Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence. For example the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the **index** of a_n, and indicates where a_n occurs in the list.

DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n},$$
$$b_n = (-1)^{n+1}\frac{1}{n},$$
$$c_n = \frac{n-1}{n},$$
$$d_n = (-1)^{n+1}$$

or by listing terms,

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$
$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$
$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$
$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$

We also sometimes write

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}.$$

Example 1 Write down the first few terms of each of the following sequences.

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

Note the inclusion of the "..." at the end! This is an important piece of notation as it is the only thing that tells us that the sequence continues on and doesn't terminate at the last term.

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n=1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$$

bounce back and forth between 1 and -1, never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value.

If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence **diverges to** $-\infty$.

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

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Example 2: Determine whether the following sequences converge or diverge:

1- $a_n = \frac{1}{n}$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{\infty} = 0$ (Converge)

2- $a_n = \frac{n^2+1}{n+2}$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{n^2+1}{n+2}\right) = \lim_{n\to\infty}\left(\frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n}{n^2}+\frac{2}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n^2}}{\frac{1}{n}+\frac{2}{n^2}}\right) = \frac{1+\frac{1}{\infty^2}}{\frac{1}{\infty}+\frac{2}{\infty^2}} = \infty$

(Diverge)

3- $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left((-1)^n\left(1-\frac{1}{n}\right)\right) = (-1)^\infty\left(1-\frac{1}{\infty}\right) = \begin{matrix}+1\\-1\end{matrix}$

The $\lim_{n\to\infty}(a_n)$ does not exist so it is (Diverge)

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule and Example 1a

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ Difference Rule and Example 1a

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule

(d) $\lim_{n\to\infty}\frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty}\frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ Sum and Quotient Rules ■

THEOREM 2 **The Sandwich Theorem for Sequences**

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\frac{\cos n}{n} \to 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$;

(b) $\frac{1}{2^n} \to 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$;

(c) $(-1)^n \frac{1}{n} \to 0$ because $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$. ■

THEOREM 3 The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE 5 Applying Theorem 3

Show that $\sqrt{(n+1)/n} \to 1$.

Solution We know that $(n+1)/n \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 3 gives $\sqrt{(n+1)/n} \to \sqrt{1} = 1$. ■

EXAMPLE 6 The Sequence $\{2^{1/n}\}$

The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 ■

Using l'Hôpital's Rule

The next theorem enables us to use l'Hôpital's Rule to find the limits of some sequences. It formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$.

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

EXAMPLE 7 Applying L'Hôpital's Rule

Show that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Solution The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 5, $\lim_{n\to\infty}(\ln n)/n$ will equal $\lim_{x\to\infty}(\ln x)/x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty}(\ln n)/n = 0$. ■

When we use l'Hôpital's Rule to find the limit of a sequence, we often treat n as a continuous real variable and differentiate directly with respect to n. This saves us from having to rewrite the formula for a_n as we did in Example 7.

EXAMPLE 8 Applying L'Hôpital's Rule

Find

$$\lim_{n\to\infty} \frac{2^n}{5n}.$$

Solution By l'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$$
$$= \infty.$$

■

EXAMPLE 9 Applying L'Hôpital's Rule to Determine Convergence

Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$
$$= n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2. ■

Convergent sequences

Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Factorial Notation
The notation $n!$ ("n factorial") means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to n. Notice that $(n+1)! = (n+1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define 0! to be 1. Factorials grow even faster than exponentials, as the table suggests.

n	e^n (rounded)	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	4.9×10^8	2.4×10^{18}

EXAMPLE 10 Applying Theorem 5

(a) $\frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \to 2 \cdot 0 = 0$ Formula 1

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ Formula 2

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ Formula 3 with $x = 3$ and Formula 2

(d) $\left(-\frac{1}{2}\right)^n \to 0$ Formula 4 with $x = -\frac{1}{2}$

(e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ Formula 5 with $x = -2$

(f) $\frac{100^n}{n!} \to 0$ Formula 6 with $x = 100$ ■

Example Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\frac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}$$

So, the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{\frac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x) = \frac{\mathbf{e}^{2x}}{x}$$

and note that,

$$f(n) = \frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty}\frac{e^{2n}}{n}=\lim_{x\to\infty}\frac{e^{2x}}{x}=\lim_{x\to\infty}\frac{2e^{2x}}{1}=\infty$$

So, the sequence in this part diverges (to ∞).

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

We will also need to be careful with this sequence. We might be tempted to just say that the limit of the sequence terms is zero (and we'd be correct). However, technically we can't take the limit of sequences whose terms alternate in sign, because we don't know how to do limits of functions that exhibit that same behavior. Also, we want to be very careful to not rely too much on intuition with these problems. As we will see in the next section, and in later sections, our intuition can lead us astray in these problems if we aren't careful.

So, let's work this one by the book. We will need to use Theorem 2 on this problem. To this we'll first need to compute,

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using $r=-1$. So, by Theorem 3 this sequence diverges.

Examples: Show whether the following sequences are convergence or divergence:

1- $a_n=\frac{n+1}{2n+1}$

Sol: $\lim_{n\to\infty}\left(\frac{n+1}{2n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right)=\frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}}=\frac{1}{2}$ (Conv.)

2- $a_n = \frac{2n-1}{n+1}$

Sol: $\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}-\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{2-\frac{1}{n}}{1+\frac{1}{n}}\right) = \frac{2-\frac{1}{\infty}}{1+\frac{1}{\infty}} = 2$ (Conv.)

3- $a_n = \frac{2^n-1}{2^n}$

Sol: $\lim_{n\to\infty}\left(\frac{2^n-1}{2^n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2^n}{2^n}-\frac{1}{2^n}}{\frac{2^n}{2^n}}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{1}{2^n}}{1}\right) = \frac{1-\frac{1}{2^\infty}}{1} = \frac{1-0}{1} = 1$ (Conv.)

4- $a_n = 1 + \frac{(-1)^n}{2^n}$

Sol: $\lim_{n\to\infty}\left(1+\frac{(-1)^n}{2^n}\right) = \lim_{n\to\infty}(1) + \lim_{n\to\infty}\left(\frac{(-1)^n}{2^n}\right) = 1 + \frac{(-1)^\infty}{2^\infty} = 1 + 0 = 1$ (Conv.)

5- $a_n = \cos\frac{n\pi}{2}$

Sol: $\lim_{n\to\infty}\left(\cos\frac{n\pi}{2}\right) = \cos\frac{\infty\pi}{2} = \begin{matrix} 0 \\ -1 \\ +1 \end{matrix}$ The limit does not exist (Div.)

6- $a_n = \sin\frac{n\pi}{2}$

Sol: $\lim_{n\to\infty}\left(\sin\frac{n\pi}{2}\right) = \sin\frac{\infty\pi}{2} = \begin{matrix} 0 \\ -1 \\ +1 \end{matrix}$ The limit does not exist (Div.)

7- $a_n = \frac{1}{10n}$

Sol: $\lim_{n\to\infty}\left(\frac{1}{10n}\right) = \frac{1}{10}\lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{10}\times\frac{1}{\infty} = \frac{1}{10}\times 0 = 0$ (Conv.)

8- $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

Sol: $\lim_{n\to\infty}\left((-1)^n\left(1-\frac{1}{n}\right)\right) = (-1)^\infty\left(1-\frac{1}{\infty}\right) = \begin{matrix} +1 \\ -1 \end{matrix}$ The limit does not exist (Dev.)

9- $a_n = \frac{2n+1}{1-3n}$

Sol: $\lim_{n\to\infty}\left(\frac{2n+1}{1-3n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}+\frac{1}{n}}{\frac{1}{n}-\frac{3n}{n}}\right) = \lim_{n\to\infty}\left(\frac{2+\frac{1}{n}}{\frac{1}{n}-3}\right) = -\frac{2}{3}$ (Conv.)

10- $a_n = \sqrt{\frac{2n}{n+1}}$

Sol: $\lim_{n\to\infty}\left(\sqrt{\frac{2n}{n+1}}\right) = \sqrt{\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)} = \sqrt{\lim_{n\to\infty}\left(\frac{\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)} = \sqrt{\lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2}$ (Conv.)

11- $a_n = \sin n\pi$

Sol: $\lim_{n\to\infty}(\sin n\pi) = 0$ (Conv.)

12- $a_n = n\pi\cos n\pi$

Sol: $\lim_{n\to\infty}(n\pi\cos n\pi) = \lim_{n\to\infty}(n\pi)\cdot\lim_{n\to\infty}(\cos n\pi) = \infty\times\binom{-1}{+1} = \begin{matrix}-\infty\\+\infty\end{matrix}$ (Dev.)

13- $a_n = \tanh n$

Sol: $\lim_{n\to\infty}(\tanh n) = \lim_{n\to\infty}\left(\frac{e^n-e^{-n}}{e^n+e^{-n}}\right) = \lim_{n\to\infty}\frac{e^n}{e^n}\left(\frac{1-\frac{e^{-n}}{e^n}}{1+\frac{e^{-n}}{e^n}}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{1}{e^{2n}}}{1+\frac{1}{e^{2n}}}\right) = 1$ (Conv.)

14- $a_n = \ln n - \ln(n+1)$

Sol: $\lim_{n\to\infty}(\ln n - \ln(n+1)) = \lim_{n\to\infty}\left(\ln\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\ln\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right) =$

$\lim_{n\to\infty}\left(\ln\frac{1}{1+\frac{1}{n}}\right) = \ln 1 = 0$ (Conv.)

15- $a_n = n\sin\frac{1}{n}$

Sol: $\lim_{n\to\infty}\left(n\sin\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = 1$ (Conv.) $Rule$: $\left\{\lim_{n\to 0}\left(\frac{\sin x}{x}\right) = 1\right\}$

16- $a_n = \frac{4-7n^6}{n^6+3}$

Sol: $\lim\limits_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{4}{n^6}-\frac{7n^6}{n^6}}{\frac{n^6}{n^6}+\frac{3}{n^6}}\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) = \frac{0-7}{1+0} = -7$ (Conv.)

17- $a_n = \frac{\ln(3n+5)}{n}$

Sol: $\lim\limits_{n\to\infty}\left(\frac{\ln(3n+5)}{n}\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{3}{3n+5}}{1}\right) = \lim\limits_{n\to\infty}\left(\frac{3}{3n+5}\right) = 0$ (Conv.)

Homework:

1- $a_n = \frac{1-2n}{1+2n}$

2- $a_n = \frac{2^n}{2^{n+1}}$

3- $a_n = \frac{1-n}{n^2}$

4- $a_n = \left(\frac{1}{3}\right)^n$

5- $a_n = \frac{(-1)^{n+1}}{2n-1}$

6- $a_n = 1 + \frac{(-1)^n}{n}$

7- $a_n = \frac{(n)^2}{(n+1)^2}$

8- $a_n = \frac{1-5n^4}{n^4+8n^3}$

9- $a_n = \frac{2(n+1)+1}{2n+1}$

10- $a_n = \frac{n^n}{(n+1)^{n+1}}$

11- $a_n = \sqrt{2 - \frac{1}{n}}$

12- $a_n = \frac{n^2-2n+1}{n-1}$

13- $a_n = \frac{3^n}{n^3}$

14- $a_n = \frac{n^2}{2n-1} \sin \frac{1}{n}$

15- $a_n = (0.5)^n$

16- $a_n = \sqrt[n]{3n + 5}$

17- $a_n = \frac{\ln n}{n^3}$

18- $a_n = \frac{\ln n^2}{n}$

19- $a_n = \frac{\ln n}{n^{1/n}}$

20- $a_n = \sqrt[n]{n^2}$

21- $a_n = \sqrt[n]{3n}$

22- $a_n = \left(\frac{n-2}{n}\right)^n$

Series

An *infinite series* is the sum of an infinite sequence of numbers.

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

DEFINITIONS **Infinite Series, *n*th Term, Partial Sum, Converges, Sum**

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Indeed there is a pattern. The partial sums form a sequence whose nth term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence of partial sums converges to 2 because $\lim_{n\to\infty}(1/2^n) = 0$. We say

"the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is 2."

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

If the sequence of partial sums is a convergent sequence (*i.e.* its limit exists and is finite) then the series is also called **convergent** and in this case if $\lim_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (*i.e.* its limit doesn't exist or is plus or minus infinity) then the series is also called **divergent**.

Example 1 Determine if the following series is convergent or divergent. If it converges determine its value.

$$\sum_{n=1}^{\infty} n$$

Solution

To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums.

$$s_n = \sum_{i=1}^{n} i$$

This is a known series and its value can be shown to be,

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Don't worry if you didn't know this formula (we'd be surprised if anyone knew it…) as you won't be required to know it in my course.

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$$

is convergent or divergent. That's not terribly difficult in this case. The limit of the sequence terms is,

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

The formula $a/(1 - r)$ for the sum of a geometric series applies *only* when the summation index begins with $n = 1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ (or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$).

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}.$$ ■

EXAMPLE 2 Index Starts with $n = 0$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{5}{1+(1/4)} = 4.$$ ■

Example 3:

$$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^{n-1}$$

$a = 4; \quad r = \frac{1}{2} < 1$ .... (Conv.)

$$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$$

Example 4:

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^{n-1} - 2^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{2^{n-1}}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$$

$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$; $\mathrm{a} = 1 , \quad \mathrm{r} = \frac{1}{2} < 1$ ... ... (Conv.)

$\sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$; $\mathrm{a} = 1 , \quad \mathrm{r} = \frac{1}{3} < 1$ ... ... (Conv.)

$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$; *It is convergent*

$$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$

Example 5: Determine whether each of the following series convergence or divergence. If it is convergent, find the sum.

(1) $$\sum_{n=0}^{\infty}(\frac{1}{\sqrt{2}})^{n}=\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2}})^{n-1}$$

$a=1;\quad r=\frac{1}{\sqrt{2}}<1$ .... (Conv.)

$$\sum_{n=1}^{\infty}(\frac{1}{\sqrt{2}})^{n-1}=\frac{a}{1-r}=\frac{1}{1-\frac{1}{\sqrt{2}}}=3.414$$

(2) $$\sum_{n=0}^{\infty}\frac{5}{2^{n}}-\frac{1}{3^{n}}=\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}-\frac{1}{3^{n-1}}=\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$$

$$\sum_{n=1}^{\infty}5(\frac{1}{2})^{n-1}\quad ;\ \mathrm{a}=5\,,\ \mathrm{r}=\frac{1}{2}<1\ \ldots\ldots\ldots\ldots\ (\mathrm{Conv.})$$

$$\sum_{n=1}^{\infty}(\frac{1}{3})^{n-1}\quad ;\ \mathrm{a}=1\,,\ \mathrm{r}=\frac{1}{3}<1\ \ldots\ldots\ldots\ldots\ (\mathrm{Conv.})$$

$$\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}=\frac{a}{1-r}=\frac{5}{1-\frac{1}{2}}-\frac{1}{1-\frac{1}{3}}=\frac{17}{2}$$

(3) $$\sum_{n=0}^{\infty}e^{-2n}=\sum_{n=1}^{\infty}(e^{-2})^{n-1}$$

$$\sum_{n=1}^{\infty}(e^{-2})^{n-1}\ ;\quad a=1\,,\ r=e^{-2}<1\ \ldots\ldots\ldots\ (Conv.)$$

$$\sum_{n=1}^{\infty}(e^{-2})^{n-1}=\frac{a}{1-r}=\frac{1}{1-e^{-2}}=1.15$$

The *n*th-Term Test for Divergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

EXAMPLE 7 Applying the *n*th-Term Test

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0.$

(e) $\sum_{n=1}^{\infty} 2(\cos\frac{\pi}{3})^n = \sum_{n=1}^{\infty} 2(\frac{1}{2})^n$; $\lim_{n\to\infty}(2(\frac{1}{2})^n) = 2(0) = 0 \ldots\ldots$ $(Conv.)$

(f) $\sum_{n=0}^{\infty} (\tan\frac{\pi}{4})^n = \sum_{n=0}^{\infty} (1)^n$; $\lim_{n\to\infty}(1)^n = 1 \ldots\ldots$ $(Div.)$

(g) $\sum_{n=1}^{\infty} \frac{5(-1)^n}{4^n} = \sum_{n=1}^{\infty} 5(\frac{-1}{4})^n$; $\lim_{n\to\infty} 5(\frac{-1}{4})^n = 5(0) = 0 \ldots\ldots$ $(Conv.)$

(h) $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$; $\lim_{n\to\infty} \frac{n!}{1000^n} = \lim_{n\to\infty} \frac{1}{\frac{1000^n}{n!}} = \frac{1}{0} = \infty \ \ldots\ldots$ $(Div.)$

$$(\boldsymbol{i})\ \sum_{n=0}^{\infty} \cos n\pi \ = Divergnece\ because\ of\ \lim_{n\to\infty}(\cos n\pi) = \begin{matrix} +1 \\ -1 \end{matrix}\quad (does\ not\ exist)$$

$$(\boldsymbol{j})\ \sum_{n=1}^{\infty}(-1)^{n+1}\cdot n\ ;\quad \lim_{n\to\infty}(-1)^{n+1}\cdot n = \begin{matrix} +\infty \\ -\infty \end{matrix}\quad (does\ not\ exist);\ \dots\dots\dots(Div.)$$

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k\sum a_n = kA$ (Any number k).

EXAMPLE 9 Find the sums of the following series.

(a)
$$\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} - \sum_{n=1}^{\infty}\frac{1}{6^{n-1}}$$ Difference Rule
$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)}$$ Geometric series with $a = 1$ and $r = 1/2, 1/6$
$$= 2 - \frac{6}{5}$$
$$= \frac{4}{5}$$

(b)
$$\sum_{n=0}^{\infty}\frac{4}{2^n} = 4\sum_{n=0}^{\infty}\frac{1}{2^n}$$ Constant Multiple Rule
$$= 4\left(\frac{1}{1-(1/2)}\right)$$ Geometric series with $a = 1, r = 1/2$
$$= 8$$ ■

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots.$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots.$$

It works like a horizontal shift. We saw this in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

EXAMPLE 10 Reindexing a Geometric Series

We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose. ■

The *p*-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Example: Use the p-Series

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$; $p = 1 \ldots \ldots \ldots$ $(Div.)$

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $p = 2 \ldots \ldots \ldots$ $(Con.)$

(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$; $p = \frac{1}{2} \ldots \ldots \ldots$ $(Div.)$

The Integral Test

EXAMPLE Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Solution

By using p-series ………$p = 2 > 1 \ldots \ldots (Conv.)$

By using Integral test to see whether it is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \ldots \ldots \ldots \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1} = 0 + 1 = 1$$

The integral test is convergence……..the series converges.

EXAMPLE 3 The p-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution:

$*$ $By\ using\ p - series\ \ldots..p = 1\ \ldots\ldots.(Div.)$

$*$ $By\ using\ integral\ test$

$$\sum_{n=1}^{\infty} \frac{1}{n} \ldots\ldots\ldots.\int_1^{\infty} \frac{1}{x}\,dx = [\ln x]_1^{\infty} = \ln\infty - \ln 1 = \infty$$

The series is divergent because the integral test diverges.

Example: $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \ldots\ldots \int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \int_1^{\infty} x e^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_1^{\infty} = -\frac{1}{2}e^{-\infty^2} + \frac{1}{2}e^{-1^2}$$

$$= \frac{1}{2e} - \frac{1}{2e^{\infty}} = \frac{1}{2e}$$

The integral test converges ……… The series converges.

Comparison Tests

We can test the convergence of many more series by comparing their terms to those of a series whose convergence is known.

Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1+2\ln 2}{9} + \frac{1+3\ln 3}{14} + \frac{1+4\ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - (1/2^n)}$$

$$= 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

(c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we take $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5}$$

$$= \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

Example:

(1) $$a_n = \sum_{n=1}^{\infty} \frac{1}{2n^2 - n}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad ; \ using\ p - series \ldots \ldots p = 2 > 1 \ldots \ldots .. (Conv.)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{2n^2 - n} = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2} > 0 \ldots \ldots \ldots \ldots \ a_n = Conv.$$

$$(2)\ a_n = \sum_{n=1}^{\infty} \frac{3n^3 - 2n^2 + 4}{n^5 - n^2 + 2}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad ;\ using\ p - series \dots\dots p = 2 > 1 \dots\dots..(Conv.)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^5 - 2n^4 + 4n^2}{n^5 - n^2 + 2} = \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{4}{n^3}}{1 - \frac{1}{n^3} + \frac{2}{n^5}} = 3 > 0 \dots\dots\ a_n = Conv.$$

$$(3)\ a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2 - 3n}}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}} \qquad ;\ using\ p - series \dots\dots p = \frac{2}{3} < 1 \dots\dots..(Div.)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{2}{3}}}{(8n^2 - 3n)^{\frac{1}{3}}} = \lim_{n\to\infty} \left(\frac{n^2}{8n^2 - 3n}\right)^{\frac{1}{3}} = \lim_{n\to\infty} \left(\frac{1}{8 - \frac{3}{n}}\right)^{\frac{1}{3}} = 0.5 > 0 \dots\dots$$

$a_n = diverge\ because\ b_n\ is\ divergent$

$$(4)\ a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + 1}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}} \qquad ;\ using\ p - series \dots\dots p = \frac{5}{2} > 1 \dots\dots..(Conv.)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}} \cdot n^{\frac{1}{2}}}{n^3 + 1} = \lim_{n\to\infty} \frac{n^3}{n^3 + 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^3}} = 1 > 0 \dots\dots$$

$a_n = Converge\ as\ b_n\ is\ convergent$

Homework:

(1) $\sum_{n=0}^{\infty} \frac{2^n}{5^n} \ldots \ldots . . (Geometric\ series)$

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \ldots \ldots . (Integral\ test)$

(3) $\sum_{n=1}^{\infty} \frac{1}{n-2} \ldots \ldots \ldots$ (Comparison test)

(4) $\sum_{n=1}^{\infty} \frac{1}{(n-1)(\sqrt{n}-1)} \ldots \ldots \ldots .$ (Comparison test)

(5) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \ldots \ldots \ldots$ (Comparison test)

Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence. For example the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the **index** of a_n, and indicates where a_n occurs in the list.

DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n},$$
$$b_n = (-1)^{n+1}\frac{1}{n},$$
$$c_n = \frac{n-1}{n},$$
$$d_n = (-1)^{n+1}$$

or by listing terms,

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$
$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$
$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$
$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$

We also sometimes write

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}.$$

Example 1 Write down the first few terms of each of the following sequences.

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

Note the inclusion of the "..." at the end! This is an important piece of notation as it is the only thing that tells us that the sequence continues on and doesn't terminate at the last term.

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n=1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$$

bounce back and forth between 1 and -1, never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value.

If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence **diverges to** $-\infty$.

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

---

Example 2: Determine whether the following sequences converge or diverge:

1- $a_n = \frac{1}{n}$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{\infty} = 0$ (Converge)

2- $a_n = \frac{n^2+1}{n+2}$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{n^2+1}{n+2}\right) = \lim_{n\to\infty}\left(\frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n}{n^2}+\frac{2}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n^2}}{\frac{1}{n}+\frac{2}{n^2}}\right) = \frac{1+\frac{1}{\infty^2}}{\frac{1}{\infty}+\frac{2}{\infty^2}} = \infty$

(Diverge)

3- $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

Sol: $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left((-1)^n\left(1-\frac{1}{n}\right)\right) = (-1)^\infty\left(1-\frac{1}{\infty}\right) = \begin{matrix}+1\\-1\end{matrix}$

The $\lim_{n\to\infty}(a_n)$ does not exist so it is (Diverge)

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

EXAMPLE 3 Applying Theorem 1

By combining Theorem 1 with the limits of Example 1, we have:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule and Example 1a

(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ Difference Rule and Example 1a

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule

(d) $\lim_{n\to\infty}\frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty}\frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ Sum and Quotient Rules ■

THEOREM 2 **The Sandwich Theorem for Sequences**

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

EXAMPLE 4 Applying the Sandwich Theorem

Since $1/n \to 0$, we know that

(a) $\frac{\cos n}{n} \to 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$;

(b) $\frac{1}{2^n} \to 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$;

(c) $(-1)^n\frac{1}{n} \to 0$ because $-\frac{1}{n} \leq (-1)^n\frac{1}{n} \leq \frac{1}{n}$. ■

THEOREM 3 The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE 5 Applying Theorem 3

Show that $\sqrt{(n+1)/n} \to 1$.

Solution We know that $(n+1)/n \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 3 gives $\sqrt{(n+1)/n} \to \sqrt{1} = 1$. ■

EXAMPLE 6 The Sequence $\{2^{1/n}\}$

The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 ■

Using l'Hôpital's Rule

The next theorem enables us to use l'Hôpital's Rule to find the limits of some sequences. It formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$.

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

EXAMPLE 7 Applying L'Hôpital's Rule

Show that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Solution The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 5, $\lim_{n\to\infty}(\ln n)/n$ will equal $\lim_{x\to\infty}(\ln x)/x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty}(\ln n)/n = 0$. ■

When we use l'Hôpital's Rule to find the limit of a sequence, we often treat n as a continuous real variable and differentiate directly with respect to n. This saves us from having to rewrite the formula for a_n as we did in Example 7.

EXAMPLE 8 Applying L'Hôpital's Rule

Find

$$\lim_{n\to\infty} \frac{2^n}{5n}.$$

Solution By l'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$$
$$= \infty.$$

■

EXAMPLE 9 Applying L'Hôpital's Rule to Determine Convergence

Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$
$$= n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0$$
$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0}$$
$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{l'Hôpital's Rule}$$
$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2. ■

Convergent sequences

Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Factorial Notation
The notation $n!$ ("n factorial") means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to n. Notice that $(n+1)! = (n+1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define 0! to be 1. Factorials grow even faster than exponentials, as the table suggests.

n	e^n (rounded)	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	4.9×10^8	2.4×10^{18}

EXAMPLE 10 Applying Theorem 5

(a) $\frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \to 2 \cdot 0 = 0$ — Formula 1

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ — Formula 2

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ — Formula 3 with $x = 3$ and Formula 2

(d) $\left(-\frac{1}{2}\right)^n \to 0$ Formula 4 with $x = -\frac{1}{2}$

(e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ Formula 5 with $x = -2$

(f) $\frac{100^n}{n!} \to 0$ Formula 6 with $x = 100$ ■

Example Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}$$

So, the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x) = \frac{e^{2x}}{x}$$

and note that,

$$f(n) = \frac{e^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty}\frac{e^{2n}}{n}=\lim_{x\to\infty}\frac{e^{2x}}{x}=\lim_{x\to\infty}\frac{2e^{2x}}{1}=\infty$$

So, the sequence in this part diverges (to ∞).

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

We will also need to be careful with this sequence. We might be tempted to just say that the limit of the sequence terms is zero (and we'd be correct). However, technically we can't take the limit of sequences whose terms alternate in sign, because we don't know how to do limits of functions that exhibit that same behavior. Also, we want to be very careful to not rely too much on intuition with these problems. As we will see in the next section, and in later sections, our intuition can lead us astray in these problems if we aren't careful.

So, let's work this one by the book. We will need to use Theorem 2 on this problem. To this we'll first need to compute,

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using $r=-1$. So, by Theorem 3 this sequence diverges.

Examples: Show whether the following sequences are convergence or divergence:

1- $a_n=\frac{n+1}{2n+1}$

Sol: $\lim_{n\to\infty}\left(\frac{n+1}{2n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right)=\frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}}=\frac{1}{2}$ (Conv.)

2- $a_n = \frac{2n-1}{n+1}$

Sol: $\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}-\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{2-\frac{1}{n}}{1+\frac{1}{n}}\right) = \frac{2-\frac{1}{\infty}}{1+\frac{1}{\infty}} = 2$ (Conv.)

3- $a_n = \frac{2^n-1}{2^n}$

Sol: $\lim_{n\to\infty}\left(\frac{2^n-1}{2^n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2^n}{2^n}-\frac{1}{2^n}}{\frac{2^n}{2^n}}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{1}{2^n}}{1}\right) = \frac{1-\frac{1}{2^\infty}}{1} = \frac{1-0}{1} = 1$ (Conv.)

4- $a_n = 1 + \frac{(-1)^n}{2^n}$

Sol: $\lim_{n\to\infty}\left(1+\frac{(-1)^n}{2^n}\right) = \lim_{n\to\infty}(1) + \lim_{n\to\infty}\left(\frac{(-1)^n}{2^n}\right) = 1 + \frac{(-1)^\infty}{2^\infty} = 1 + 0 = 1$ (Conv.)

5- $a_n = \cos\frac{n\pi}{2}$

Sol: $\lim_{n\to\infty}\left(\cos\frac{n\pi}{2}\right) = \cos\frac{\infty\pi}{2} = \begin{matrix} 0 \\ -1 \\ +1 \end{matrix}$ The limit does not exist (Div.)

6- $a_n = \sin\frac{n\pi}{2}$

Sol: $\lim_{n\to\infty}\left(\sin\frac{n\pi}{2}\right) = \sin\frac{\infty\pi}{2} = \begin{matrix} 0 \\ -1 \\ +1 \end{matrix}$ The limit does not exist (Div.)

7- $a_n = \frac{1}{10n}$

Sol: $\lim_{n\to\infty}\left(\frac{1}{10n}\right) = \frac{1}{10}\lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{10}\times\frac{1}{\infty} = \frac{1}{10}\times 0 = 0$ (Conv.)

8- $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

Sol: $\lim_{n\to\infty}\left((-1)^n\left(1-\frac{1}{n}\right)\right) = (-1)^\infty\left(1-\frac{1}{\infty}\right) = \begin{matrix} +1 \\ -1 \end{matrix}$ The limit does not exist (Dev.)

9- $a_n = \frac{2n+1}{1-3n}$

Sol: $\lim_{n\to\infty}\left(\frac{2n+1}{1-3n}\right) = \lim_{n\to\infty}\left(\frac{\frac{2n}{n}+\frac{1}{n}}{\frac{1}{n}-\frac{3n}{n}}\right) = \lim_{n\to\infty}\left(\frac{2+\frac{1}{n}}{\frac{1}{n}-3}\right) = -\frac{2}{3}$ (Conv.)

10- $a_n = \sqrt{\frac{2n}{n+1}}$

Sol: $\lim_{n\to\infty}\left(\sqrt{\frac{2n}{n+1}}\right) = \sqrt{\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)} = \sqrt{\lim_{n\to\infty}\left(\frac{\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)} = \sqrt{\lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2}$ (Conv.)

11- $a_n = \sin n\pi$

Sol: $\lim_{n\to\infty}(\sin n\pi) = 0$ (Conv.)

12- $a_n = n\pi\cos n\pi$

Sol: $\lim_{n\to\infty}(n\pi\cos n\pi) = \lim_{n\to\infty}(n\pi)\cdot\lim_{n\to\infty}(\cos n\pi) = \infty\times\binom{-1}{+1} = \begin{matrix}-\infty\\+\infty\end{matrix}$ (Dev.)

13- $a_n = \tanh n$

Sol: $\lim_{n\to\infty}(\tanh n) = \lim_{n\to\infty}\left(\frac{e^n-e^{-n}}{e^n+e^{-n}}\right) = \lim_{n\to\infty}\frac{e^n}{e^n}\left(\frac{1-\frac{e^{-n}}{e^n}}{1+\frac{e^{-n}}{e^n}}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{1}{e^{2n}}}{1+\frac{1}{e^{2n}}}\right) = 1$ (Conv.)

14- $a_n = \ln n - \ln(n+1)$

Sol: $\lim_{n\to\infty}(\ln n - \ln(n+1)) = \lim_{n\to\infty}\left(\ln\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\ln\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right) =$

$\lim_{n\to\infty}\left(\ln\frac{1}{1+\frac{1}{n}}\right) = \ln 1 = 0$ (Conv.)

15- $a_n = n\sin\frac{1}{n}$

Sol: $\lim_{n\to\infty}\left(n\sin\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = 1$ (Conv.) $Rule$: $\left\{\lim_{n\to 0}\left(\frac{\sin x}{x}\right) = 1\right\}$

16- $a_n = \frac{4-7n^6}{n^6+3}$

Sol: $\lim_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = \lim_{n\to\infty}\left(\frac{\frac{4}{n^6}-\frac{7n^6}{n^6}}{\frac{n^6}{n^6}+\frac{3}{n^6}}\right) = \lim_{n\to\infty}\left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) = \frac{0-7}{1+0} = -7$ (Conv.)

17- $a_n = \frac{\ln(3n+5)}{n}$

Sol: $\lim_{n\to\infty}\left(\frac{\ln(3n+5)}{n}\right) = \lim_{n\to\infty}\left(\frac{\frac{3}{3n+5}}{1}\right) = \lim_{n\to\infty}\left(\frac{3}{3n+5}\right) = 0$ (Conv.)

Homework:

1- $a_n = \frac{1-2n}{1+2n}$

2- $a_n = \frac{2^n}{2^{n+1}}$

3- $a_n = \frac{1-n}{n^2}$

4- $a_n = \left(\frac{1}{3}\right)^n$

5- $a_n = \frac{(-1)^{n+1}}{2n-1}$

6- $a_n = 1 + \frac{(-1)^n}{n}$

7- $a_n = \frac{(n)^2}{(n+1)^2}$

8- $a_n = \frac{1-5n^4}{n^4+8n^3}$

9- $a_n = \frac{2(n+1)+1}{2n+1}$

10- $a_n = \frac{n^n}{(n+1)^{n+1}}$

11- $a_n = \sqrt{2-\frac{1}{n}}$

12- $a_n = \frac{n^2-2n+1}{n-1}$

13- $a_n = \frac{3^n}{n^3}$

14- $a_n = \frac{n^2}{2n-1} \sin \frac{1}{n}$

15- $a_n = (0.5)^n$

16- $a_n = \sqrt[n]{3n + 5}$

17- $a_n = \frac{\ln n}{n^3}$

18- $a_n = \frac{\ln n^2}{n}$

19- $a_n = \frac{\ln n}{n^{1/n}}$

20- $a_n = \sqrt[n]{n^2}$

21- $a_n = \sqrt[n]{3n}$

22- $a_n = \left(\frac{n-2}{n}\right)^n$

Series

An *infinite series* is the sum of an infinite sequence of numbers.

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

DEFINITIONS Infinite Series, *n*th Term, Partial Sum, Converges, Sum

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Indeed there is a pattern. The partial sums form a sequence whose nth term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence of partial sums converges to 2 because $\lim_{n\to\infty}(1/2^n) = 0$. We say

"the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is 2."

Partial sum		Suggestive expression for partial sum	Value
First:	$s_1 = 1$	$2 - 1$	1
Second:	$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$	$\frac{3}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4}$	$\frac{7}{4}$
⋮	⋮	⋮	⋮
nth:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

If the sequence of partial sums is a convergent sequence (*i.e.* its limit exists and is finite) then the series is also called **convergent** and in this case if $\lim_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (*i.e.* its limit doesn't exist or is plus or minus infinity) then the series is also called **divergent**.

Example 1 Determine if the following series is convergent or divergent. If it converges determine its value.

$$\sum_{n=1}^{\infty} n$$

Solution

To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums.

$$s_n = \sum_{i=1}^{n} i$$

This is a known series and its value can be shown to be,

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Don't worry if you didn't know this formula (we'd be surprised if anyone knew it…) as you won't be required to know it in my course.

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$$

is convergent or divergent. That's not terribly difficult in this case. The limit of the sequence terms is,

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

The formula $a/(1 - r)$ for the sum of a geometric series applies *only* when the summation index begins with $n = 1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ (or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$).

EXAMPLE 1 Index Starts with $n = 1$

The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$ ■

EXAMPLE 2 Index Starts with $n = 0$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{5}{1 + (1/4)} = 4.$$ ■

Example 3:

$$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^{n-1}$$

$a = 4; \quad r = \frac{1}{2} < 1$ .... (Conv.)

$$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$$

Example 4:

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^{n-1} - 2^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{2^{n-1}}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$$

$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$; $\text{a} = 1, \quad \text{r} = \frac{1}{2} < 1$ ... ... (Conv.)

$\sum_{n=1}^{\infty} (\frac{1}{3})^{n-1}$; $\text{a} = 1, \quad \text{r} = \frac{1}{3} < 1$ ... ... (Conv.)

$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$; *It is convergent*

$$\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$

Example 5: Determine whether each of the following series convergence or divergence. If it is convergent, find the sum.

(1) $$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{n}=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

$a=1; \quad r=\frac{1}{\sqrt{2}}<1$ .... (Conv.)

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{n-1}=\frac{a}{1-r}=\frac{1}{1-\frac{1}{\sqrt{2}}}=3.414$$

(2) $$\sum_{n=0}^{\infty}\frac{5}{2^n}-\frac{1}{3^n}=\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}-\frac{1}{3^{n-1}}=\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$$

$$\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1} \quad ; \; \mathrm{a}=5, \; \mathrm{r}=\frac{1}{2}<1 \; \dots\dots\dots\dots \; (\mathrm{Conv.})$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1} \quad ; \; \mathrm{a}=1, \; \mathrm{r}=\frac{1}{3}<1 \; \dots\dots\dots\dots \; (\mathrm{Conv.})$$

$$\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}=\frac{a}{1-r}=\frac{5}{1-\frac{1}{2}}-\frac{1}{1-\frac{1}{3}}=\frac{17}{2}$$

(3) $$\sum_{n=0}^{\infty}e^{-2n}=\sum_{n=1}^{\infty}(e^{-2})^{n-1}$$

$$\sum_{n=1}^{\infty}(e^{-2})^{n-1} \quad ; \quad a=1, \; r=e^{-2}<1 \dots\dots\dots (Conv.)$$

$$\sum_{n=1}^{\infty}(e^{-2})^{n-1}=\frac{a}{1-r}=\frac{1}{1-e^{-2}}=1.15$$

The *n*th-Term Test for Divergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

EXAMPLE 7 Applying the *n*th-Term Test

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

(e) $\sum_{n=1}^{\infty} 2(\cos\frac{\pi}{3})^n = \sum_{n=1}^{\infty} 2(\frac{1}{2})^n \; ; \; \lim_{n\to\infty}(2(\frac{1}{2})^n) = 2(0) = 0 \ldots\ldots \; (Conv.)$

(f) $\sum_{n=0}^{\infty} (\tan\frac{\pi}{4})^n = \sum_{n=0}^{\infty} (1)^n \; ; \; \lim_{n\to\infty}(1)^n = 1 \ldots\ldots \; (Div.)$

(g) $\sum_{n=1}^{\infty} \frac{5(-1)^n}{4^n} = \sum_{n=1}^{\infty} 5(\frac{-1}{4})^n \; ; \; \lim_{n\to\infty} 5(\frac{-1}{4})^n = 5(0) = 0 \ldots\ldots \; (Conv.)$

(h) $\sum_{n=0}^{\infty} \frac{n!}{1000^n} \; ; \; \lim_{n\to\infty} \frac{n!}{1000^n} = \lim_{n\to\infty} \frac{1}{\frac{1000^n}{n!}} = \frac{1}{0} = \infty \ldots\ldots \; (Div.)$

$$(\boldsymbol{i})\ \sum_{n=0}^{\infty} \cos n\pi \ = Divergnece\ because\ of\ \lim_{n\to\infty}(\cos n\pi) = \begin{matrix} +1 \\ -1 \end{matrix}\ \ (does\ not\ exist)$$

$$(\boldsymbol{j})\ \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n\ \ ;\ \ \lim_{n\to\infty}(-1)^{n+1} \cdot n = \begin{matrix} +\infty \\ -\infty \end{matrix}\ \ (does\ not\ exist);\ \ldots\ldots\ldots(Div.)$$

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA \qquad$ (Any number k).

EXAMPLE 9 Find the sums of the following series.

(a)

$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \qquad \text{Difference Rule}$$

$$= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} \qquad \text{Geometric series with } a = 1 \text{ and } r = 1/2,\ 1/6$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

(b)

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{1 - (1/2)} \right) \qquad \text{Geometric series with } a = 1,\ r = 1/2$$

$$= 8$$

■

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots.$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots.$$

It works like a horizontal shift. We saw this in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

EXAMPLE 10 Reindexing a Geometric Series

We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose. ■

The *p*-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Example: Use the p-Series

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$; $p = 1$ ... ... $(Div.)$

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $p = 2$ ... ... $(Con.)$

(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$; $p = \frac{1}{2}$ ... ... $(Div.)$

The Integral Test

EXAMPLE Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Solution

By using p-series .... $p = 2 > 1$ ... . $(Conv.)$

By using Integral test to see whether it is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ ... ... } \int_1^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1} = 0 + 1 = 1$$

The integral test is convergence........the series converges.

EXAMPLE 3 The p-Series

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution:

$*$ $By\ using\ p - series\ \ldots..p = 1 \ldots \ldots.(Div.)$

$*$ $By\ using\ integral\ test$

$$\sum_{n=1}^{\infty} \frac{1}{n} \ldots\ldots\ldots. \int_1^{\infty} \frac{1}{x}\,dx = [\ln x]_1^{\infty} = \ln\infty - \ln 1 = \infty$$

The series is divergent because the integral test diverges.

Example: $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} \ldots\ldots \int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \int_1^{\infty} x e^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_1^{\infty} = -\frac{1}{2}e^{-\infty^2} + \frac{1}{2}e^{-1^2}$$

$$= \frac{1}{2e} - \frac{1}{2e^{\infty}} = \frac{1}{2e}$$

The integral test converges ……… The series converges.

Comparison Tests

We can test the convergence of many more series by comparing their terms to those of a series whose convergence is known.

Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE Using the Limit Comparison Test

Which of the following series converge, and which diverge?

(a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

(b) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

(c) $\frac{1+2\ln 2}{9} + \frac{1+3\ln 3}{14} + \frac{1+4\ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

Solution

(a) Let $a_n = (2n+1)/(n^2+2n+1)$. For large n, we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

(b) Let $a_n = 1/(2^n - 1)$. For large n, we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

(c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we take $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

Example:

(1) $a_n = \sum_{n=1}^{\infty} \frac{1}{2n^2 - n}$

$b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$; $using\ p - series \ldots\ldots p = 2 > 1 \ldots\ldots..(Conv.)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{2n^2 - n} = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2} > 0 \ldots\ldots\ldots\ldots a_n = Conv.$$

$(2)\ a_n = \sum_{n=1}^{\infty} \frac{3n^3 - 2n^2 + 4}{n^5 - n^2 + 2}$

$b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$; $using\ p - series \dots \dots p = 2 > 1 \dots \dots . . (Conv.)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^5 - 2n^4 + 4n^2}{n^5 - n^2 + 2} = \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{4}{n^3}}{1 - \frac{1}{n^3} + \frac{2}{n^5}} = 3 > 0 \dots \dots\ a_n = Conv.$$

$(3)\ a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2 - 3n}}$

$b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$; $using\ p - series \dots \dots p = \frac{2}{3} < 1 \dots \dots . . (Div.)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{2}{3}}}{(8n^2 - 3n)^{\frac{1}{3}}} = \lim_{n\to\infty} \left(\frac{n^2}{8n^2 - 3n}\right)^{\frac{1}{3}} = \lim_{n\to\infty} \left(\frac{1}{8 - \frac{3}{n}}\right)^{\frac{1}{3}} = 0.5 > 0 \dots \dots$$

$a_n = diverge\ because\ b_n\ is\ divergent$

$(4)\ a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + 1}$

$b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{2}}}$; $using\ p - series \dots \dots p = \frac{5}{2} > 1 \dots \dots . . (Conv.)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}} \cdot n^{\frac{1}{2}}}{n^3 + 1} = \lim_{n\to\infty} \frac{n^3}{n^3 + 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^3}} = 1 > 0 \dots \dots$$

$a_n = Converge\ as\ b_n\ is\ convergent$

Homework:

(1) $$\sum_{n=0}^{\infty} \frac{2^n}{5^n} \dots \dots .. (Geometric\ series)$$

(2) $$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \dots \dots . (Integral\ test)$$

(3) $$\sum_{n=1}^{\infty} \frac{1}{n-2} \dots \dots \dots (\text{Comparison test})$$

(4) $$\sum_{n=1}^{\infty} \frac{1}{(n-1)(\sqrt{n}-1)} \dots \dots \dots . (\text{Comparison test})$$

(5) $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \dots \dots \dots \quad (\text{Comparison test})$$

Differential Equations

1- First Order Differential Equations ......$f(x, y, y') = 0$

2- Second Order Differential Equations ...$f(x, y, y', y'') = 0$

3- Higher Order Differential Equations ...$f(x, y, y', y'', y''', y'''', \ldots) = 0$

First Order Differential Equations

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

Types of First-order Differential Equations

1- Separable,

2- Homogeneous,

3- Exact,

4- Linear,

5- Bernoulli

1- Separable Differential Equations

Suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

where f is a function of *both* the independent and dependent variables. A **solution** of the equation is a differentiable function $y = y(x)$ defined on an interval of x-values

Equation (1) is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

g is a function of x;
H is a function of y.

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

$H(y) = \dfrac{1}{h(y)}$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{2}$$

After completing the integrations, we obtain the solution y defined implicitly as a function of x.

EXAMPLE 1 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y)e^x$$

$$dy = (1 + y)e^x\,dx$$

Treat dy/dx as a quotient of differentials and multiply both sides by dx.

$$\frac{dy}{1 + y} = e^x\,dx$$

Divide by $(1 + y)$.

$$\int \frac{dy}{1 + y} = \int e^x\,dx$$

Integrate both sides.

$$\ln(1 + y) = e^x + C$$

C represents the combined constants of integration.

The last equation gives y as an implicit function of x. ■

EXAMPLE 2 Solve the equation $y(x+1)\frac{dy}{dx} = x(y^2+1)$.

Solution We change to differential form, separate the variables, and integrate:

$$y(x+1)\,dy = x(y^2+1)\,dx$$

$$\frac{y\,dy}{y^2+1} = \frac{x\,dx}{x+1} \qquad x \neq -1$$

$$\int \frac{y\,dy}{1+y^2} = \int \left(1 - \frac{1}{x+1}\right) dx \qquad \text{Divide } x \text{ by } x+1.$$

$$\frac{1}{2}\ln(1+y^2) = x - \ln|x+1| + C.$$

The last equation gives the solution y as an implicit function of x. ■

Examples: Find the solution of the following differential equations:

1- $\sin x\,dx + y^2dy = 0$

$$\int \sin x\,dx + \int y^2dy = 0$$

$$-\cos x + \frac{y^3}{3} = c$$

2- $xdx - y^2dy = 0$

$$\int xdx - \int y^2dy = 0$$

$$\frac{x^2}{2} - \frac{y^3}{3} = c$$

3- $\frac{dy}{dx} = y^2x^3$

$$\frac{dy}{y^2} = x^3dx$$

$$\int \frac{dy}{y^2} = \int x^3dx$$

$$-\frac{1}{y} = \frac{x^4}{4} + c$$

$$y = \frac{-4}{x^4 + c}$$

4- $\frac{dy}{dx} = -\frac{x}{y} \quad ; y(1) = 4$

$$\int y dy = \int -x dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 + x^2 = 2c = c$$

$$4^2 + 1^2 = c \;\rightarrow\; c = 17$$

$$y^2 + x^2 = 17$$

5- $\frac{dy}{dx} = (1 + y^2)e^x$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + c$$

$$y = \tan(e^x + c)$$

6- $\ln(y^2 + 1)\, dx + \frac{2y(x-1)}{y^2+1} dy = 0$

$$\ln(y^2 + 1)\, dx = -\frac{2y(x-1)}{y^2 + 1} dy$$

$$\int \frac{dx}{(x-1)} = -\int \frac{2y}{(y^2 + 1)\ln(y^2 + 1)} dy$$

$$\int \frac{dx}{(x-1)} = -\int \frac{\frac{2y}{(y^2 + 1)}}{\ln(y^2 + 1)} dy$$

$$\ln(x-1) = -\ln(\ln(y^2 + 1)) + c$$

7- $\frac{dy}{dx} = \cos(x+y)$

We cannot separate this equation, so we change it to

Let: $x + y = u \quad \rightarrow \quad \frac{du}{dx} = 1 + \frac{dy}{dx} \qquad (Note: y = f(x))$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{dy}{dx} = \cos(x+y) \quad \rightarrow \quad \frac{du}{dx} - 1 = \cos(u)$$

$$\frac{du}{dx} = 1 + \cos(u)$$

$$\frac{du}{1+\cos(u)} = dx$$

$$\int \frac{du}{1+\cos(u)} = \int dx$$

$$\int \frac{(1-\cos u)}{(1-\cos u)(1+\cos u)} du = \int dx$$

$$\int \frac{(1-\cos u)}{(1-\cos^2 u)} du = \int dx \quad \rightarrow \quad \int \frac{(1-\cos u)}{\sin^2 u} du = \int dx$$

$$\int \frac{1}{\sin^2 u} du - \int \frac{\cos u}{\sin^2 u} du = \int dx$$

$$\int \csc^2 u \, du - \int \cot u \csc u \, du = \int dx$$

$$-\cot u + \csc u = x + c$$

$$-\cot(x+y) + \csc(x+y) = x + c$$

2- Homogeneous Differential Equations

A first-order differential equation of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is called *homogeneous*. It can be transformed into an equation whose variables are separable by defining the new variable $v = y/x$. Then, $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution into the original differential equation and collecting terms with like variables then gives the separable equation

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0.$$

After solving this separable equation, the solution of the original equation is obtained when we replace v by y/x.

Examples: Find the solution of the following differential equations:

1- $\frac{dy}{dx} = \frac{y+x}{x}$

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

$$Let: \frac{y}{x} = v \quad \rightarrow \quad y = vx \quad \rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = v + 1 \quad \rightarrow \quad x\frac{dv}{dx} = 1 \quad \rightarrow \quad dv = \frac{dx}{x}$$

$$\int dv = \int \frac{dx}{x} \quad \rightarrow \quad v = \ln x + c \quad \rightarrow \quad \frac{y}{x} = \ln x + c$$

2- $\left(xe^{\frac{y}{x}} + y\right) dx - xdy = 0$

$$\left(xe^{\frac{y}{x}} + y\right) dx = xdy$$

$$\frac{dy}{dx} = \frac{xe^{\frac{y}{x}} + y}{x} = e^{\frac{y}{x}} + \frac{y}{x}$$

$$Let: \frac{y}{x} = v \quad \rightarrow \quad y = vx \quad \rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = e^v + v \quad \rightarrow \quad x\frac{dv}{dx} = e^v \quad \rightarrow \quad \frac{dv}{e^v} = \frac{dx}{x}$$

$$\int \frac{dv}{e^v} = \int \frac{dx}{x} \quad \rightarrow \quad \int e^{-v}\, dv = \int \frac{dx}{x}$$

$$-e^{-v} = \ln x + c \qquad \rightarrow \qquad -e^{-\frac{y}{x}} = \ln x + c$$

3- $\frac{dy}{dx} = \frac{x-y}{x+y}$

$$\frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \qquad (Note:\ \div x)$$

$$Let: \frac{y}{x} = v \quad \rightarrow \quad y = vx \quad \rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$$

$$\frac{dx}{x} = \frac{1+v}{1-2v-v^2}dv$$

$$\int \frac{dx}{x} = \int \frac{1+v}{1-2v-v^2}dv$$

$$\ln x = -\frac{1}{2}\ln(1-2v-v^2) + c$$

$$\ln x = -\frac{1}{2}\ln\left(1-2\frac{y}{x} - (\frac{y}{x})^2\right) + c$$

Homework:

1- $(x+1)\frac{dy}{dx} = x(y^2+1)$

2- $\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{1+x^2} \quad ; \quad y(0) = 1$

3- $xdy - ydx - x^2y^3dy = 0$

4- $\frac{dy}{dx} = \frac{x+y}{x-y}$

3- Exact Differential Equations

$M(x,y)dx + N(x,y)dy = 0$ $\quad$ *is exact differentical equation if* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$Note:\ M = \dfrac{\partial f}{\partial x} \quad ; \quad N = \dfrac{\partial f}{\partial y}$

Examples: Solve the following differential equations:

1- $2xydx + (1 + x^2)dy = 0$

$M = 2xy \quad \rightarrow \quad \dfrac{\partial M}{\partial y} = 2x$

$N = (1 + x^2) \quad \rightarrow \quad \dfrac{\partial N}{\partial x} = 2x$

$\therefore\ The\ D.E. is\ exact\ beccause\ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$2xydx + (1 + x^2)dy = 0$

$2xydx + dy + x^2dy = 0 \quad \rightarrow \quad 2xydx + x^2dy = -dy$

$d(x^2y) = -dy$

$\int d(x^2y) = \int -dy$

$x^2y = -y + c$

2- $(x + \sin y)dx + (x\cos y - 2y)dy = 0$

$M = x + \sin y \quad \rightarrow \quad \dfrac{\partial M}{\partial y} = \cos y$

$N = x\cos y - 2y \quad \rightarrow \quad \dfrac{\partial N}{\partial x} = \cos y$

$\therefore\ The\ D.E. is\ exact\ beccause\ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$(x+\sin y)dx+(x\cos y-2y)dy=0$$

$$xdx+\sin y\,dx+x\cos y\,dy-2ydy=0$$

$$d(x\sin y)=2ydy-xdx$$

$$\int d(x\sin y)=\int 2ydy-\int xdx$$

$$x\sin y=y^2-\frac{x^2}{2}+c$$

3- $(5y^4x^3-2y^7)dy-(x^7-3x^2y^5)dx=0$

$$(5y^4x^3-2y^7)dy+(3x^2y^5-x^7)dx=0$$

$$M=3x^2y^5-x^7 \quad\rightarrow\quad \frac{\partial M}{\partial y}=15x^2y^4$$

$$N=5y^4x^3-2y^7 \quad\rightarrow\quad \frac{\partial N}{\partial x}=15x^2y^4$$

$$\therefore The\ D.E. is\ exact\ beccause\ \ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$(5y^4x^3-2y^7)dy+(3x^2y^5-x^7)dx=0$$

$$5y^4x^3dy-2y^7dy+3x^2y^5dx-x^7dx=0$$

$$d(x^3y^5)=2y^7dy+x^7dx$$

$$\int d(x^3y^5)=\int 2y^7dy+\int x^7dx$$

$$x^3y^5=\frac{y^8}{4}+\frac{x^8}{8}+c$$

4- $\left(\frac{y^2}{1+x^2}-2y\right)dx+(2y\tan^{-1}x-2x+\sinh y)dy=0$

$$M = \frac{y^2}{1+x^2} - 2y \quad \rightarrow \quad \frac{\partial M}{\partial y} = \frac{2y}{1+x^2} - 2$$

$$N = 2y\tan^{-1}x - 2x + \sinh y \quad \rightarrow \quad \frac{\partial N}{\partial x} = \frac{2y}{1+x^2} - 2$$

$$\therefore \textit{The D.E. is exact beccause } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\left(\frac{y^2}{1+x^2} - 2y\right)dx + (2y\tan^{-1}x - 2x)dy + \sinh y\, dy = 0$$

$$d(y^2\tan^{-1}x - 2xy) + \sinh y\, dy = 0$$

$$\int d(y^2\tan^{-1}x - 2xy) + \int \sinh y\, dy = 0$$

$$y^2\tan^{-1}x - 2xy + \cosh y = c$$

5- $3x^2\sinh^{-1}y\, dx + \frac{dy}{1+y^2} + \frac{x^3}{\sqrt{1+y^2}}dy = 0$

$$3x^2\sinh^{-1}y\, dx + \left(\frac{1}{1+y^2} + \frac{x^3}{\sqrt{1+y^2}}\right)dy = 0$$

$$M = 3x^2\sinh^{-1}y \quad \rightarrow \quad \frac{\partial M}{\partial y} = \frac{3x^2}{\sqrt{1+y^2}}$$

$$N = \frac{1}{1+y^2} + \frac{x^3}{\sqrt{1+y^2}} \quad \rightarrow \quad \frac{\partial N}{\partial x} = \frac{3x^2}{\sqrt{1+y^2}}$$

$$\therefore \textit{The D.E. is exact beccause } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$d(x^3\sinh^{-1}y) + \frac{dy}{1+y^2} = 0$$

$$\int d(x^3\sinh^{-1}y) + \int \frac{dy}{1+y^2} = 0$$

$$x^3\sinh^{-1}y + \tan^{-1}y = c$$

4- Linear Differential Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad (1)$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**.

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

EXAMPLE Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left-hand side is } \frac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

EXAMPLE Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left-hand side is } vy.$$

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

In solving the linear equation in Example , we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example, by remembering that the left-hand side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation this means that

$$v(x)y = \int v(x)Q(x)\,dx.$$

We need only integrate the product of the integrating factor $v(x)$ with $Q(x)$ on the right-hand side of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable and can be solved by the method of **separable method**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) = 0$$

$$\frac{dy}{y} = -P(x)\,dx \qquad \text{Separating the variables}$$

Examples:

1- $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$$

$$I.F. = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$x^3\frac{dy}{dx} + 3x^2y = \sin x$$

$$x^3dy + 3x^2ydx = \sin x\, dx$$

$$d(x^3y) = \sin x\, dx$$

$$\int d(x^3y) = \int \sin x\, dx$$

$$x^3y = -\cos x + c$$

$$y = \frac{-\cos x}{x^3} + \frac{c}{x^3}$$

2- $t(1+t^2)dx = (x + xt^2 - t^2)dt \qquad (Note: x = f(t))$

$$\frac{dx}{dt} = \frac{x + xt^2 - t^2}{t(1+t^2)} = \frac{x(1+t^2) - t^2}{t(1+t^2)} = \frac{x(1+t^2)}{t(1+t^2)} - \frac{t^2}{t(1+t^2)} = \frac{x}{t} - \frac{t}{(1+t^2)}$$

$$\frac{dx}{dt} - \frac{1}{t}x = -\frac{t}{(1+t^2)}$$

$$I.F. = e^{\int P(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln t^{-1}} = \frac{1}{t}$$

$$\frac{1}{t} \cdot \frac{dx}{dt} - \frac{1}{t^2}x = -\frac{1}{(1+t^2)}$$

$$\frac{1}{t} \cdot dx - \frac{1}{t^2}xdt = -\frac{1}{(1+t^2)}dt$$

$$d(\frac{x}{t}) = -\frac{1}{(1+t^2)}dt$$

$$\int d(\frac{x}{t}) = \int -\frac{1}{(1+t^2)}dt$$

$$\frac{x}{t} = -\tan^{-1} t + c$$

$$x = t(-\tan^{-1} t + c)$$

3- $\frac{dy}{dx} = \frac{y}{3x-y^5}$

$$\frac{dx}{dy} = \frac{3x - y^5}{y} = \frac{3x}{y} - y^4$$

$$\frac{dx}{dy} - \frac{3}{y}x = -y^4$$

$$I.F. = e^{\int P(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = \frac{1}{y^3}$$

$$\frac{1}{y^3} \cdot \frac{dx}{dy} - \frac{3}{y^4}x = -y$$

$$\frac{1}{y^3}\cdot dx - \frac{3}{y^4}xdy = -ydy$$

$$d(\frac{x}{y^3}) = -ydy$$

$$\int d(\frac{x}{y^3}) = \int -ydy$$

$$\frac{x}{y^3} = -\frac{y^2}{2} + c$$

$$x = -\frac{y^5}{2} + cy^3$$

Homework:

1- $(x^2 + y^2)dx + (2xy + \cos y)dy = 0$

2- $\frac{dy}{dx} - 3y = 6$

3- $\frac{dy}{dx} + x^2y = x^2$

5- Bernoulli Differential Equations

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Examples:

1. $\frac{dy}{dx} - y = e^{-x}y^2$

$n = 2 \quad \rightarrow \quad u = y^{1-n} = y^{1-2} = y^{-1} \quad \rightarrow \quad y = u^{-1}$

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2} \qquad (\div\ -u^{-2})$$

$$\frac{du}{dx} + u = -e^{-x} \qquad (Is\ linear\ D.E.)$$

$$I.F. = e^{\int P(x)dx} = e^{\int dx} = e^{x}$$

$$e^{x}\frac{du}{dx} + e^{x}u = -e^{x}e^{-x}$$

$$e^{x}du + e^{x}udx = -dx$$

$$d(e^{x}u) = -dx$$

$$\int d(e^{x}u) = \int -dx$$

$$e^{x}u = -x + c$$

$$e^x \frac{1}{y} = -x + c \quad \rightarrow \quad y = \frac{e^x}{-x + c}$$

2. $\frac{dy}{dx} = \frac{x^3 y^4 - 2y}{x}$

$$\frac{dy}{dx} + \frac{2}{x} y = x^2 y^4$$

$$n = 4 \quad \rightarrow \quad u = y^{1-n} = y^{1-4} = y^{-3}$$

$$-3y^{-4} \frac{dy}{dx} = \frac{du}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{-3y^{-4}} \frac{du}{dx}$$

Substitution into the original equation gives

$$\frac{1}{-3y^{-4}} \frac{du}{dx} + \frac{2}{x} y = x^2 y^4 \qquad (* \ -3y^{-4})$$

$$\frac{du}{dx} - \frac{6}{x} y^{-3} = -3x^2$$

$$\frac{du}{dx} - \frac{6}{x} u = -3x^2 \qquad (Is\ linear\ D.E.)$$

$$I.F. = e^{\int P(x)dx} = e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = e^{\ln x^{-6}} = \frac{1}{x^6}$$

$$\frac{1}{x^6} \frac{du}{dx} - \frac{6}{x^7} u = -\frac{3}{x^4}$$

$$\frac{1}{x^6} du - \frac{6}{x^7} u dx = -\frac{3}{x^4} dx$$

$$d\left(\frac{u}{x^6}\right) = -\frac{3}{x^4} dx$$

$$\int d\left(\frac{u}{x^6}\right) = \int -\frac{3}{x^4} dx$$

$$\frac{u}{x^6} = x^{-3} + c \quad \rightarrow \quad \frac{y^{-3}}{x^6} = x^{-3} + c$$

$$y^{-3} = x^3(1 + cx^3)$$

3. $x\frac{dy}{dx} + y = y^{-2}$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$

$$n = -2 \quad \rightarrow \quad u = y^{1-n} = y^{1-(-2)} = y^3 \quad \rightarrow \quad y = u^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3}u^{-2/3}\frac{du}{dx}$$

Substitution into the original equation gives

$$\frac{1}{3}u^{-2/3}\frac{du}{dx} + \frac{1}{x}u^{1/3} = \frac{1}{x}u^{-2/3}$$

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x}$$

$$I.F. = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$x^3\frac{du}{dx} + 3x^2u = 3x^2$$

$$x^3du + 3x^2udx = 3x^2dx$$

$$d(x^3u) = 3x^2dx$$

$$\int d(x^3u) = \int 3x^2dx$$

$$x^3u = x^3 + c$$

$$x^3y^3 = x^3 + c$$

Homework:

1- $\frac{dy}{dx} - y = -y^2$

2- $\frac{dy}{dx} - y = xy^2$

Second – Order Differential Equations

An equation of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \qquad (1)$$

which is linear in y and its derivatives, is called a **second-order linear differential equation**.

1. If $G(x) = 0$, the equation is said to be **Homogeneous** as following

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (2)$$

2. If $G(x) \neq 0$, the equation is called **Nonhomogeneous**

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

---

1. Second – Order Linear Homogeneous Differential Equations

Two fundamental results are important to solving Equation (2). The first of these says that if we know two solutions y_1 and y_2 of the linear homogeneous equation, then any **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

THEOREM 1—The Superposition Principle If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants c_1 and c_2, the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0, \qquad (3)$$

where a, b, and c are constants. To solve Equation (3), we seek a function which when multiplied by a constant and added to a constant times its first derivative plus a constant times its second derivative sums identically to zero. One function that behaves this way is the exponential function $y = e^{rx}$, when r is a constant. Two differentiations of this exponential function give $y' = re^{rx}$ and $y'' = r^2e^{rx}$, which are just constant multiples of the original exponential. If we substitute $y = e^{rx}$ into Equation (3), we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since the exponential function is never zero, we can divide this last equation through by e^{rx}. Thus, $y = e^{rx}$ is a solution to Equation (3) if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0. \qquad (4)$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider which depend on the value of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2. Then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions to Equation (3) because e^{r_2x} is not a constant multiple of e^{r_1x} (see Exercise 61). From Theorem 2 we conclude the following result.

THEOREM 1 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 \neq r_2$ → *Then the solution is*: $y = c_1e^{r_1x} + c_2e^{r_2x}$

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1e^{3x} + c_2e^{-2x}.$$ ■

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$. To simplify the notation, let $r = -b/2a$. Then we have one solution $y_1 = e^{rx}$ with $2ar + b = 0$. Since multiplication of e^{rx} by a constant fails to produce a second linearly independent solution, suppose we try multiplying by a *function* instead. The simplest such function would be $u(x) = x$, so let's see if $y_2 = xe^{rx}$ is also a solution. Substituting y_2 into the differential equation gives

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + (0)xe^{rx} = 0. \end{aligned}$$

The first term is zero because $r = -b/2a$; the second term is zero because r solves the auxiliary equation. The functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent (see Exercise 62). From Theorem 2 we conclude the following result.

THEOREM 2 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 = r_2 = r$ → *Then the solution is*: $y = c_1e^{rx} + c_2xe^{rx}$

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}.$$ ■

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.) These two complex roots then give rise to two linearly independent solutions

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

(The expressions involving the sine and cosine terms follow from Euler's identity in Section 8.9.) However, the solutions y_1 and y_2 are *complex valued* rather than real valued. Nevertheless, because of the superposition principle (Theorem 1), we can obtain from them the two real-valued solutions

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\alpha x}\cos\beta x \quad \text{and} \quad y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = e^{\alpha x}\sin\beta x.$$

The functions y_3 and y_4 are linearly independent (see Exercise 63). From Theorem 2 we conclude the following result.

THEOREM 3 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta \rightarrow$ Then the solution is:

$$y = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)$$

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$ ■

Examples: Find the general solution of the following differential equations:

1. $y'' - y' - 2y = 0$

Let $y = e^{rx} \rightarrow y' = re^{rx} \rightarrow y'' = r^2e^{rx}$ *sub in original Eq.*

$r^2e^{rx} - re^{rx} - 2e^{rx} = 0 \qquad (\div e^{rx})$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \quad and \quad r_2 = -1 \qquad (r_1 \neq r_2)$

$y = c_1 e^{2x} + c_2 e^{-x}$

2. $y'' - 6y' + 9y = 0$

Let $y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx} \quad sub\ in\ original\ Eq.$

$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0 \qquad (\div e^{rx})$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = 3 \quad and \quad r_2 = 3 \qquad (r_1 = r_2 = r)$

$y = c_1 e^{3x} + c_2 x e^{3x}$

3. $y'' + 4y' + 5y = 0$

Let $y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx} \quad sub\ in\ original\ Eq.$

$r^2 e^{rx} + 4re^{rx} + 5e^{rx} = 0 \qquad (\div e^{rx})$

$r^2 + 4r + 5 = 0$

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad r = \frac{-4 \mp \sqrt{4^2 - 4 * 1 * 5}}{2 * 1}$$

$r_1 = -2 + i \quad and \quad r_2 = -2 - i \quad \rightarrow \quad \alpha = -2 \quad and \quad \beta = 1$

$y = e^{-2x}(c_1 \sin x + c_2 \cos x)$

Homework:

1. $y'' + 4y = 0$

Initial Value and Boundary Value Problems

The general solution to a second-order equation contains two arbitrary constants, it is necessary to specify two conditions. These conditions are called **initial conditions**.

THEOREM If P, Q, R, and G are continuous throughout an open interval I, then there exists one and only one function $y(x)$ satisfying both the differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x)$$

on the interval I, and the initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

at the specified point $x_0 \in I$.

EXAMPLE Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1e^x + c_2xe^x.$$

Then,

$$y' = c_1e^x + c_2(x + 1)e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \quad \text{and} \quad -1 = c_1 + c_2 \cdot 1.$$

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

Another approach to determine the values of the two arbitrary constants in the general solution to a second-order differential equation is to specify the values of the solution function at *two different points* in the interval I. That is, we solve the differential equation subject to the **boundary values**

EXAMPLE Solve the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

Solution The auxiliary equation is $r^2 + 4 = 0$, which has the complex roots $r = \pm 2i$. The general solution to the differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The boundary conditions are satisfied if

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

It follows that $c_1 = 0$ and $c_2 = 2$. The solution to the boundary value problem is

$$y = 2 \sin 2x.$$ ■

Nonhomogeneous Linear Differential Equations

Two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \tag{1}$$

where a, b, and c are constants and G is continuous over some open interval I. Let $y_c = c_1 y_1 + c_2 y_2$ be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0. \tag{2}$$

(We learned how to find y_c in Section 16.1.) Now suppose we could somehow come up with a particular function y_p that solves the nonhomogeneous equation (1). Then the sum

$$y = y_c + y_p \tag{3}$$

also solves the nonhomogeneous equation (1) because

$$
\begin{aligned}
a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p) \\
&= (ay_c'' + by_c' + cy_c) + (ay_p'' + by_p' + cy_p) \\
&= 0 + G(x) \qquad y_c \text{ solves Eq. (2) and } y_p \text{ solves Eq. (1)} \\
&= G(x).
\end{aligned}
$$

THEOREM The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2) and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

This method for finding a particular solution y_p to nonhomogeneous equation (1) applies to special cases for which $G(x)$ is a sum of terms of various polynomials $p(x)$ multiplying an exponential with possibly sine or cosine factors. That is, $G(x)$ is a sum of terms of the following forms:

$$p_1(x)e^{rx}, \qquad p_2(x)e^{\alpha x}\cos\beta x, \qquad p_3(x)e^{\alpha x}\sin\beta x.$$

For instance, $1 - x$, e^{2x}, xe^x, $\cos x$, and $5e^x - \sin 2x$ represent functions in this category.

You may find the following table helpful in solving the problems at the end of this section.

TABLE 16.1 The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}

$\sin kx$, $\cos kx$	k is not a root of the auxiliary equation	$B \cos kx + C \sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

***Example*:** Solve the following nonhomogeneous differential equation:

$y'' - 4y = \sin x$

Step 1: Find y_c $\quad\rightarrow\quad y'' - 4y = 0$

$r^2 - 4 = 0$

$(r-2)(r+2) = 0$

$r_1 = 2 \;\; ; \quad r_2 = -2$

$\therefore \;\; y_c = c_1 e^{2x} + c_2 e^{-2x}$

Step 2: Find y_p

$G(x) = \sin x \quad$ *and there is no similarity between* y_c *and* $G(x)$, *therefore*:

$y_p = A \sin x + B \cos x$

$y_p' = A \cos x - B \sin x$

$y_p'' = -A \sin x - B \cos x$

Substituting y_p, y_p', y_p'' *in the original equation* $(y'' - 4y = \sin x)$, *we get*

$-A \sin x - B \cos x - 4(A \sin x + B \cos x) = \sin x$

$-5A \sin x - 5B \cos x = \sin x$

- $\sin x \quad \rightarrow \quad -5A = 1 \quad \rightarrow \quad A = -\frac{1}{5}$
- $\cos x \quad \rightarrow \quad -5B = 0 \quad \rightarrow \quad B = 0$

$\therefore y_p = -\frac{1}{5}\sin x$

$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5}\sin x$

***Example*:** Solve the following nonhomogeneous differential equation:

$y'' - y' - 2y = e^{2x}$

Step 1: Find $y_c \quad \rightarrow \quad y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \;\; ; \quad r_2 = -1$

$\therefore \; y_c = c_1 e^{2x} + c_2 e^{-x}$

Step 2: Find y_p

$G(x) = e^{2x}$ *and there is similarity between* y_c *and* $G(x)$, *therefore*:

Note that: If we assume

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

Substituting y_p, y_p', y_p'' ***in the original equation*** $(y'' - y' - 2y = e^{2x})$, ***we get***

$4Ae^{2x} - 2Ae^{2x} - 2Ae^{2x} = e^{2x}$

$0 = e^{2x}$ *the assumption is not correct, thus, we assume* $y_p = Axe^{2x}$

$y_p = Axe^{2x}$

$y_p' = 2Axe^{2x} + Ae^{2x}$

$y_p'' = 4Axe^{2x} + 4Ae^{2x}$

Substituting $\boldsymbol{y_p, y_p', y_p''}$ ***in the original equation*** $(\boldsymbol{y'' - y' - 2y = e^{2x}})$***, we get***

$4Axe^{2x} + 4Ae^{2x} - (2Axe^{2x} + Ae^{2x}) - 2(Axe^{2x}) = e^{2x}$

$3Ae^{2x} = e^{2x} \quad \rightarrow \quad A = \frac{1}{3}$

$\therefore y_p = \frac{1}{3}xe^{2x}$

$y = y_c + y_p = c_1e^{2x} + c_2e^{-x} + \frac{1}{3}xe^{2x}$

***Example*:** Solve the following nonhomogeneous differential equation:

$y'' + 2y' - 3y = 6$

Step 1: Find $y_c \quad \rightarrow \quad y'' + 2y' - 3y = 0$

$r^2 + 2r - 3 = 0$

$(r - 1)(r + 3) = 0$

$r_1 = 1 \;\; ; \quad r_2 = -3$

$\therefore \;\; y_c = c_1e^{x} + c_2e^{-3x}$

Step 2: Find y_p

$G(x) = 6 \;\; \textit{and there is no similarity between } y_c \textit{ and } G(x), \textit{therefore}:$

$y_p = A$

$y_p' = 0$

$y_p'' = 0$

Substituting y_p, y_p', y_p'' *in the original equation* $(y'' + 2y' - 3y = 6)$, *we get*

$$0 + 2 * 0 - 3A = 6$$

$$A = -2$$

$$\therefore y_p = -2$$

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-3x} - 2$$

Example: Solve the following nonhomogeneous differential equation:

$$y'' - 6y' + 9y = e^{3x}$$

Step 1: Find y_c $\rightarrow$ $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = r_2 = 3$

$\therefore$ $y_c = c_1 e^{3x} + c_2 x e^{3x}$

Step 2: Find y_p

$G(x) = e^{2x}$ *and there is similarity between* y_c *and* $G(x)$, *therefore*:

has $r = 3$ as a repeated root. The appropriate choice for y_p in this case is neither Ae^{3x} nor Axe^{3x} because the complementary solution contains both of those terms already. Thus, we choose a term containing the next higher power of x as a factor. When we substitute

$$y_p = Ax^2e^{3x}$$

and its derivatives in the given differential equation, we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$$

or

$$2Ae^{3x} = e^{3x}.$$

Thus, $A = 1/2$, and the particular solution is

$$y_p = \frac{1}{2}x^2e^{3x}. \quad ■$$

$$y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{2}x^2 e^{3x}$$

Solving nonhomogeneous D.E by the Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation (1) once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring (in a way to be explained) that the

resulting expression satisfy the nonhomogeneous equation (1). There are two functions to be determined, and requiring that Equation (1) be satisfied is only one condition. As a second condition, we also require that

$$v_1'y_1 + v_2'y_2 = 0. \tag{4}$$

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with the Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation
$$ay'' + by' + cy = 0$$
to find the functions y_1 and y_2.
2. Solve the equations
$$v_1'y_1 + v_2'y_2 = 0,$$
$$v_1'y_1' + v_2'y_2' = \frac{G(x)}{a}$$
simultaneously for the derivative functions v_1' and v_2'.
3. Integrate v_1' and v_2' to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.
4. Write down the particular solution to nonhomogeneous equation (1) as
$$y_p = v_1y_1 + v_2y_2.$$

Example: Solve $y'' - y' - 2y = e^{2x}$

Step 1: Find y_c $\rightarrow$ $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \ ; \quad r_2 = -1$

$\therefore \ y_c = c_1 e^{2x} + c_2 e^{-x}$

Step 2: Find y_p

$y_p = v_1 e^{2x} + v_2 e^{-x}$

$v_1' e^{2x} + v_2' e^{-x} = 0 \qquad \dots \dots \dots (1)$

$v_1' 2e^{2x} + v_2'(-e^{-x}) = e^{2x} \qquad \dots \dots \dots (2) \qquad Note\ (a = 1)$

$3v_1' e^{2x} = e^{2x}$

$v_1' = \frac{1}{3} \qquad \rightarrow \qquad v_1 = \int \frac{1}{3} dx = \frac{x}{3}$

$\frac{1}{3} e^{2x} + v_2' e^{-x} = 0 \qquad \rightarrow \qquad v_2' = -\frac{1}{3} e^{3x}$

$v_2 = \int -\frac{1}{3} e^{3x}\, dx = -\frac{1}{9} e^{3x}$

$y_p = \frac{x}{3} e^{2x} - \frac{1}{9} e^{3x} e^{-x}$

$y_p = \frac{x}{3} e^{2x} - \frac{1}{9} e^{2x}$

$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{x}{3} e^{2x} - \frac{1}{9} e^{2x}$

Example: Solve $y'' + y = \sec x$

Step 1: Find $y_c \qquad \rightarrow \qquad y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r_1 = i \ ; \ r_2 = -i \qquad (\alpha = 0 \ ; \ \beta = 1)$

$\therefore \ y_c = e^0 (c_1 \sin x + c_2 \cos x) = c_1 \sin x + c_2 \cos x$

Step 2: Find y_p

$$y_p = v_1 \sin x + v_2 \cos x$$

$$v_1' \sin x + v_2' \cos x = 0 \qquad \dots\dots\dots(1)$$

$$v_1' \cos x + v_2'(-\sin x) = \sec x \qquad \dots\dots\dots(2) \qquad \color{red}{Note\ (a = 1)}$$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \sec x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$$v_1 = \int 1\, dx = x$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \sec x}{-\sin^2 x - \cos^2 x} = \frac{-\sin x}{\cos x}$$

$$v_2 = \int \frac{-\sin x}{\cos x} dx = \ln|\cos x|$$

$$y_p = x \sin x + \cos x \ln|\cos x|$$

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|$$

$$y = (c_1 + x) \sin x + \cos x\,(c_2 + \ln|\cos x|)$$

EXAMPLE Find the general solution to the equation

$$y'' + y = \tan x.$$

Solution The solution of the homogeneous equation

$$y'' + y = 0$$

is given by

$$y_c = c_1 \cos x + c_2 \sin x.$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$, the conditions to be satisfied in Equations (4) and (5) are

$$v_1' \cos x + v_2' \sin x = 0,$$

$$-v_1' \sin x + v_2' \cos x = \tan x. \qquad a = 1$$

Solution of this system gives

$$v_1{}' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}.$$

Likewise,

$$v_2{}' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

After integrating $v_1{}'$ and $v_2{}'$, we have

$$\begin{aligned} v_1(x) &= \int \frac{-\sin^2 x}{\cos x}\,dx \\ &= -\int (\sec x - \cos x)\,dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2(x) = \int \sin x\,dx = -\cos x.$$

Note that we have omitted the constants of integration in determining v_1 and v_2. They would merely be absorbed into the arbitrary constants in the complementary solution.

Substituting v_1 and v_2 into the expression for y_p in Step 4 gives

$$\begin{aligned} y_p &= [-\ln|\sec x + \tan x| + \sin x]\cos x + (-\cos x)\sin x \\ &= (-\cos x)\ln|\sec x + \tan x|. \end{aligned}$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x)\ln|\sec x + \tan x|.$$ ■

EXAMPLE Solve the nonhomogeneous equation

$$y'' + y' - 2y = xe^x.$$

Solution The auxiliary equation is

$$r^2 + r - 2 = (r + 2)(r - 1) = 0$$

giving the complementary solution

$$y_c = c_1 e^{-2x} + c_2 e^x.$$

The conditions to be satisfied in Equations (4) and (5) are

$$v_1' e^{-2x} + v_2' e^{x} = 0,$$
$$-2v_1' e^{-2x} + v_2' e^{x} = xe^{x}. \qquad a = 1$$

Solving the above system for v_1' and v_2' gives

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}} = \frac{-xe^{2x}}{3e^{-x}} = -\frac{1}{3}xe^{3x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix}}{3e^{-x}} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}.$$

Integrating to obtain the parameter functions, we have

$$v_1(x) = \int -\frac{1}{3}xe^{3x}\,dx$$
$$= -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx\right)$$
$$= \frac{1}{27}(1 - 3x)e^{3x},$$

and

$$v_2(x) = \int \frac{x}{3}\,dx = \frac{x^2}{6}.$$

Therefore,

$$y_p = \left[\frac{(1 - 3x)e^{3x}}{27}\right]e^{-2x} + \left(\frac{x^2}{6}\right)e^{x}$$
$$= \frac{1}{27}e^{x} - \frac{1}{9}xe^{x} + \frac{1}{6}x^2e^{x}.$$

The general solution to the differential equation is

$$y = c_1e^{-2x} + c_2e^{x} - \frac{1}{9}xe^{x} + \frac{1}{6}x^2e^{x},$$

where the term $(1/27)e^x$ in y_p has been absorbed into the term c_2e^x in the complementary solution. ■

EXAMPLE Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$ giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A, B, and C. When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

or, collecting terms with like powers of x,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$ ■

EXAMPLE Find a particular solution of $y'' - y' = 2 \sin x$.

Solution If we try to find a particular solution of the form

$$y_p = A \sin x$$

and substitute the derivatives of y_p in the given equation, we find that A must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of x. Since this requires A to equal both -2 and 0 at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form $A \sin x$.

It turns out that the required form is the sum

$$y_p = A \sin x + B \cos x.$$

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x$$

or

$$(B - A) \sin x - (A + B) \cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \quad \text{and} \quad A + B = 0.$$

Simultaneous solution of these two equations gives $A = -1$ and $B = 1$. Our particular solution is

$$y_p = \cos x - \sin x.$$ ■

EXAMPLE Find a particular solution of $y'' - 3y' + 2y = 5e^x$.

Solution If we substitute

$$y_p = Ae^x$$

and its derivatives in the differential equation, we find that

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

or

$$0 = 5e^x.$$

However, the exponential function is never zero. The trouble can be traced to the fact that $y = e^x$ is already a solution of the related homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0,$$

which has $r = 1$ as a root. So we would expect Ae^x to become zero when substituted into the left-hand side of the differential equation.

The appropriate way to modify the trial solution in this case is to multiply Ae^x by x. Thus, our new trial solution is

$$y_p = Axe^x.$$

The result of substituting the derivatives of this new candidate into the differential equation is

$$(Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2Axe^x = 5e^x$$

or

$$-Ae^x = 5e^x.$$

Thus, $A = -5$ gives our sought-after particular solution

$$y_p = -5xe^x.$$ ■

EXAMPLE Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the auxiliary equation

$$r^2 - r = 0.$$

Its roots are $r = 1$ and $r = 0$. Therefore, the complementary solution to the associated homogeneous equation is

$$y_c = c_1e^x + c_2.$$

We now seek a particular solution y_p. That is, we seek a function that will produce $5e^x - \sin 2x$ when substituted into the left-hand side of the given differential equation. One part of y_p is to produce $5e^x$, the other $-\sin 2x$.

Since any function of the form c_1e^x is a solution of the associated homogeneous equation, we choose our trial solution y_p to be the sum

$$y_p = Axe^x + B\cos 2x + C\sin 2x,$$

including xe^x where we might otherwise have included only e^x. When the derivatives of y_p are substituted into the differential equation, the resulting equation is

$$(Axe^x + 2Ae^x - 4B\cos 2x - 4C\sin 2x)$$
$$- (Axe^x + Ae^x - 2B\sin 2x + 2C\cos 2x) = 5e^x - \sin 2x$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x.$$

This equation will hold if

$$A = 5, \qquad 4B + 2C = 0, \qquad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$. Our particular solution is

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

The general solution to the differential equation is

$$y = y_c + y_p = c_1e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$ ■

Second – Order Differential Equations

An equation of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \qquad (1)$$

which is linear in y and its derivatives, is called a **second-order linear differential equation**.

1. If $G(x) = 0$, the equation is said to be **Homogeneous** as following

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (2)$$

2. If $G(x) \neq 0$, the equation is called **Nonhomogeneous**

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

---

1. Second – Order Linear Homogeneous Differential Equations

Two fundamental results are important to solving Equation (2). The first of these says that if we know two solutions y_1 and y_2 of the linear homogeneous equation, then any **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

THEOREM 1—The Superposition Principle If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants c_1 and c_2, the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0, \qquad (3)$$

where a, b, and c are constants. To solve Equation (3), we seek a function which when multiplied by a constant and added to a constant times its first derivative plus a constant times its second derivative sums identically to zero. One function that behaves this way is the exponential function $y = e^{rx}$, when r is a constant. Two differentiations of this exponential function give $y' = re^{rx}$ and $y'' = r^2e^{rx}$, which are just constant multiples of the original exponential. If we substitute $y = e^{rx}$ into Equation (3), we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since the exponential function is never zero, we can divide this last equation through by e^{rx}. Thus, $y = e^{rx}$ is a solution to Equation (3) if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0. \tag{4}$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider which depend on the value of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2. Then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions to Equation (3) because $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$ (see Exercise 61). From Theorem 2 we conclude the following result.

THEOREM 1 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 \neq r_2$ → *Then the solution is*: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}.$$

■

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$. To simplify the notation, let $r = -b/2a$. Then we have one solution $y_1 = e^{rx}$ with $2ar + b = 0$. Since multiplication of e^{rx} by a constant fails to produce a second linearly independent solution, suppose we try multiplying by a *function* instead. The simplest such function would be $u(x) = x$, so let's see if $y_2 = xe^{rx}$ is also a solution. Substituting y_2 into the differential equation gives

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + (0)xe^{rx} = 0. \end{aligned}$$

The first term is zero because $r = -b/2a$; the second term is zero because r solves the auxiliary equation. The functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent (see Exercise 62). From Theorem 2 we conclude the following result.

THEOREM 2 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 = r_2 = r$ → *Then the solution is*: $y = c_1e^{rx} + c_2xe^{rx}$

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}.$$

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.) These two complex roots then give rise to two linearly independent solutions

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

(The expressions involving the sine and cosine terms follow from Euler's identity in Section 8.9.) However, the solutions y_1 and y_2 are *complex valued* rather than real valued. Nevertheless, because of the superposition principle (Theorem 1), we can obtain from them the two real-valued solutions

$$y_3 = \frac{1}{2} y_1 + \frac{1}{2} y_2 = e^{\alpha x} \cos \beta x \qquad \text{and} \qquad y_4 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2 = e^{\alpha x} \sin \beta x.$$

The functions y_3 and y_4 are linearly independent (see Exercise 63). From Theorem 2 we conclude the following result.

THEOREM 3 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

Note: If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ $\rightarrow$ *Then the solution is*:

$$y = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$ ■

Examples: Find the general solution of the following differential equations:

1. $y'' - y' - 2y = 0$

Let $y = e^{rx}$ $\rightarrow$ $y' = re^{rx}$ $\rightarrow$ $y'' = r^2 e^{rx}$ *sub in original Eq.*

$r^2 e^{rx} - re^{rx} - 2e^{rx} = 0$ $\qquad (\div e^{rx})$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \quad and \quad r_2 = -1 \qquad (r_1 \neq r_2)$

$y = c_1 e^{2x} + c_2 e^{-x}$

2. $y'' - 6y' + 9y = 0$

Let $y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx} \quad sub\ in\ original\ Eq.$

$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0 \qquad (\div\, e^{rx})$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = 3 \quad and \quad r_2 = 3 \qquad (r_1 = r_2 = r)$

$y = c_1 e^{3x} + c_2 x e^{3x}$

3. $y'' + 4y' + 5y = 0$

Let $y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx} \quad sub\ in\ original\ Eq.$

$r^2 e^{rx} + 4re^{rx} + 5e^{rx} = 0 \qquad (\div\, e^{rx})$

$r^2 + 4r + 5 = 0$

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad r = \frac{-4 \mp \sqrt{4^2 - 4 * 1 * 5}}{2 * 1}$$

$r_1 = -2 + i \quad and \quad r_2 = -2 - i \quad \rightarrow \quad \alpha = -2 \quad and \quad \beta = 1$

$y = e^{-2x}(c_1 \sin x + c_2 \cos x)$

Homework:

1. $y'' + 4y = 0$

Initial Value and Boundary Value Problems

The general solution to a second-order equation contains two arbitrary constants, it is necessary to specify two conditions. These conditions are called **initial conditions**.

THEOREM If P, Q, R, and G are continuous throughout an open interval I, then there exists one and only one function $y(x)$ satisfying both the differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x)$$

on the interval I, and the initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

at the specified point $x_0 \in I$.

EXAMPLE Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1 e^x + c_2 x e^x.$$

Then,

$$y' = c_1 e^x + c_2(x + 1)e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \quad \text{and} \quad -1 = c_1 + c_2 \cdot 1.$$

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

Another approach to determine the values of the two arbitrary constants in the general solution to a second-order differential equation is to specify the values of the solution function at *two different points* in the interval I. That is, we solve the differential equation subject to the **boundary values**

EXAMPLE Solve the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

Solution The auxiliary equation is $r^2 + 4 = 0$, which has the complex roots $r = \pm 2i$. The general solution to the differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The boundary conditions are satisfied if

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

It follows that $c_1 = 0$ and $c_2 = 2$. The solution to the boundary value problem is

$$y = 2 \sin 2x.$$

Nonhomogeneous Linear Differential Equations

Two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \tag{1}$$

where a, b, and c are constants and G is continuous over some open interval I. Let $y_c = c_1 y_1 + c_2 y_2$ be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0. \tag{2}$$

(We learned how to find y_c in Section 16.1.) Now suppose we could somehow come up with a particular function y_p that solves the nonhomogeneous equation (1). Then the sum

$$y = y_c + y_p \tag{3}$$

also solves the nonhomogeneous equation (1) because

$$\begin{aligned} a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p) &= (ay_c'' + by_c' + cy_c) + (ay_p'' + by_p' + cy_p) \\ &= 0 + G(x) \qquad y_c \text{ solves Eq. (2) and } y_p \text{ solves Eq. (1)} \\ &= G(x). \end{aligned}$$

THEOREM The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2) and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

This method for finding a particular solution y_p to nonhomogeneous equation (1) applies to special cases for which $G(x)$ is a sum of terms of various polynomials $p(x)$ multiplying an exponential with possibly sine or cosine factors. That is, $G(x)$ is a sum of terms of the following forms:

$$p_1(x)e^{rx}, \qquad p_2(x)e^{\alpha x}\cos\beta x, \qquad p_3(x)e^{\alpha x}\sin\beta x.$$

For instance, $1 - x$, e^{2x}, xe^x, $\cos x$, and $5e^x - \sin 2x$ represent functions in this category.

You may find the following table helpful in solving the problems at the end of this section.

TABLE 16.1 The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}

$\sin kx$, $\cos kx$	k is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

***Example*:** Solve the following nonhomogeneous differential equation:

$y'' - 4y = \sin x$

Step 1: Find y_c $\quad \rightarrow \quad y'' - 4y = 0$

$r^2 - 4 = 0$

$(r-2)(r+2) = 0$

$r_1 = 2 \;\; ; \;\; r_2 = -2$

$\therefore \;\; y_c = c_1 e^{2x} + c_2 e^{-2x}$

Step 2: Find y_p

$G(x) = \sin x \quad$ *and there is no similarity between* y_c *and* $G(x)$, *therefore*:

$y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$

$y_p'' = -A\sin x - B\cos x$

Substituting y_p, y_p', y_p'' *in the original equation* $(y'' - 4y = \sin x)$, *we get*

$-A\sin x - B\cos x - 4(A\sin x + B\cos x) = \sin x$

$-5A\sin x - 5B\cos x = \sin x$

- $\sin x \quad \rightarrow \quad -5A = 1 \quad \rightarrow \quad A = -\frac{1}{5}$
- $\cos x \quad \rightarrow \quad -5B = 0 \quad \rightarrow \quad B = 0$

$\therefore y_p = -\frac{1}{5}\sin x$

$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5}\sin x$

***Example*:** Solve the following nonhomogeneous differential equation:

$y'' - y' - 2y = e^{2x}$

Step 1: Find $y_c \quad \rightarrow \quad y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \;\; ; \quad r_2 = -1$

$\therefore \; y_c = c_1 e^{2x} + c_2 e^{-x}$

Step 2: Find y_p

$G(x) = e^{2x} \quad and\ there\ is\ similarity\ between\ y_c\ and\ G(x), therefore:$

$Note\ that: If\ we\ assume$

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$\boldsymbol{Substituting\ \ y_p, y_p', y_p''\ in\ the\ original\ equation\ (y'' - y' - 2y = e^{2x}), we\ get}$

$4Ae^{2x} - 2Ae^{2x} - 2Ae^{2x} = e^{2x}$

$0 = e^{2x} \qquad the\ assumption\ is\ not\ correct, thus, we\ assume\ y_p = Axe^{2x}$

$y_p = Axe^{2x}$

$y_p' = 2Axe^{2x} + Ae^{2x}$

$y_p'' = 4Axe^{2x} + 4Ae^{2x}$

Substituting $\boldsymbol{y_p, y_p', y_p''}$ ***in the original equation*** $(\boldsymbol{y'' - y' - 2y = e^{2x}})$, ***we get***

$4Axe^{2x} + 4Ae^{2x} - (2Axe^{2x} + Ae^{2x}) - 2(Axe^{2x}) = e^{2x}$

$3Ae^{2x} = e^{2x} \quad \rightarrow \quad A = \dfrac{1}{3}$

$\therefore y_p = \dfrac{1}{3}xe^{2x}$

$y = y_c + y_p = c_1e^{2x} + c_2e^{-x} + \dfrac{1}{3}xe^{2x}$

Example: Solve the following nonhomogeneous differential equation:

$y'' + 2y' - 3y = 6$

Step 1: Find $y_c \quad \rightarrow \quad y'' + 2y' - 3y = 0$

$r^2 + 2r - 3 = 0$

$(r - 1)(r + 3) = 0$

$r_1 = 1 \;\; ; \quad r_2 = -3$

$\therefore \;\; y_c = c_1e^{x} + c_2e^{-3x}$

Step 2: Find y_p

$G(x) = 6$ *and there is no similarity between* y_c *and* $G(x)$, *therefore*:

$y_p = A$

$y_p' = 0$

$y_p'' = 0$

Substituting y_p, y_p', y_p'' *in the original equation* $(y'' + 2y' - 3y = 6)$, *we get*

$$0 + 2 * 0 - 3A = 6$$

$$A = -2$$

$$\therefore y_p = -2$$

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-3x} - 2$$

Example: Solve the following nonhomogeneous differential equation:

$$y'' - 6y' + 9y = e^{3x}$$

Step 1: Find y_c $\rightarrow$ $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$$r_1 = r_2 = 3$$

$$\therefore \; y_c = c_1 e^{3x} + c_2 x e^{3x}$$

Step 2: Find y_p

$G(x) = e^{2x}$ *and there is similarity between* y_c *and* $G(x)$, *therefore*:

has $r = 3$ as a repeated root. The appropriate choice for y_p in this case is neither Ae^{3x} nor Axe^{3x} because the complementary solution contains both of those terms already. Thus, we choose a term containing the next higher power of x as a factor. When we substitute

$$y_p = Ax^2 e^{3x}$$

and its derivatives in the given differential equation, we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$$

or

$$2Ae^{3x} = e^{3x}.$$

Thus, $A = 1/2$, and the particular solution is

$$y_p = \frac{1}{2}x^2e^{3x}. \quad ■$$

$$y = y_c + y_p = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{2}x^2 e^{3x}$$

Solving nonhomogeneous D.E by the Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation (1) once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring (in a way to be explained) that the

resulting expression satisfy the nonhomogeneous equation (1). There are two functions to be determined, and requiring that Equation (1) be satisfied is only one condition. As a second condition, we also require that

$$v_1'y_1 + v_2'y_2 = 0. \tag{4}$$

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with the Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation
$$ay'' + by' + cy = 0$$
to find the functions y_1 and y_2.
2. Solve the equations
$$v_1'y_1 + v_2'y_2 = 0,$$
$$v_1'y_1' + v_2'y_2' = \frac{G(x)}{a}$$
simultaneously for the derivative functions v_1' and v_2'.
3. Integrate v_1' and v_2' to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.
4. Write down the particular solution to nonhomogeneous equation (1) as
$$y_p = v_1y_1 + v_2y_2.$$

Example: Solve $y'' - y' - 2y = e^{2x}$

Step 1: Find y_c $\rightarrow$ $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \;\; ; \;\; r_2 = -1$

$\therefore \quad y_c = c_1 e^{2x} + c_2 e^{-x}$

Step 2: Find y_p

$y_p = v_1 e^{2x} + v_2 e^{-x}$

$v_1' e^{2x} + v_2' e^{-x} = 0 \qquad \dots\dots\dots(1)$

$v_1' 2e^{2x} + v_2'(-e^{-x}) = e^{2x} \qquad \dots\dots\dots(2) \qquad Note\ (a = 1)$

$3v_1' e^{2x} = e^{2x}$

$$v_1' = \frac{1}{3} \quad \rightarrow \quad v_1 = \int \frac{1}{3} dx = \frac{x}{3}$$

$$\frac{1}{3} e^{2x} + v_2' e^{-x} = 0 \quad \rightarrow \quad v_2' = -\frac{1}{3} e^{3x}$$

$$v_2 = \int -\frac{1}{3} e^{3x}\, dx = -\frac{1}{9} e^{3x}$$

$$y_p = \frac{x}{3} e^{2x} - \frac{1}{9} e^{3x} e^{-x}$$

$$y_p = \frac{x}{3} e^{2x} - \frac{1}{9} e^{2x}$$

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{x}{3} e^{2x} - \frac{1}{9} e^{2x}$$

Example: Solve $y'' + y = \sec x$

Step 1: Find $y_c \quad \rightarrow \quad y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r_1 = i \ ; \ r_2 = -i \qquad (\alpha = 0\ ;\ \beta = 1)$

$\therefore \quad y_c = e^0 (c_1 \sin x + c_2 \cos x) = c_1 \sin x + c_2 \cos x$

Step 2: Find y_p

$$y_p = v_1 \sin x + v_2 \cos x$$

$$v_1' \sin x + v_2' \cos x = 0 \qquad \dots\dots\dots (1)$$

$$v_1' \cos x + v_2'(-\sin x) = \sec x \qquad \dots\dots\dots (2) \qquad \textit{Note } (a = 1)$$

$$v_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \sec x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$$v_1 = \int 1\, dx = x$$

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \sec x}{-\sin^2 x - \cos^2 x} = \frac{-\sin x}{\cos x}$$

$$v_2 = \int \frac{-\sin x}{\cos x} dx = \ln|\cos x|$$

$$y_p = x \sin x + \cos x \ln|\cos x|$$

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|$$

$$y = (c_1 + x) \sin x + \cos x\,(c_2 + \ln|\cos x|)$$

EXAMPLE Find the general solution to the equation

$$y'' + y = \tan x.$$

Solution The solution of the homogeneous equation

$$y'' + y = 0$$

is given by

$$y_c = c_1 \cos x + c_2 \sin x.$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$, the conditions to be satisfied in Equations (4) and (5) are

$$v_1' \cos x + v_2' \sin x = 0,$$

$$-v_1' \sin x + v_2' \cos x = \tan x. \qquad a = 1$$

Solution of this system gives

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

After integrating v_1' and v_2', we have

$$\begin{aligned} v_1(x) &= \int \frac{-\sin^2 x}{\cos x}\,dx \\ &= -\int (\sec x - \cos x)\,dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2(x) = \int \sin x\,dx = -\cos x.$$

Note that we have omitted the constants of integration in determining v_1 and v_2. They would merely be absorbed into the arbitrary constants in the complementary solution.

Substituting v_1 and v_2 into the expression for y_p in Step 4 gives

$$\begin{aligned} y_p &= [-\ln|\sec x + \tan x| + \sin x]\cos x + (-\cos x)\sin x \\ &= (-\cos x)\ln|\sec x + \tan x|. \end{aligned}$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x)\ln|\sec x + \tan x|.$$ ■

EXAMPLE Solve the nonhomogeneous equation

$$y'' + y' - 2y = xe^x.$$

Solution The auxiliary equation is

$$r^2 + r - 2 = (r + 2)(r - 1) = 0$$

giving the complementary solution

$$y_c = c_1 e^{-2x} + c_2 e^x.$$

The conditions to be satisfied in Equations (4) and (5) are

$$v_1{}'e^{-2x} + v_2{}'e^{x} = 0,$$
$$-2v_1{}'e^{-2x} + v_2{}'e^{x} = xe^{x}. \qquad a = 1$$

Solving the above system for $v_1{}'$ and $v_2{}'$ gives

$$v_1{}' = \frac{\begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}} = \frac{-xe^{2x}}{3e^{-x}} = -\frac{1}{3}xe^{3x}.$$

Likewise,

$$v_2{}' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix}}{3e^{-x}} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}.$$

Integrating to obtain the parameter functions, we have

$$v_1(x) = \int -\frac{1}{3}xe^{3x}\,dx$$
$$= -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx\right)$$
$$= \frac{1}{27}(1 - 3x)e^{3x},$$

and

$$v_2(x) = \int \frac{x}{3}\,dx = \frac{x^2}{6}.$$

Therefore,

$$y_p = \left[\frac{(1 - 3x)e^{3x}}{27}\right]e^{-2x} + \left(\frac{x^2}{6}\right)e^{x}$$
$$= \frac{1}{27}e^{x} - \frac{1}{9}xe^{x} + \frac{1}{6}x^2e^{x}.$$

The general solution to the differential equation is

$$y = c_1e^{-2x} + c_2e^{x} - \frac{1}{9}xe^{x} + \frac{1}{6}x^2e^{x},$$

where the term $(1/27)e^x$ in y_p has been absorbed into the term c_2e^x in the complementary solution. ■

EXAMPLE Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$ giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A, B, and C. When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

or, collecting terms with like powers of x,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$ ■

EXAMPLE Find a particular solution of $y'' - y' = 2 \sin x$.

Solution If we try to find a particular solution of the form

$$y_p = A \sin x$$

and substitute the derivatives of y_p in the given equation, we find that A must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of x. Since this requires A to equal both -2 and 0 at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form $A \sin x$.

It turns out that the required form is the sum

$$y_p = A \sin x + B \cos x.$$

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x$$

or

$$(B - A) \sin x - (A + B) \cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \quad \text{and} \quad A + B = 0.$$

Simultaneous solution of these two equations gives $A = -1$ and $B = 1$. Our particular solution is

$$y_p = \cos x - \sin x.$$ ■

EXAMPLE Find a particular solution of $y'' - 3y' + 2y = 5e^x$.

Solution If we substitute

$$y_p = Ae^x$$

and its derivatives in the differential equation, we find that

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

or

$$0 = 5e^x.$$

However, the exponential function is never zero. The trouble can be traced to the fact that $y = e^x$ is already a solution of the related homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0,$$

which has $r = 1$ as a root. So we would expect Ae^x to become zero when substituted into the left-hand side of the differential equation.

The appropriate way to modify the trial solution in this case is to multiply Ae^x by x. Thus, our new trial solution is

$$y_p = Axe^x.$$

The result of substituting the derivatives of this new candidate into the differential equation is

$$(Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2Axe^x = 5e^x$$

or

$$-Ae^x = 5e^x.$$

Thus, $A = -5$ gives our sought-after particular solution

$$y_p = -5xe^x.$$ ■

EXAMPLE Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the auxiliary equation

$$r^2 - r = 0.$$

Its roots are $r = 1$ and $r = 0$. Therefore, the complementary solution to the associated homogeneous equation is

$$y_c = c_1e^x + c_2.$$

We now seek a particular solution y_p. That is, we seek a function that will produce $5e^x - \sin 2x$ when substituted into the left-hand side of the given differential equation. One part of y_p is to produce $5e^x$, the other $-\sin 2x$.

Since any function of the form c_1e^x is a solution of the associated homogeneous equation, we choose our trial solution y_p to be the sum

$$y_p = Axe^x + B\cos 2x + C\sin 2x,$$

including xe^x where we might otherwise have included only e^x. When the derivatives of y_p are substituted into the differential equation, the resulting equation is

$$(Axe^x + 2Ae^x - 4B\cos 2x - 4C\sin 2x) - (Axe^x + Ae^x - 2B\sin 2x + 2C\cos 2x) = 5e^x - \sin 2x$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x.$$

This equation will hold if

$$A = 5, \qquad 4B + 2C = 0, \qquad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$. Our particular solution is

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

The general solution to the differential equation is

$$y = y_c + y_p = c_1e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$ ■