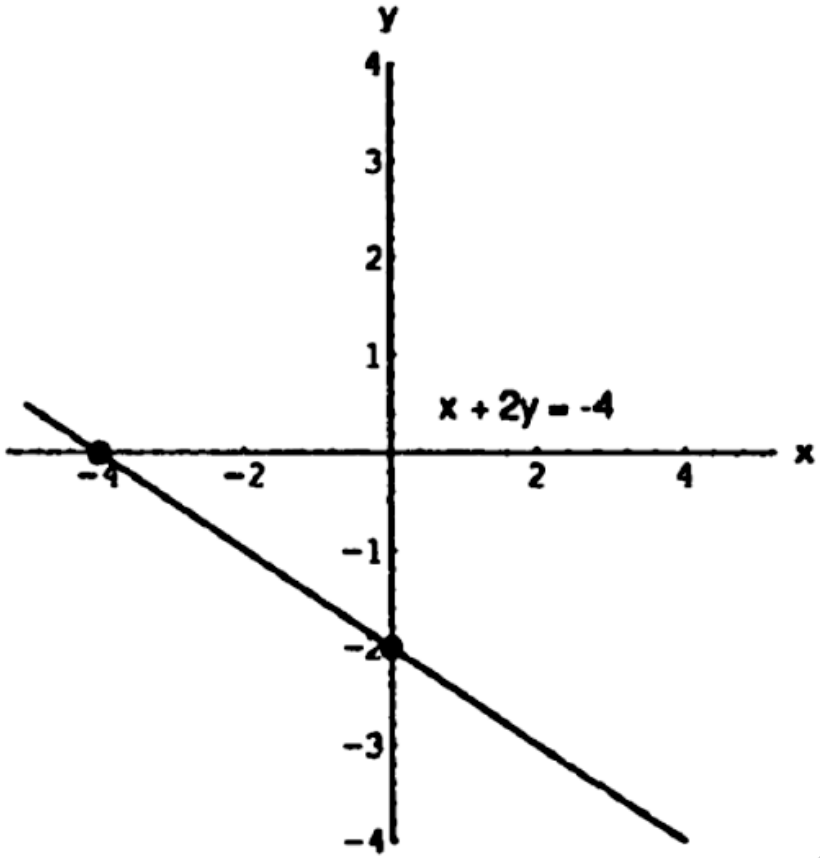
31. $3x + 4y = 12$

32. $x + 2y = -4$

31. x-intercept = 4, y-intercept = 3



32. x-intercept = −4, y-intercept = −2



Parabolas

Graph the parabolas in Exercises 53–60. Label the vertex, axis, and intercepts in each case.

53. $y = x^2 - 2x - 3$ **54.** $y = x^2 + 4x + 3$

The Graph of $y = ax^2 + bx + c$, $a \neq 0$

$$x = -\frac{b}{2a}.$$

53. $x = -\frac{b}{2a} = -\frac{-2}{2(1)} = 1$

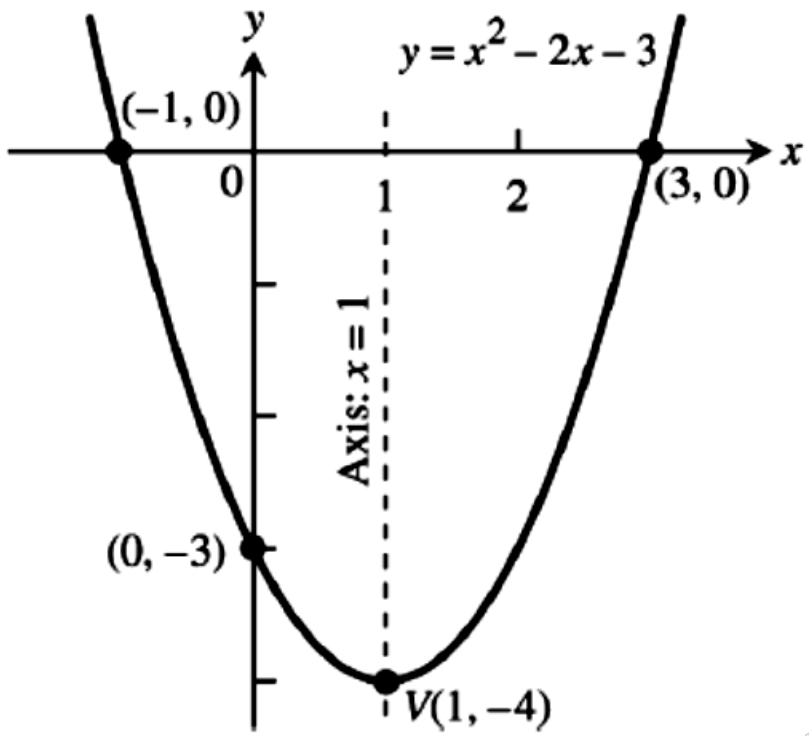
$\Rightarrow y = (1)^2 - 2(1) - 3 = -4$

$\Rightarrow V = (1, -4)$. If $x = 0$ then $y = -3$.

Also, $y = 0 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = 3$ or

$x = -1$. Axis of parabola is $x = 1$.



Parabolas

Graph the parabolas in Exercises 53–60. Label the vertex, axis, and intercepts in each case.

53. $y = x^2 - 2x - 3$ **54.** $y = x^2 + 4x + 3$

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$

$$x = -\frac{b}{2a}.$$

54. $x = -\frac{b}{2a} = -\frac{4}{2(1)} = -2$

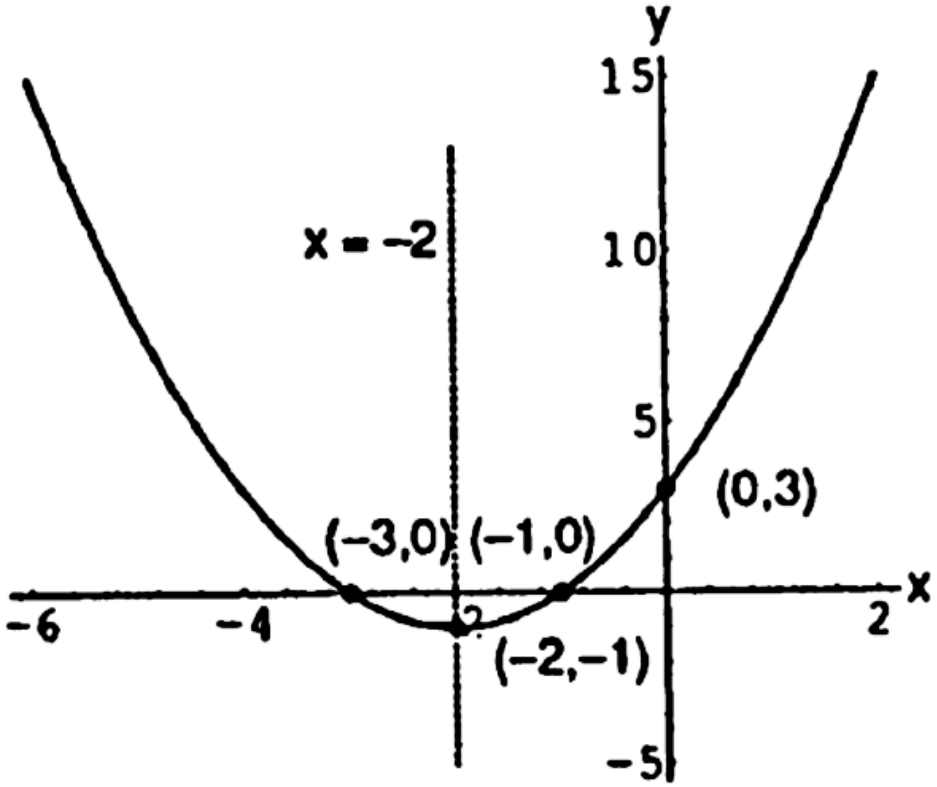
$\Rightarrow y = (-2)^2 + 4(-2) + 3 = -1$

$\Rightarrow V = (-2, -1)$. If $x = 0$ then $y = 3$.

Also, $y = 0 \Rightarrow x^2 + 4x + 3 = 0$

$\Rightarrow (x + 1)(x + 3) = 0 \Rightarrow x = -1$ or

$x = -3$. Axis of parabola is $x = -2$.



Recognizing Functions

In Exercises 1–4, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

1. a. $f(x) = 7 - 3x$ **b.** $g(x) = \sqrt[5]{x}$

c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ **d.** $r(x) = 8^x$

2. a. $F(t) = t^4 - t$ **b.** $G(t) = 5^t$

c. $H(z) = \sqrt{z^3 + 1}$ **d.** $R(z) = \sqrt[3]{z^7}$

Constant functions result when the slope $m = 0$

$f(x) = mx + b$, for constants m and b,

$f(x) = x^a$, where a is a constant

$f(x) = x^n$, for $n = 1, 2, 3, 4, 5$

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$p(x) = ax^2 + bx + c$, are called **quadratic functions.**

$p(x) = ax^3 + bx^2 + cx + d$ of degree 3

$f(x) = \dfrac{p(x)}{q(x)}$

Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking

$f(x) = a^x$ $f(x) = \log_a x$

1. (a) linear, polynomial of degree 1, algebraic. (b) power, algebraic.
   (c) rational, algebraic. (d) exponential.

2. (a) polynomial of degree 4, algebraic. (b) exponential.
   (c) algebraic. (d) power, algebraic.

Constant functions result when the slope $m = 0$

$f(x) = mx + b$, for constants m and b,

$f(x) = x^a$, where a is a constant

$f(x) = x^n$, for $n = 1, 2, 3, 4, 5$

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$p(x) = ax^2 + bx + c$, are called **quadratic functions**.

$p(x) = ax^3 + bx^2 + cx + d$ of degree 3

$f(x) = \dfrac{p(x)}{q(x)}$

Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking

$f(x) = a^x$ $\qquad$ $f(x) = \log_a x$

3. a. $y = \dfrac{3 + 2x}{x - 1}$ $\qquad$ **b.** $y = x^{5/2} - 2x + 1$

c. $y = \tan \pi x$ $\qquad$ **d.** $y = \log_7 x$

4. a. $y = \log_5 \left(\dfrac{1}{t}\right)$ $\qquad$ **b.** $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$

c. $g(x) = 2^{1/x}$ $\qquad$ **d.** $w = 5 \cos \left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$

3. (a) rational, algebraic. (b) algebraic.
   (c) trigonometric. (d) logarithmic.

4. (a) logarithmic. (b) algebraic.
   (c) exponential. (d) trigonometric.

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

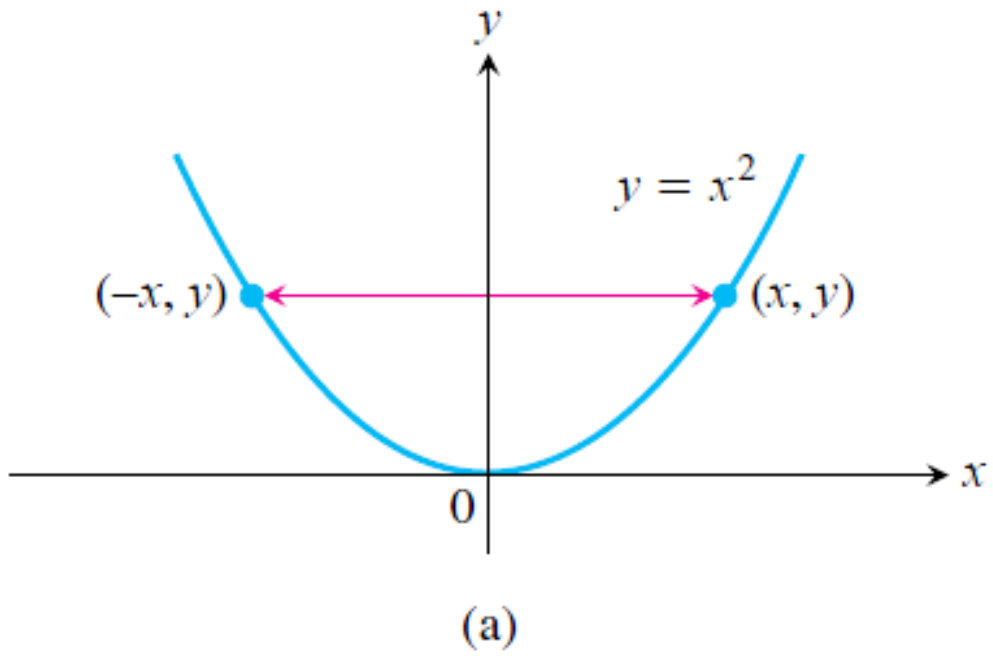
DEFINITIONS **Even Function, Odd Function**

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.



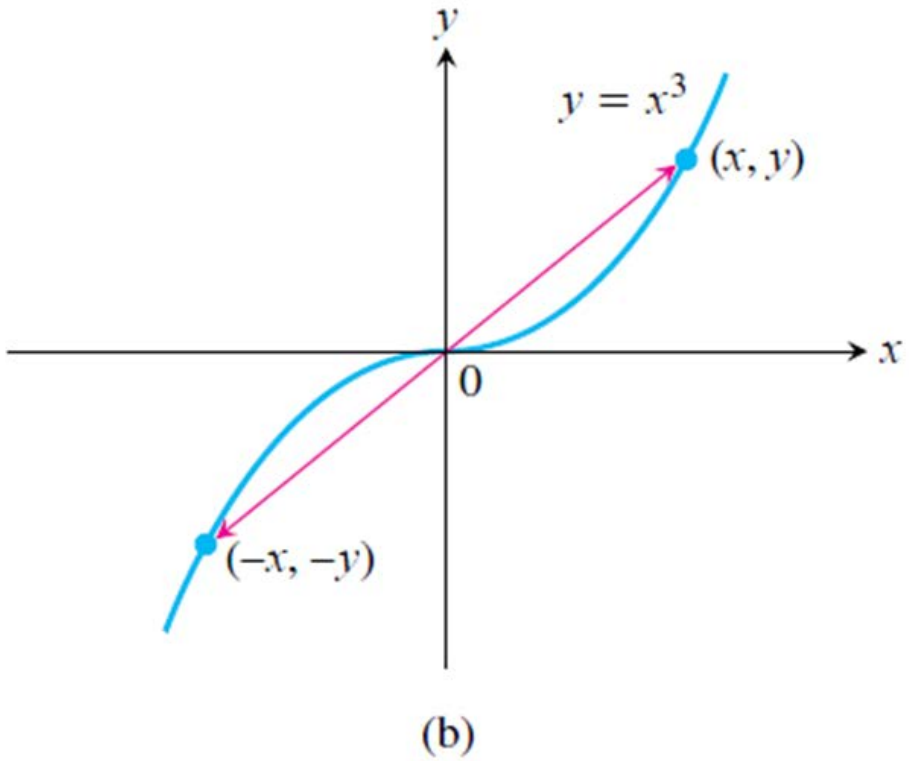


FIGURE 1.46 In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

EXAMPLE 2 Recognizing Even and Odd Functions

$f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.47a).

$f(x) = x$ Odd function: $(-x) = -x$ for all x; symmetry about the origin.

$f(x) = x + 1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.47b).

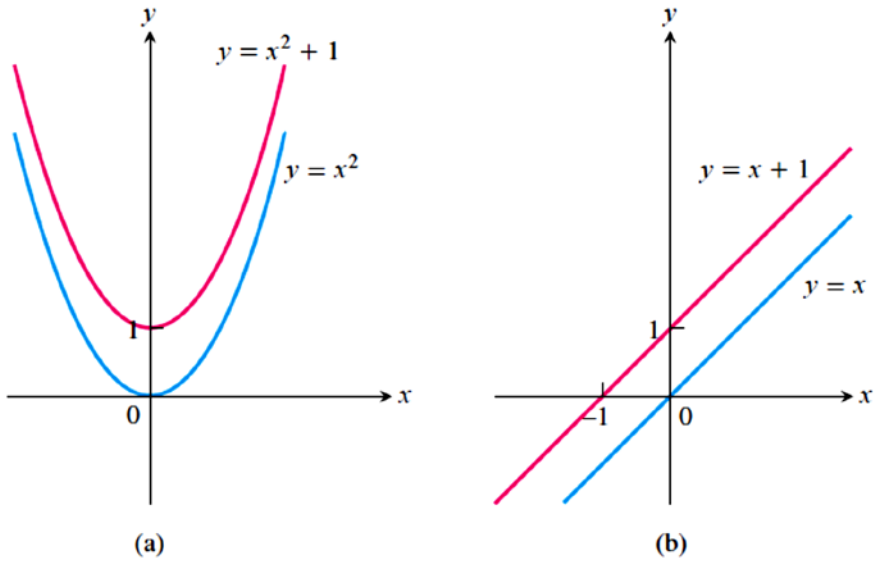
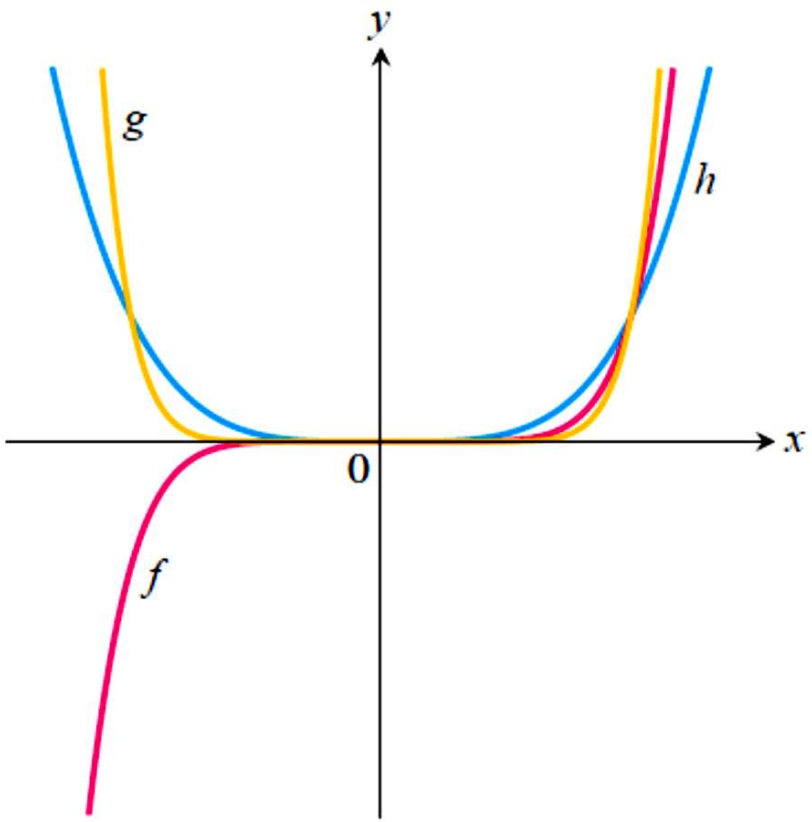


FIGURE 1.47 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 2).

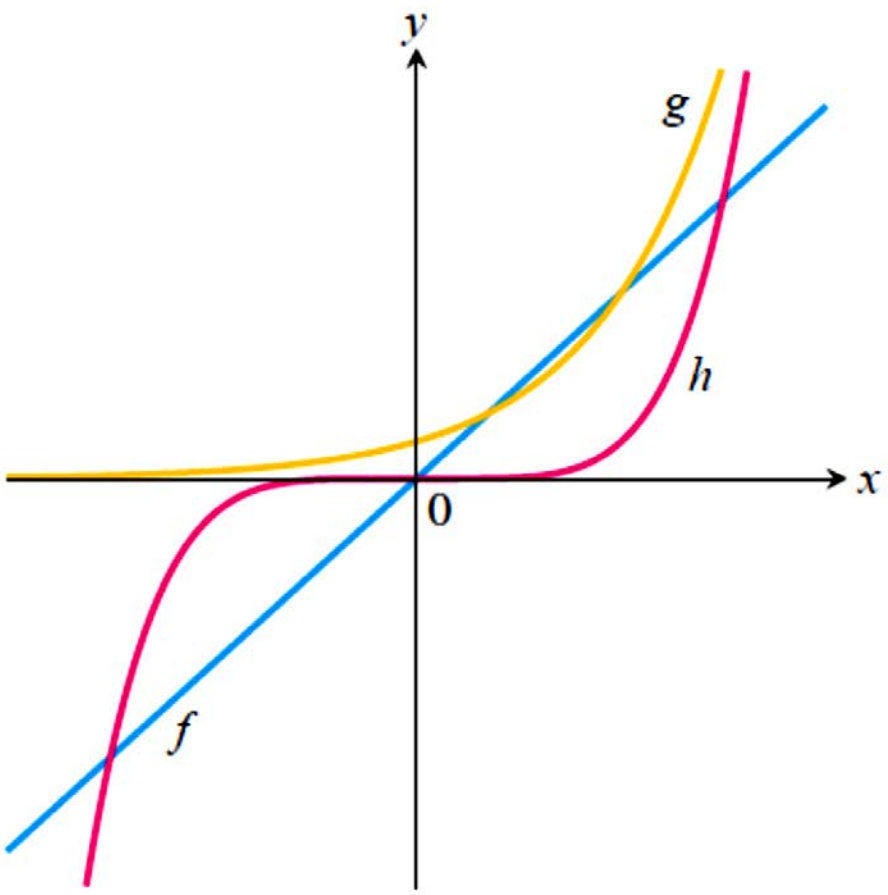
In Exercises 5 and 6, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

5. a. $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



5. (a) Graph h because it is an even function and rises less rapidly than does Graph g.
   (b) Graph f because it is an odd function.
   (c) Graph g because it is an even function and rises more rapidly than does Graph h.

6. a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



6. (a) Graph f because it is linear.

(b) Graph g because it contains $(0, 1)$.

(c) Graph h because it is a nonlinear odd function.

Even and Odd Functions

In Exercises 19-24, say whether the function is even, odd, or neither. Give reasons for your answer.

19. $f(x) = 3$

20. $f(x) = x^{-5}$

21. $f(x) = x^2 + 1$

22. $f(x) = x^2 + x$

23. $g(x) = x^3 + x$

24. $g(x) = x^4 + 3x^2 - 1$

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

19. Since a horizontal line not through the origin is symmetric with respect to the y-axis, but not with respect to the origin, the function is even.

20. $f(x) = x^{-5} = \frac{1}{x^5}$ and $f(-x) = (-x)^{-5} = \frac{1}{(-x)^5} = -\left(\frac{1}{x^5}\right) = -f(x)$. Thus the function is odd.

21. Since $f(x) = x^2 + 1 = (-x)^2 + 1 = -f(x)$. The function **odd** .

22. Since $[f(x) = x^2 + x] \neq [f(-x) = (-x)^2 - x]$ and $[f(x) = x^2 + x] \neq [-f(x) = -(x)^2 - x]$ the function is neither even nor odd.

23. Since $g(x) = x^3 + x$, $g(-x) = -x^3 - x = -(x^3 + x) = -g(x)$. So the function is odd.

24. $g(x) = x^4 + 3x^2 + 1 = (-x)^4 + 3(-x)^2 - 1 = g(-x)$, thus the function is even.

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$ **b.** $g(f(0))$

c. $f(g(x))$ **d.** $g(f(x))$

e. $f(f(-5))$ **f.** $g(g(2))$

g. $f(f(x))$ **h.** $g(g(x))$

5. (a) $f(g(0)) = f(-3) = 2$

(b) $g(f(0)) = g(5) = 22$

(c) $f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2$

(d) $g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 22$

(e) $f(f(-5)) = f(0) = 5$

(f) $g(g(2)) = g(1) = -2$

(g) $f(f(x)) = f(x + 5) = (x + 5) + 5 = x + 10$

(h) $g(g(x)) = g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$

7. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a. $u(v(f(x)))$ **b.** $u(f(v(x)))$

c. $v(u(f(x)))$ **d.** $v(f(u(x)))$

e. $f(u(v(x)))$ **f.** $f(v(u(x)))$

7. (a) $u(v(f(x))) = u\left(v\left(\frac{1}{x}\right)\right) = u\left(\frac{1}{x^2}\right) = 4\left(\frac{1}{x}\right)^2 - 5 = \frac{4}{x^2} - 5$

(b) $u(f(v(x))) = u\left(f\left(x^2\right)\right) = u\left(\frac{1}{x^2}\right) = 4\left(\frac{1}{x^2}\right) - 5 = \frac{4}{x^2} - 5$

(c) $v(u(f(x))) = v\left(u\left(\frac{1}{x}\right)\right) = v\left(4\left(\frac{1}{x}\right) - 5\right) = \left(\frac{4}{x} - 5\right)^2$

(d) $v(f(u(x))) = v(f(4x - 5)) = v\left(\frac{1}{4x-5}\right) = \left(\frac{1}{4x-5}\right)^2$

(e) $f(u(v(x))) = f\left(u\left(x^2\right)\right) = f\left(4\left(x^2\right) - 5\right) = \frac{1}{4x^2-5}$

(f) $f(v(u(x))) = f(v(4x - 5)) = f\left((4x-5)^2\right) = \frac{1}{(4x-5)^2}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 9 and 10 as a composite involving one or more of f, g, h, and j.

9. **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$

c. $y = x^{1/4}$ **d.** $y = 4x$

e. $y = \sqrt{(x-3)^3}$ **f.** $y = (2x-6)^3$

10. **a.** $y = 2x - 3$ **b.** $y = x^{3/2}$

c. $y = x^9$ **d.** $y = x - 6$

e. $y = 2\sqrt{x-3}$ **f.** $y = \sqrt{x^3 - 3}$

9. (a) $y = f(g(x))$ (b) $y = j(g(x))$

(c) $y = g(g(x))$ (d) $y = j(j(x))$

(e) $y = g(h(f(x)))$ (f) $y = h(j(f(x)))$

10. (a) $y = f(j(x))$ (b) $y = h(g(x)) = g(h(x))$

(c) $y = h(h(x))$ (d) $y = f(f(x))$

(e) $y = j(g(f(x)))$ (f) $y = g(f(h(x)))$

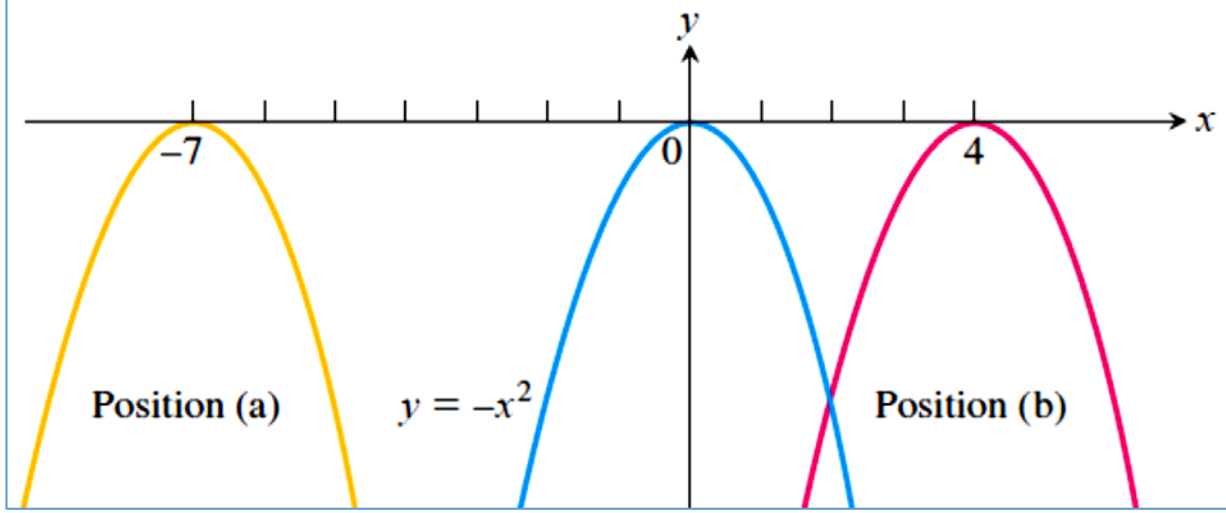
11. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	
b.	$x + 2$	$3x$	
c.		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d.	$\frac{x}{x - 1}$	$\frac{x}{x - 1}$	
e.		$1 + \frac{1}{x}$	x
f.	$\frac{1}{x}$		x

11.

	g(x)	f(x)	(f ∘ g)(x)
(a)	$x - 7$	$\sqrt{x}$	$\sqrt{x - 7}$
(b)	$x + 2$	$3x$	$3(x + 2) = 3x + 6$
(c)	x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(d)	$\frac{x}{x-1}$	$\frac{x}{x-1}$	$\frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x-(x-1)} = x$
(e)	$\frac{1}{x-1}$	$1 + \frac{1}{x}$	x
(f)	$\frac{1}{x}$	$\frac{1}{x}$	x

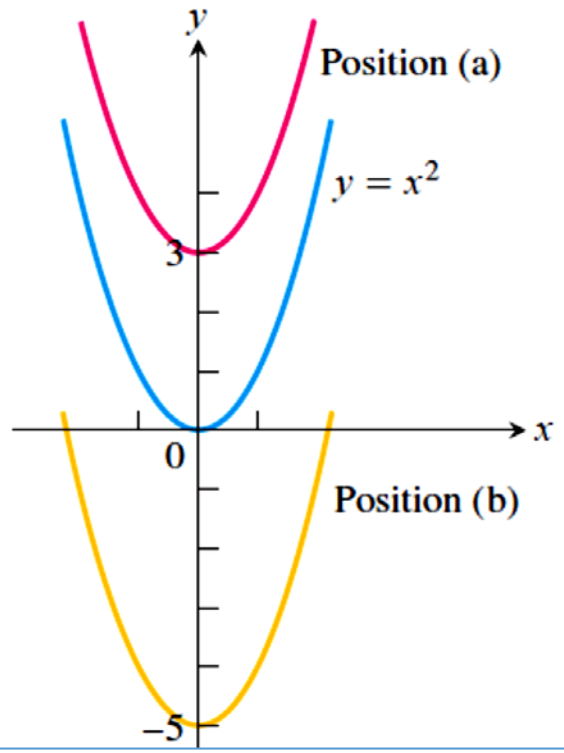
The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



(a) $y = -(x + 7)^2$

(b) $y = -(x - 4)^2$

The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



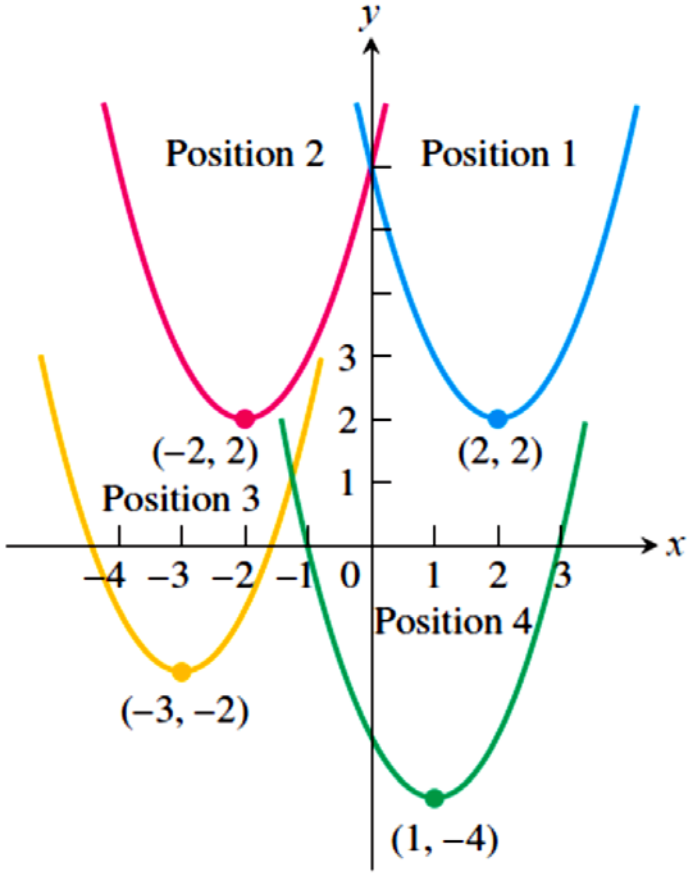
(a) $y = x^2 + 3$

(b) $y = x^2 - 5$

Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

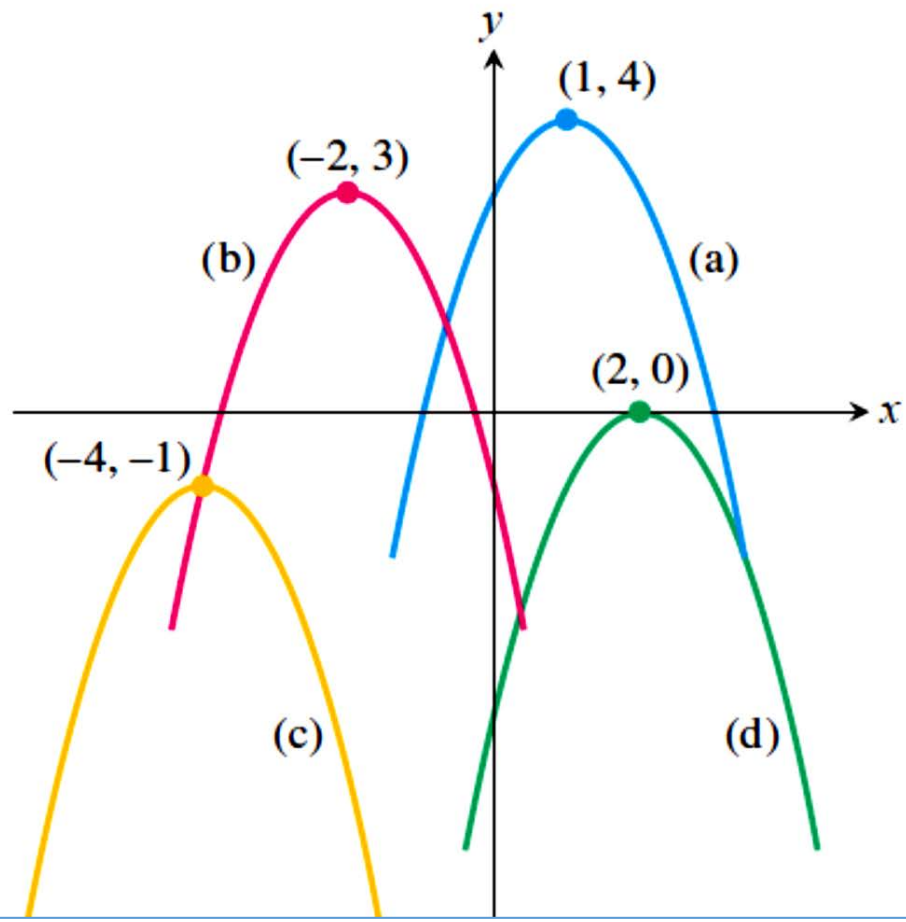
a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$



(a) Position 4 (b) Position 1 (c) Position 2 (d) Position 3

The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



(a) $y = -(x - 1)^2 + 4$ (b) $y = -(x + 2)^2 + 3$ (c) $y = -(x + 4)^2 - 1$ (d) $y = -(x - 2)^2$

Limits by Substitution

In Exercises 21–28, find the limits by substitution. *Support your answers with a computer or calculator if available.*

21. $\lim_{x \to 2} 2x$

22. $\lim_{x \to 0} 2x$

23. $\lim_{x \to 1/3} (3x - 1)$

24. $\lim_{x \to 1} \frac{-1}{(3x - 1)}$

25. $\lim_{x \to -1} 3x(2x - 1)$

26. $\lim_{x \to -1} \frac{3x^2}{2x - 1}$

27. $\lim_{x \to \pi/2} x \sin x$

28. $\lim_{x \to \pi} \frac{\cos x}{1 - \pi}$

21. $\lim_{x \to 2} 2x = 2(2) = 4$

22. $\lim_{x \to 0} 2x = 2(0) = 0$

23. $\lim_{x \to \frac{1}{3}} (3x - 1) = 3\left(\frac{1}{3}\right) - 1 = 0$

24. $\lim_{x \to 1} \frac{-1}{3x-1} = \frac{-1}{3(1)-1} = -\frac{1}{2}$

26. $\lim_{x \to -1} \frac{3x^2}{2x-1} = \frac{3(-1)^2}{2(-1)-1} = \frac{3}{-3} = -1$

25. $\lim_{x \to -1} 3x(2x - 1) = 3(-1)(2(-1) - 1) = 9$

28. $\lim_{x \to \pi} \frac{\cos x}{1-\pi} = \frac{\cos \pi}{1-\pi} = \frac{-1}{1-\pi} = \frac{1}{\pi-1}$

27. $\lim_{x \to \frac{\pi}{2}} x \sin x = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}$

2.2 CALCULATING LIMITS USING THE LIMIT LAWS

1. $\lim_{x \to -7} (2x + 5) = 2(-7) + 5 = -14 + 5 = -9$

2. $\lim_{x \to 12} (10 - 3x) = 10 - 3(12) = 10 - 36 = -26$

3. $\lim_{x \to 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$

4. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

5. $\lim_{t \to 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$

6. $\lim_{s \to \frac{2}{3}} 3s(2s - 1) = 3\left(\frac{2}{3}\right)\left[2\left(\frac{2}{3}\right) - 1\right] = 2\left(\frac{4}{3} - 1\right) = \frac{2}{3}$

7. $\lim_{x \to 2} \frac{x+3}{x+6} = \frac{2+3}{2+6} = \frac{5}{8}$

8. $\lim_{x \to 5} \frac{4}{x-7} = \frac{4}{5-7} = \frac{4}{-2} = -2$

$$\cdot \lim_{h \to 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h \to 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \lim_{h \to 0} \frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h \to 0} \frac{5}{\sqrt{5h+4}+2}$$

$$= \frac{5}{\sqrt{4}+2} = \frac{5}{4}$$

$$\cdot \lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$$

$$\cdot \lim_{x \to -3} \frac{x+3}{x^2+4x+3} = \lim_{x \to -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x \to -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$$

$$\cdot \lim_{x \to -5} \frac{x^2+3x-10}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \to -5} (x-2) = -5-2 = -7$$

$$\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16} = \lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v \to 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$$

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$$\lim_{x \to 4} \frac{4x - x^2}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x \to 4} x\left(2+\sqrt{x}\right) = 4(2+2) = 16$$

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)} = \lim_{x \to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{(x+3)-4} = \lim_{x \to 1} \left(\sqrt{x+3}+2\right)$$
$$= \sqrt{4}+2 = 4$$

. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{2/3}} = \frac{\lim_{x\to 0} (2f(x) - g(x))}{\lim_{x\to 0} (f(x) + 7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} \left(f(x) + 7\right)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2 \lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1) - (-5)}{(1 + 7)^{2/3}} = \frac{7}{4}$$

(a) quotient rule

(b) difference and power rules

(c) sum and constant multiple rules

. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim\limits_{x\to c} f(x)g(x)$ **b.** $\lim\limits_{x\to c} 2f(x)g(x)$

c. $\lim\limits_{x\to c} (f(x) + 3g(x))$ **d.** $\lim\limits_{x\to c} \dfrac{f(x)}{f(x) - g(x)}$

(a) $\lim\limits_{x\to c} f(x)\,g(x) = \left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right] = (5)(-2) = -10$

(b) $\lim\limits_{x\to c} 2f(x)\,g(x) = 2\left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right] = 2(5)(-2) = -20$

(c) $\lim\limits_{x\to c} [f(x) + 3g(x)] = \lim\limits_{x\to c} f(x) + 3\lim\limits_{x\to c} g(x) = 5 + 3(-2) = -1$

(d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x) - g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)} = \dfrac{5}{5-(-2)} = \dfrac{5}{7}$

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Using the Sandwich Theorem

1- If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x \to 0} f(x)$.

2- If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x \to 0} g(x)$.

1- $\lim\limits_{x \to 0} \sqrt{5 - 2x^2} = \sqrt{5 - 2(0)^2} = \sqrt{5}$ and $\lim\limits_{x \to 0} \sqrt{5 - x^2} = \sqrt{5 - (0)^2} = \sqrt{5}$; by the sandwich theorem,

$\lim\limits_{x \to 0} f(x) = \sqrt{5}$

2- $\lim\limits_{x \to 0} (2 - x^2) = 2 - 0 = 2$ and $\lim\limits_{x \to 0} 2\cos x = 2(1) = 2$; by the sandwich theorem, $\lim\limits_{x \to 0} g(x) = 2$

Centering Intervals About a Point

In Exercises **1-2**, sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$.

1. $a = 1, \quad b = 7, \quad x_0 = 5$

2. $a = 1, \quad b = 7, \quad x_0 = 2$

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

1. 

Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: $\delta + 5 = 7 \Rightarrow \delta = 2$, or $-\delta + 5 = 1 \Rightarrow \delta = 4$.

The value of δ which assures $|x - 5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

2. 

Step 1: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$

Step 2: $-\delta + 2 = 1 \Rightarrow \delta = 1$, or $\delta + 2 = 7 \Rightarrow \delta = 5$.

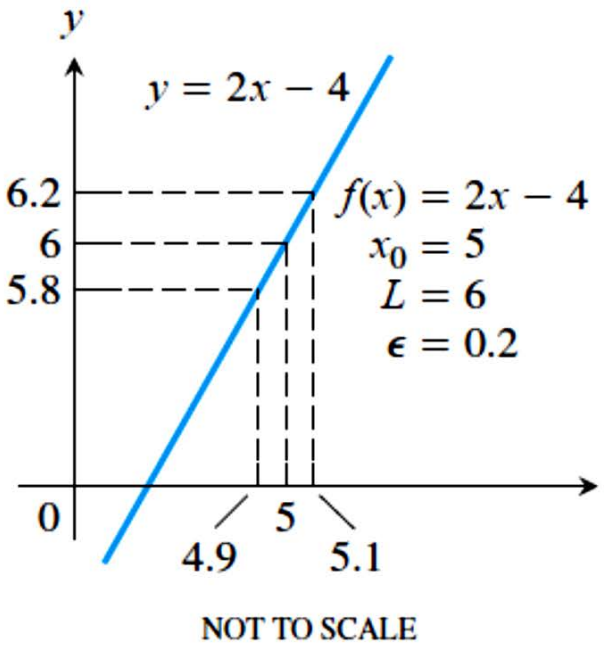
The value of δ which assures $|x - 2| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 1$.

Finding Deltas Graphically

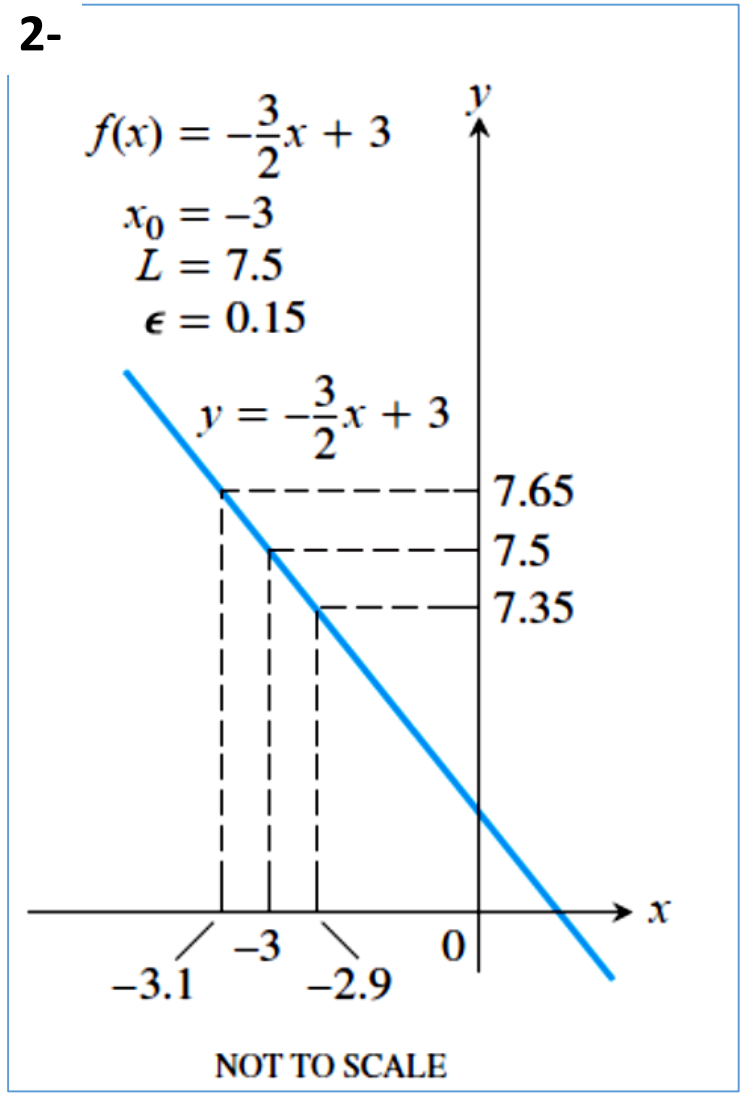
In Exercises **1-2** use the graphs to find a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

1-



2-



1- Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: From the graph, $-\delta + 5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta + 5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

2- Step 1: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$

Step 2: From the graph, $-\delta - 3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta - 3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

Finding Deltas Algebraically

Each of Exercises **1-3** gives a function $f(x)$ and numbers L, x_0 and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

1- $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$

2- $f(x) = 2x - 2, \quad L = -6, \quad x_0 = -2, \quad \epsilon = 0.02$

3- $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \epsilon = 0.1$

1- Step 1: $|(x+1) - 5| < 0.01 \Rightarrow |x - 4| < 0.01 \Rightarrow -0.01 < x - 4 < 0.01 \Rightarrow 3.99 < x < 4.01$

Step 2: $|x - 4| < \delta \Rightarrow -\delta < x - 4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4 \Rightarrow \delta = 0.01.$

2- Step 1: $|(2x - 2) - (-6)| < 0.02 \Rightarrow |2x + 4| < 0.02 \Rightarrow -0.02 < 2x + 4 < 0.02 \Rightarrow -4.02 < 2x < -3.98$

$\Rightarrow -2.01 < x < -1.99$

Step 2: $|x - (-2)| < \delta \Rightarrow -\delta < x + 2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2 \Rightarrow \delta = 0.01.$

3- Step 1: $\left|\sqrt{x+1} - 1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x+1} - 1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x + 1 < 1.21$

$\Rightarrow -0.19 < x < 0.21$

Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.

Finding One-Sided Limits Algebraically

Find the limits in Exercises **1-4**

1- $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

2- $\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}}$

3- $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

4- $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

1- $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

2- $\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}} = \sqrt{0} = 0$

3- $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

4- $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right) = \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = 1$

Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises **1-4**

1- $\lim_{\theta \to 0} \frac{\sin \sqrt{2\theta}}{\sqrt{2\theta}}$

2- $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

3- $\lim_{y \to 0} \frac{\sin 3y}{4y}$

4- $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

1- $\lim_{\theta \to 0} \frac{\sin \sqrt{2\theta}}{\sqrt{2\theta}} = \lim_{x \to 0} \frac{\sin x}{x} = 1$ (where $x = \sqrt{2\theta}$)

2- $\lim_{t \to 0} \frac{\sin kt}{t} = \lim_{t \to 0} \frac{k \sin kt}{kt} = \lim_{\theta \to 0} \frac{k \sin \theta}{\theta} = k \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = k \cdot 1 = k$ (where $\theta = kt$)

3- $\lim_{y \to 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \to 0} \frac{3 \sin 3y}{3y} = \frac{3}{4} \lim_{y \to 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{3}{4}$ (where $\theta = 3y$)

4- $\lim_{h \to 0^-} \frac{h}{\sin 3h} = \lim_{h \to 0^-} \left(\frac{1}{3} \cdot \frac{3h}{\sin 3h}\right) = \frac{1}{3} \lim_{h \to 0^-} \frac{1}{\left(\frac{\sin 3h}{3h}\right)} = \frac{1}{3} \left(\frac{1}{\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta}}\right) = \frac{1}{3} \cdot 1 = \frac{1}{3}$ (where $\theta = 3h$)

Calculating Limits as $x \to \pm\infty$

In Exercises **1- 4** , find the limit of each function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

1- $f(x) = \frac{2}{x} - 3$

2- $f(x) = \pi - \frac{2}{x^2}$

3- $g(x) = \frac{1}{2 + (1/x)}$

4- $g(x) = \frac{1}{8 - (5/x^2)}$

1- (a) -3 (b) -3

2- (a) π (b) π

3- (a) $\frac{1}{2}$ (b) $\frac{1}{2}$

4- (a) $\frac{1}{8}$ (b) $\frac{1}{8}$

Limits of Rational Functions

In Exercises **1- 4** find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

1- $f(x) = \dfrac{2x + 3}{5x + 7}$

2- $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

3- $f(x) = \dfrac{x + 1}{x^2 + 3}$

4- $f(x) = \dfrac{3x + 7}{x^2 - 2}$

1- (a) $\lim\limits_{x \to \infty} \frac{2x+3}{5x+7} = \lim\limits_{x \to \infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$

(b) $\frac{2}{5}$ (same process as part (a))

2- (a) $\lim\limits_{x \to \infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim\limits_{x \to \infty} \frac{2+\left(\frac{7}{x^3}\right)}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}} = 2$

(b) 2 (same process as part (a))

3- (a) $\lim\limits_{x \to \infty} \frac{x+1}{x^2+3} = \lim\limits_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$

(b) 0 (same process as part (a))

4- (a) $\lim\limits_{x \to \infty} \frac{3x+7}{x^2-2} = \lim\limits_{x \to \infty} \frac{\frac{3}{x}+\frac{7}{x^2}}{1-\frac{2}{x^2}} = 0$

(b) 0 (same process as part (a))