Domain and Range

Domain :- هو المجال ويمثل قيم x التي تجعل قيمة y حقيقية .

$$y = F(x)$$

Range :- هو المجال المقابل ويمثل قيم y التي تجعل قيم x حقيقية

$$x = F(y)$$

ملاحظة : المقصود بالقيم غير الحقيقية هي الاعداد السالبة التي تكون تحت الجذر والكسور التي تكون مقامها $= 0$.

Ex:- 1

Find the domain and Range of Function.

$y = x + 1$

ملاحظة :-

اذا كانت الدالة لاكسرية ولاجذرية فان المجال

او المجال المقابل هو كل الاعداد الحقيقية (R)

Sol. $D_f = R$

$x = y - 1 \Leftarrow$ ∵ الدالة لاكسرية ولاجذرية

$R_f = R$

Ex:- 2

Find the Domain and Range of the function

$y = x^2 - 1$

Sol. $D_f = R$ دالة لاكسرية ولاجذرية ،

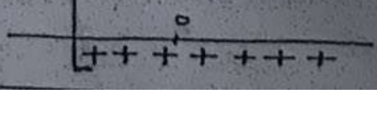
$y = x^2 - 1$

$x^2 = y + 1 \Rightarrow x = \mp\sqrt{y+1} \Rightarrow$ دالة جذرية

* في الدالة الجذرية نأخذ القيمة تحت الجذر ونجعلها اكبر او تساوي صفر

$y + 1 \geq 0 \Rightarrow y \geq -1$

$R_f = [-1, \infty)$



Ex:-

$$y = \sqrt{x^2 - 4}$$

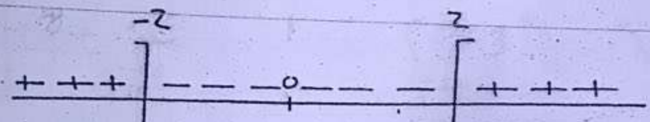
Sol.

$$x^2 - 4 \geq 0$$

$$x^2 \geq 4$$

$$x \geq \mp 2$$

$$D_f = R / (-2, 2)$$



$$y = \sqrt{x^2 - 4}$$

$$y^2 = x^2 - 4$$

$$x^2 = y^2 + 4$$

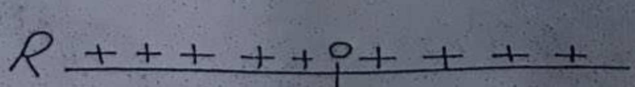
$$x = \mp \sqrt{y^2 + 4}$$

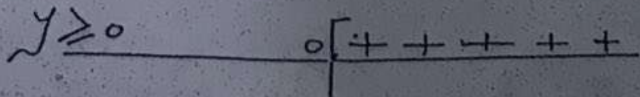
$$y^2 + 4 \geq 0$$

$$y^2 \geq -4 \Rightarrow$$ هذا لايمكن

∴ كل الاعداد الحقيقية تحقق الدالة

$$R_f = [0, \infty)$$





Ex :-

$$\underset{\sim}{y} = \frac{3x}{x-2}$$

$x - 2 = 0 \Rightarrow x = 2$

$\therefore Df = R / \{2\}$

$\therefore Df = \dfrac{3x}{x-2} \Rightarrow$ نجد (y) بدلالة x

أي نجعل x قبل المساواة

$x\underset{\sim}{y} - 2\underset{\sim}{y} = 3x$

$x\underset{\sim}{y} - 3x = 2\underset{\sim}{y}$

$x(\underset{\sim}{y} - 3) = 2\underset{\sim}{y}$

$$x = \frac{2\underset{\sim}{y}}{\underset{\sim}{y} - 3}$$

$\underset{\sim}{y} - 3 = 0 \Rightarrow \underset{\sim}{y} = 3$

$\therefore Rf = R / \{3\}$

Ex:- 6

$$y=\sqrt{x^2-3x}$$

$$x^2-3x \geq 0$$

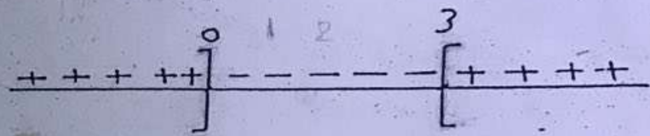
اما $x(x-3) \geq 0$ او

may: $x \geq 0$

or: $x-3 \geq 0$

$x \geq 3$

$\therefore D_f = R/(0,3)$



$$y=\sqrt{x^2-3x}$$

$$y^2=x^2-3x$$

$$\boxed{x=\frac{-B\mp\sqrt{B^2-4AC}}{2A}}$$

الدستور (سما)

$$x=\frac{+3\mp\sqrt{9+4y^2}}{2}$$

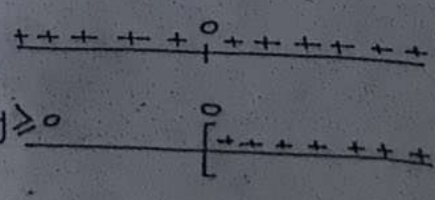
$4y^2+9 \geq 0$

$4y^2 \geq -9$ هذا لايكفي

$\therefore$ جميع قيم y تحقق المعادلة

$R_f=[0,\infty)$

$y \geq 0$



Ex:- 7

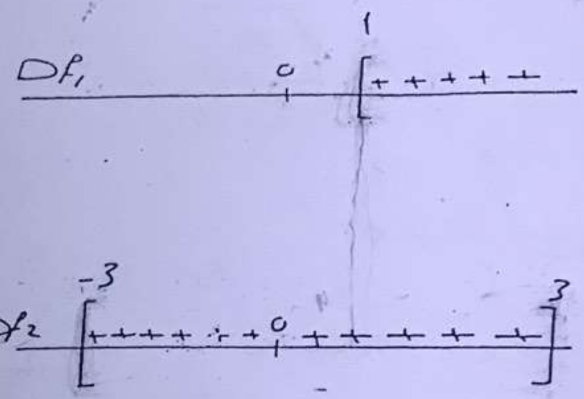
$$y = \sqrt{x-1} + \sqrt{9-x^2}$$

Find the Domain only.

$x - 1 \geq 0 \Rightarrow x \geq 1$

$9 - x^2 \geq 0 \Rightarrow x^2 \leq 9$

$x \leq \mp 3$



$Df = [3, 1]$

$[1, 3]$

Ex:- 8

$$y = -x^2$$

$Df = R \Rightarrow$ لأن الدالة ليس جذرية ولا كسرية

$x^2 = -y$

$x = \mp\sqrt{-y} \Longrightarrow$ أذا مجهول سالب تحت جذر يؤخذ لأنه تحته يكون عدد سالب

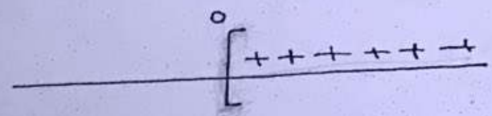
$\therefore -y \geq 0 \Rightarrow y \leq 0$

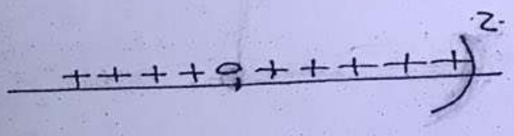
$\therefore Rf = y \leq 0$

Ex :- 9

$$y = \sqrt{\frac{x}{2-x}}$$

$$y = \frac{\sqrt{x}}{\sqrt{2-x}}$$





1- $x \geq 0$

2- $2 - x > 0$

$$D_f = [0, 2)$$

$$y = \sqrt{\frac{x}{2-x}}$$

$$y^2 = \frac{x}{2-x}$$

$$2y^2 - y^2 x = x$$

$$x + xy^2 = 2y^2$$

$$x(1+y^2) = 2y^2$$

$$x = \frac{2y^2}{1+y^2}$$

$$1 + y^2 = 0$$

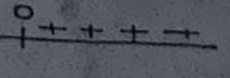
$$y^2 = -1$$

وهذا لا يمكن

∴ جميع قيم y تحقق المعادلة

$R_f = [0, \infty)$

$y \geq 0$



Ex:- 10

$$y = \frac{1}{x-2} - \frac{1}{x+2}$$

$$x-2=0 \Rightarrow x=2$$

$$x+2=0 \Rightarrow x=-2$$

$$D_f = \mathbb{R} / \{2, -2\}$$

$$y = \frac{1}{x-2} - \frac{1}{x+2}$$

$$y = \frac{(x+2)-(x-2)}{(x-2)(x+2)}$$

$$y = \frac{x+2-x+2}{x^2-4}$$

$$y = \frac{4}{x^2-4}$$

$$yx^2 - 4y = 4$$

$$yx^2 = 4 + 4y$$

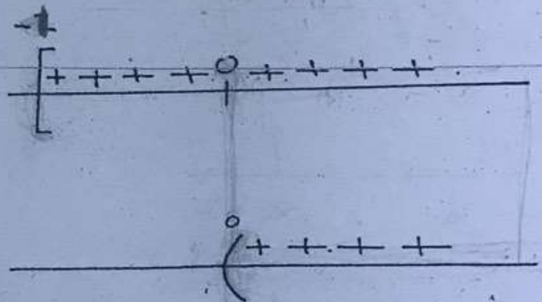
$$x^2 = \frac{4+4y}{y}$$

$$x = \mp\sqrt{\frac{4+4y}{y}}$$

$$\therefore\ 4+4y \geq 0$$

$$4y \geq -4$$

$$y \geq -1$$



$$y > 0$$

$$D_f = (0, \infty)$$

ادا كانت الدالة هي حاصل جمع او قسمة او طرح او ضرب
دالتين او اكثر فان Domain هو عبارة عن حجم
المناطق المشتركة للقيم التي تحقق الشرط
لكل الدوال

Ex :- 11

$$y = \sqrt{1 - \sqrt{x}}$$

بالحل $1 - \sqrt{x} \geq 0$ تحت الجذر الرئيسي

$1 \geq \sqrt{x}$

$x \leq 1$

$x \geq 0$ للجذر الداخلي

$D_f = [0,1]$

ويكون المجال هو تقاطع الشرطين
اي الفترة المشتركة بينهما

$y = \sqrt{1 - \sqrt{x}}$

$y^2 = 1 - \sqrt{x}$

$\sqrt{x} = 1 - y^2$

$x = (1 - y^2)^2$

$R_f = [0, 1]$

Ex:- 12

$$y = \frac{x+1}{x}$$

عندما تكون الدالة كسرية فيجب ان يكون

مقام الدالة لا يساوي صفر.

لذلك نجعل المقام يساوي الى صفر ونستثني

القيم المستخرجة.

Sol.

$x = 0$

$D_f = R / \{0\}$

$$y = \frac{x+1}{x}$$

$yx = x + 1$

$yx - x = 1$

$x(y-1) = 1$

$$x = \frac{1}{y-1}$$

$y - 1 = 0 \Rightarrow y = 1$

$R_f = R / \{1\}$.

Ex :- 13

توحيدمقامات

$$y = x + \frac{1}{x}$$

$$y = \frac{x^2+1}{x}$$

$$x = 0$$

$$D_f = R/\{0\}$$

$$y = \frac{x^2+1}{x}$$

$$yx = x^2+1$$

$$Ax^2 + Bx + C = 0$$

$$x^2 - yx + 1 = 0$$

$$x = \frac{-B \mp \sqrt{B^2-4AC}}{2A}$$

$$x = \frac{+y \mp \sqrt{y^2-4}}{2}$$

$$y^2 - 4 \geqslant 0$$

$$y^2 \geqslant 4$$

$$y \geqslant \mp 2$$

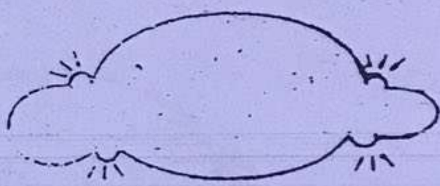
$$R_f = R/(2,-2)$$

The Absolute Value

Properties :-

1- $|-a| = |a| = a$

Ex:- $|-4| = |4| = 4$



2- $|ab| = |a| \cdot |b|$

Ex:- $|(-1)(4)| = |-1| \cdot |4| = 4$

3- $|a-b| = |b-a|$

Ex:- $|3-5| = |5-3| = 2$

4- $|a+b| \leq |a| + |b|$

Ex:- $|-4+3| = |-1| = 1 \leq |3| + |-4| = 7$

Ex:- $|-4+3| = |-1| = 1 \leq |-4| + |3| = 7$

5- If $|x| < 4$ then $-4 < x < 4$

Ex:- $|x-5| < 9$ then $-9 < x-5 < 9$

$\Rightarrow -4 < x < 14$

Ex:- $|x+3| > 4$ then $-4 > x+3 > 4$

$\Rightarrow -7 > x > 1$

Ex:- $y = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

Ex:- $y = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1 & \text{when } x \geq 0 \\ \dfrac{-x}{x} = -1 & \text{when } x < 0 \end{cases}$

x:- $y = |x-3| = \begin{cases} +(x-3) & \text{when } x \geq 3 \\ -(x-3) = 3-x & \text{when } x < 3 \end{cases}$

Ex:- $y = |x+5| - 3 = \begin{cases} +(x+5)-3 = x+2 & \text{when } x \geq 5 \\ -(x+5)-3 = -x-8 & \text{when } x < 5 \end{cases}$

Ex:- $\left|\frac{7-x}{3}\right| \geq 3$ then $-3 \geq \left(\frac{7-x}{3}\right) \geq 3$

$-9 \geq (7-x) \geq 9$ then $-16 \geq -x \geq 2$

$16 \leq x \leq -2$

* عندما نضرب المتراجحة x رقم سالب فانه يقلب .

Even and odd Function

الدوال الزوجية , الفردية

* الدالة الزوجية (Even) : هي الدالة المتناظرة حول محور y

ويكون فيها :-

$f(x) = f(-x)$

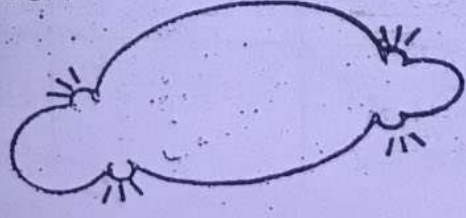
* الدالة الفردية (odd) : هي الدالة المتناظرة حول نقطة الأصل

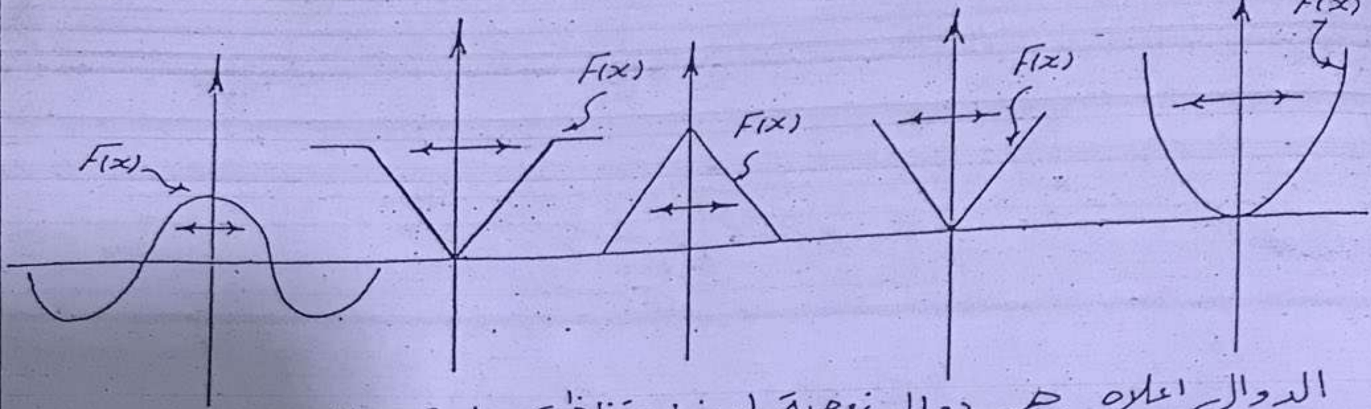
ويكون فيها :

$-f(x) = f(-x)$

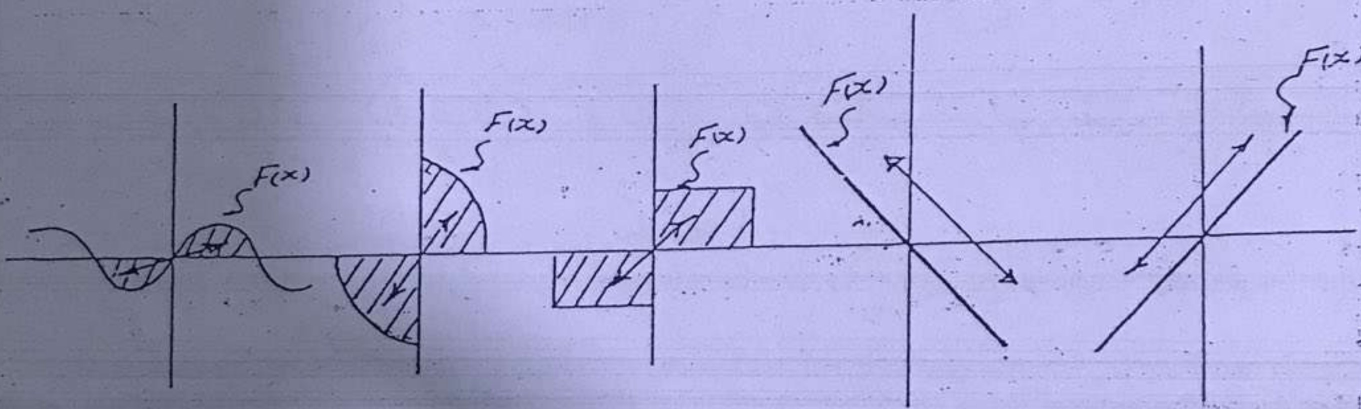
Notes :-

1. even ∓ even = even
2. odd ∓ odd = odd
3. even ∓ odd = not even , not odd
4. odd * even = odd
5. odd * odd = even
6. $\frac{odd}{even}$ = odd

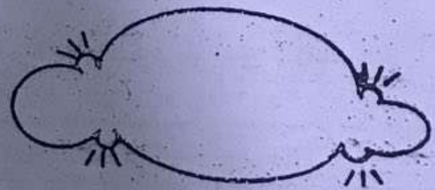


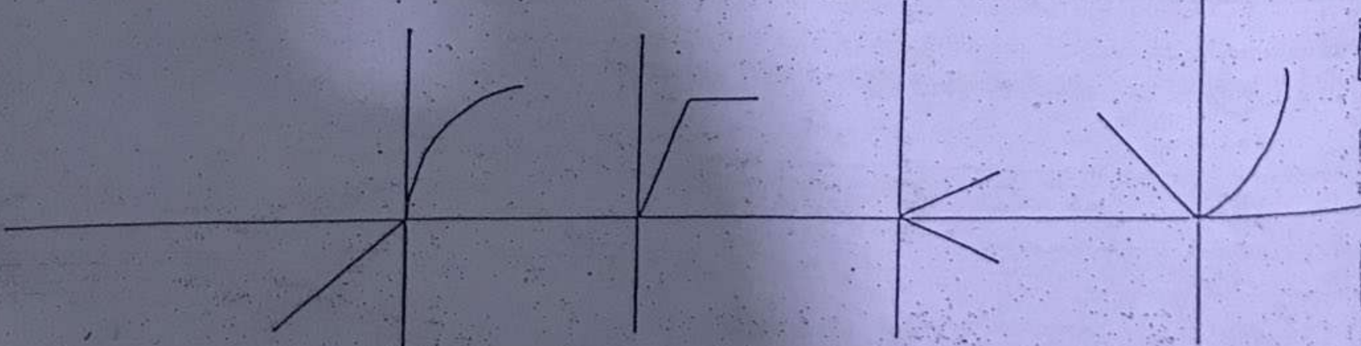


الدوال اعلاه هي دوال زوجية لانها متناظرة حول محور y.



الدوال اعلاه هي دوال فردية لانها متناظرة حول نقطة الأصل.





الدوال اعلاه غير متناظرة.

Ex:- Show that the function Even or odd or not.

① $F(x) = x^2$ م). اذا كان الاس زوجي فان الداله زوجية

Sol. $F(-x) = (-x)^2 = x^2$

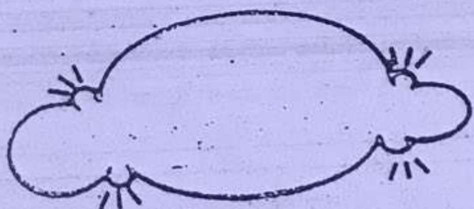
$\because F(x) = F(-x) \Rightarrow$ even.

② $F(x) = x^3$ م). اذا كان الاس فردي فان الدالة فردية

Sol. $F(-x) = (-x)^3 = -x^3$

$\because F(-x) = -F(x) \Rightarrow$ odd.

③ $F(x) = x - \frac{1}{x}$



$F(-x) = -x + \frac{1}{x} = -(x - \frac{1}{x})$

$F(-x) = -F(x) \Rightarrow$ odd

④ $y = F(x) = \frac{x+2}{x^2+1}$

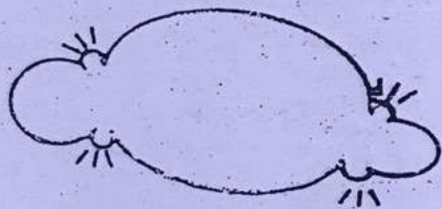
Sol. $F(-x) = \frac{-x+2}{(-x)^2+1}$

$F(-x) = \frac{-x+2}{}$ $\Rightarrow$ not even, not odd

⑤. $F(x) = \sin x$

Sol. $F(-x) = \sin(-x) = -\sin x$

$\therefore F(-x) = -F(x) \Rightarrow$ odd.



⑥. $F(x) = \cos(x)$

Sol. $F(-x) = \cos(-x) = \cos x$

$F(-x) = F(x) \Rightarrow$ even

Equation of straight Line

* The slope (m) of a straight Line through the point (x_1, y_1) and point (x_2, y_2) is :-

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

* the equations of straight Line passing through (x_0, y_0) and has aslop m is:-

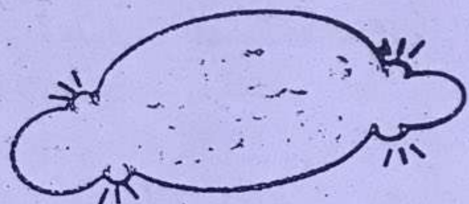
$$y - y_0 = m(x - x_0)$$

Ex:- If the point $(2, k)$ lies on the line with slope $(m=3)$ and passing through the point $(1, 6)$ Find k.

Sol.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow 3 = \frac{6-k}{1-2}$$

$$k = 9$$



or $y - y_0 = m(x - x_0)$

$$y - 6 = 3(x - 1)$$

$$y = 3x + 3$$

∵ النقطة $(2, k)$ تقع على الخط المستقيم ∴ تحقق المعادلة

$$k = 3(2) + 3$$
$$k = 9$$

Standard form equations for hyperbolas centered at the origin

Foci on the x-axis	Foci on the y axis
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Centre to focus distance	Centre to focus distance
$c = \sqrt{a^2+b^2}$	$c = \sqrt{a^2+b^2}$
Foci $(\mp c, 0)$	Foci $(0, \mp c)$
Vertices $(\mp a, 0)$	Vertices $(0, \mp a)$
Asymptotes	Asymptotes
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0$
or $y = \pm \frac{b}{a} x$	or $y = \pm \frac{a}{b} x$

Asymptotes

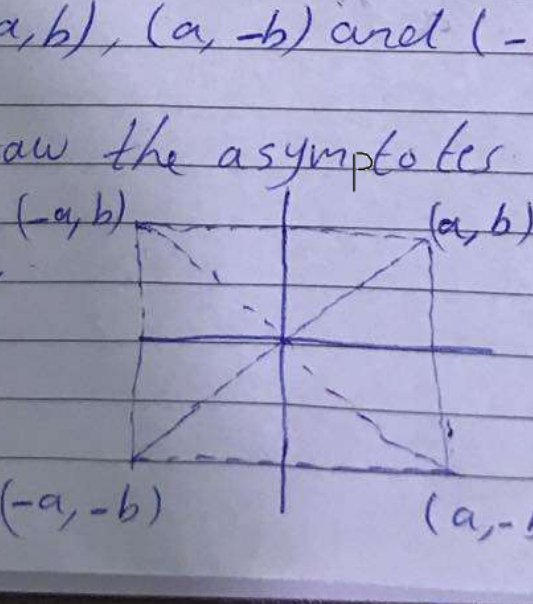
To graph a hyperbolas asymptotes, we need to draw a central rectangular with corners at (a,b), (-a,b) (a, -b) and (-a,-b). The asymptotes are extended diagonals of this rectangle. Note the center is the intersection of the diagonals.

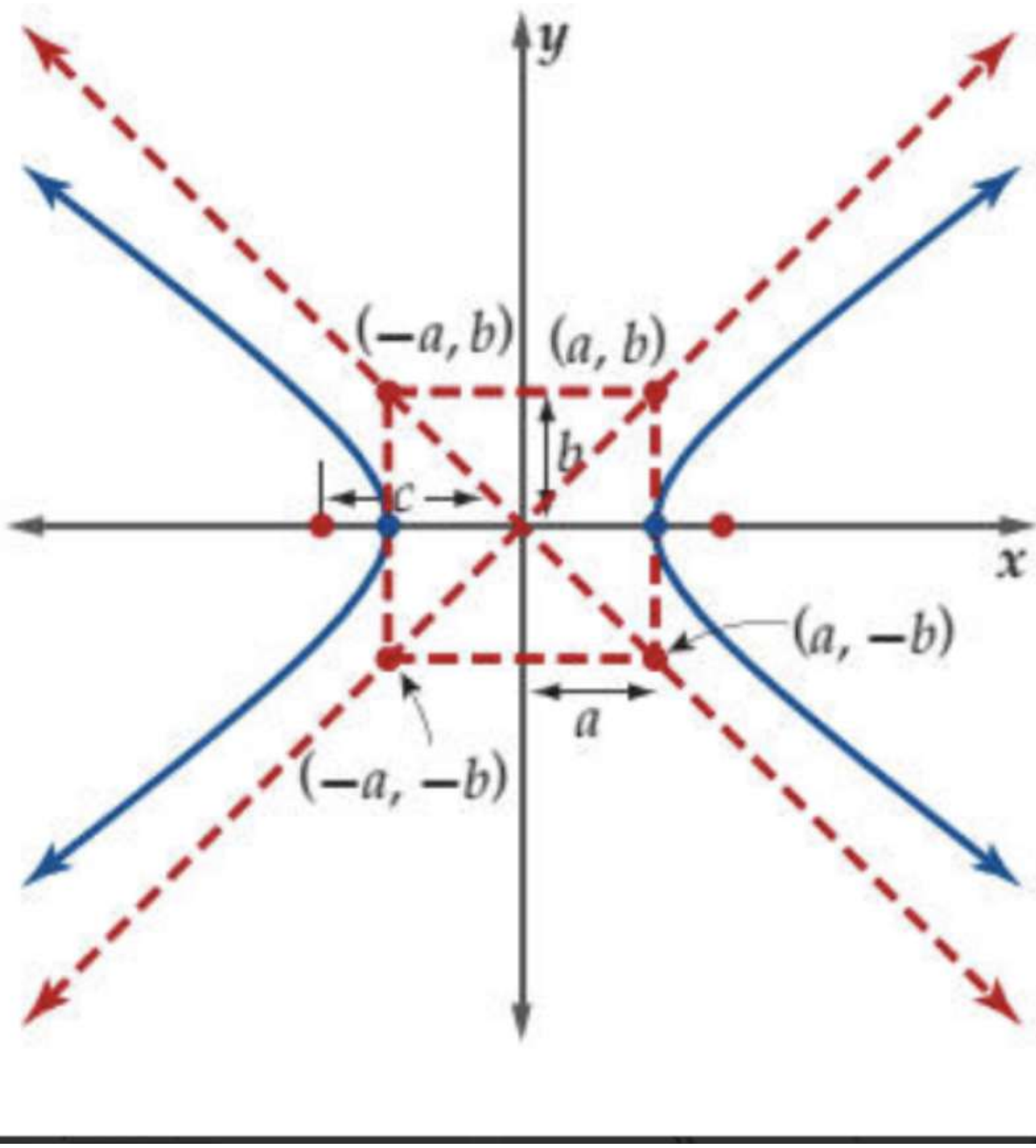
The equation of the asymptotes are

$$y = \pm \frac{b}{a} x$$

To graph a hyperbola with the centre at the origin:-

1. Find a, b, and c
2. With a dashed line, draw a central rectangle with corners of (a,b), (-a,b), (a,-b) and (-a,-b)
3. With a dashed line, draw the asymptotes
4. Sketch the hyperbola.





y
x
(−a, b)
(a, b)
b
c
(a, −b)
a
(−a, −b)

Ex. 1 sketch the graphs of the equations:-

a. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

Solution:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$a = 5$, $b = 3$

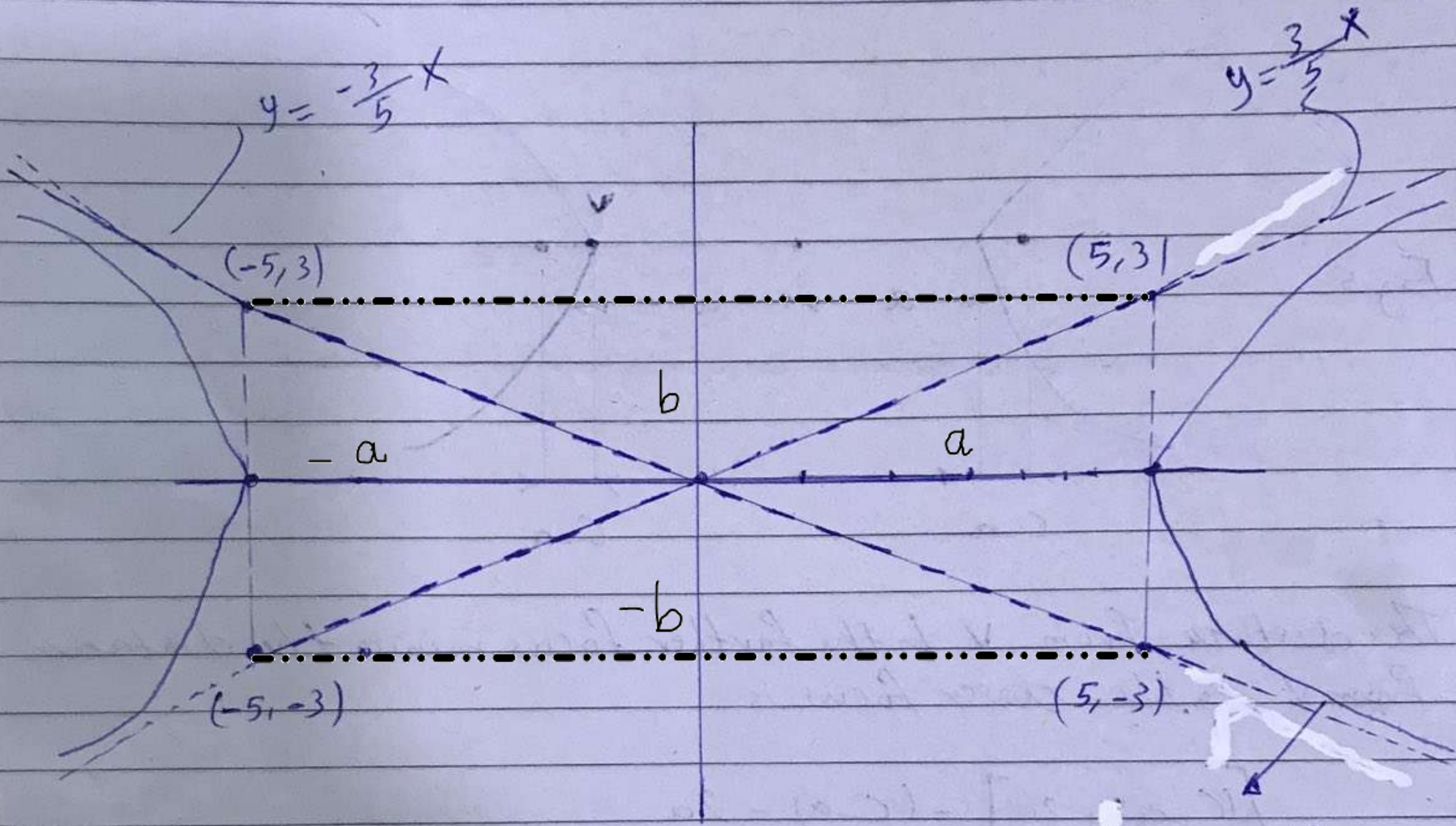
$c = \sqrt{a^2 + b^2}$ $\quad c = \sqrt{25+9} = \sqrt{34}$

find (a,b) $(-a,b)$ $(a,-b)$ & $(-a,-b)$
$(5,3)$ $(-5,3)$ $(5,-3)$ & $(-5,-3)$
to draw the rectangle

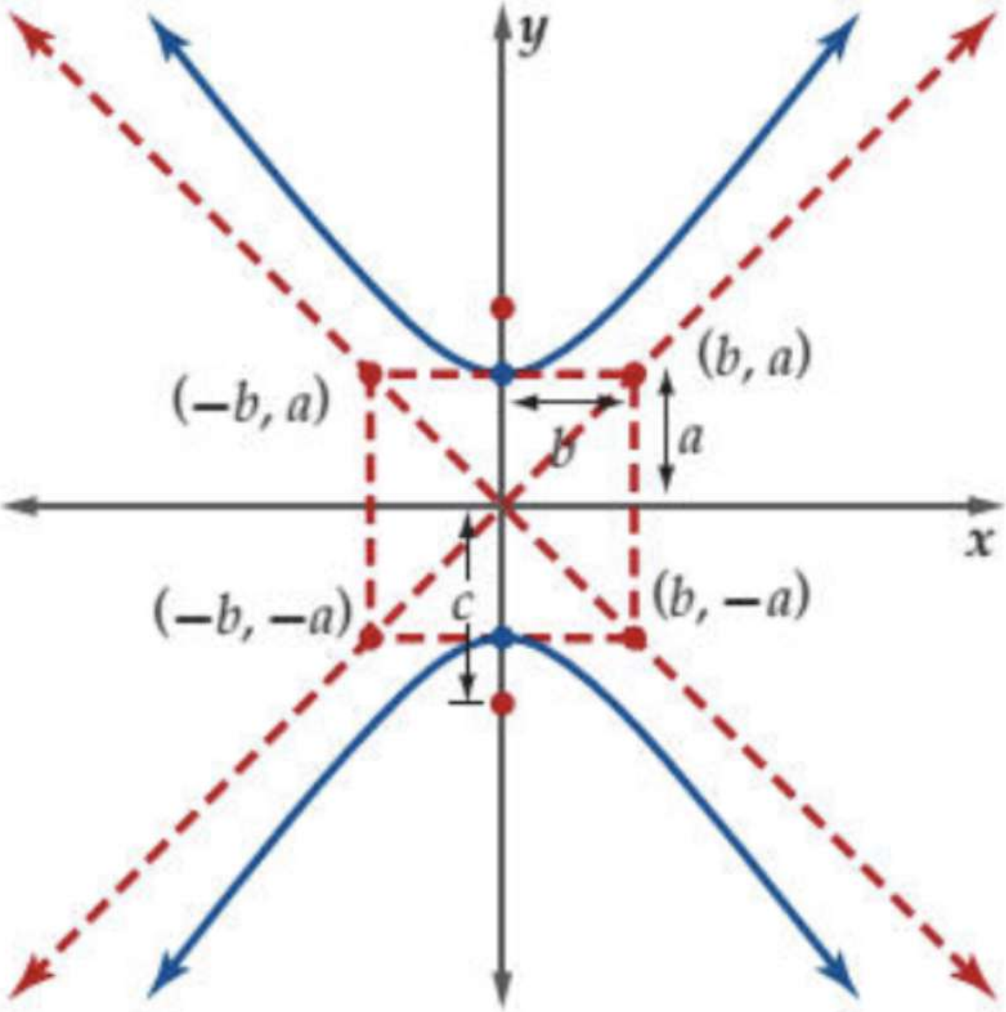
Draw the asymtotes

sketch the hyperbola

5



$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$



y
x
(b, a)
(−b, a)
a
b
(b, −a)
(−b, −a)
c

7

sketch the graph of the equation:-

$$-49x^2 + 36y^2 = 1764$$

$$\frac{-x^2}{36} + \frac{y^2}{49} = 1$$

$$\frac{y^2}{49} - \frac{x^2}{36} = 1.0$$

① Find a, b & c

$a = 7$, $b = 6$ $\quad c = \sqrt{a^2+b^2} = \sqrt{49+36} = 9.2$

Limits

Theorem about Limits

1). $\lim_{x \to a} c = c$ ١) غاية الثابت هو الثابت نفسه c = constant

). $\lim_{x \to a} x = a$

). $\lim_{x \to a} [F(x) \mp g(x)]$ ٣) الغاية تتوزع على الجمع والطرح

$= \lim_{x \to a} F(x) \mp \lim_{x \to a} g(x)$

). $\lim_{x \to a} [F(x) \cdot g(x)]$ ٤) الغاية تتوزع على الضرب

$= \lim_{x \to a} F(x) \cdot \lim_{x \to a} g(x)$

). $\lim_{x \to a} c\, F(x) = c \lim_{x \to a} F(x)$

$\lim_{x \to a} \frac{F(x)}{g(x)} = \frac{\lim_{x \to a} F(x)}{\lim_{x \to a} g(x)}$ 6) الغاية تتوزع على القسمة

7). $\lim_{x \to \infty} \frac{1}{x} = 0$

8). $\lim_{x \to 0^+} \frac{1}{x} = \infty$

9). $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

10). $\lim_{x \to 0} \sin x = 0$

11). $\lim_{x \to 0} \cos x = 1$

12). $\lim_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x \to 0} \frac{x}{\sin x} = 1$

13). $\lim_{x \to \infty} \frac{\cos x}{x} = 0$

14). $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

15). Undefined Expression تعابير غير معرفة

$\left(\frac{0}{0}, 0^0, \infty, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty, \right.$

but we can say $\underline{\infty + \infty = \infty}$ $\left. \right)$.

Proplemes' of Lim.

Ex:- 1

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$$

$$= \lim_{x \to 9} \frac{\cancel{(\sqrt{x}-3)}}{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

Ex:- 2

$$\lim_{x \to 2} \frac{x^2-4}{\sqrt{x-2}}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{\sqrt{x-2}}$$

$$= \lim_{x \to 2} \frac{\cancel{\sqrt{x-2}} \cdot \sqrt{x-2} \cdot (x+2)}{\cancel{\sqrt{x-2}}}$$

$$= \frac{\sqrt{2-2} * (2+2)}{1} = 0$$

Ex:- 3

$$\lim_{x \to 2} \frac{x^2+1}{x}$$

$$= \frac{(2)^2+1}{2}$$

$$= \frac{5}{2}$$

Ex:- 4

$$\lim_{x \to 2} \frac{4-x^2}{3-\sqrt{x^2+5}}$$

$$= \lim_{x \to 2} \frac{4-x^2}{3-\sqrt{x^2+5}} * \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}$$

$$= \lim_{x \to 2} \frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}$$

$$= \lim_{x \to 2} \frac{\cancel{(4-x^2)}(3+\sqrt{x^2+5})}{\cancel{(4-x^2)}} = 3+3 = 6$$

Ex:- 5

$$\lim_{x \to 0} \frac{5x^3+8x^2}{3x^4-16x^2}$$

$$\lim_{x \to 0} \frac{\cancel{x^2}(5x+8)}{\cancel{x^2}(3x^2-16)}$$

$$= \frac{8}{-16} = -\frac{1}{2}$$

$$= \lim_{x \to 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}\sqrt{x^2+3}+2}{\cancel{(x-1)}\,(x+1)}$$

$$= \frac{4}{2} = 2$$

Ex:- 10

$$\lim_{x \to \infty} \frac{2x^3-3x^2-5}{x^2-2}$$

$$\lim_{x \to \infty} \frac{\frac{2x^3}{x^3} - \frac{3x^2}{x^3} - \frac{5}{x^3}}{\frac{x^2}{x^3} - \frac{2}{x^3}}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{3}{x} - \frac{5}{x^3}}{\frac{1}{x} - \frac{2}{x^3}}$$

$$= \frac{2 - \cancel{\frac{3}{\infty}}^{0} - \cancel{\frac{5}{\infty}}^{0}}{\cancel{\frac{1}{\infty}}_{0} - \cancel{\frac{2}{\infty}}_{0}} = \infty$$

Ex:- 12

$$\lim_{x\to\infty} \frac{2x+3}{5x+7}$$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x}+\frac{3}{x}}{\frac{5x}{x}+\frac{7}{x}} = \frac{2}{5}$$

Ex:-

$$\lim_{x\to\infty} \frac{8x^2+7x}{4x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{8x^2}{x^2}+\frac{7x}{x^2}}{\frac{4x^2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{8+\frac{7}{x}}{4} = \frac{8}{4} = 2$$

Ex:- 14

$$\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}+\cancel{\frac{\sin x}{x}}^{0}}{\frac{x}{x}+\cancel{\frac{\cos x}{x}}_{0}} = 1$$

Ex :- 11

$$\lim_{x \to \infty} \frac{x^2 - 2}{2x^3 + 3x^2 - 5}$$

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^3} - \frac{2}{x^3}}{\frac{2x^3}{x^3} - \frac{3x^2}{x^3} - \frac{5}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{2}{x^3}}{2 - \frac{3}{x} - \frac{5}{x^3}}$$

$$= \frac{\cancel{\frac{1}{\infty}}^{0} - \cancel{\frac{2}{\infty}}^{0}}{2 - \cancel{\frac{3}{\infty}}_{0} - \cancel{\frac{5}{\infty}}_{0}} = \frac{0}{2} = 0$$

Note :-

حول الامثلة الثلاث السابقة اذا كانت الدالة كسرية نقسم البسط والمقام على اكبر اس موجود . فاذا كان اكبر اس موجود في المقام يكون الناتج (صفر) اما اذا تساوت الاسس في البسط والمقام تكون الناتج عدد حقيقي غير الصفر ويجب ان تكون الدالة عدد حقيقي غير الصفر ويجب ان تكون الدالة تقترب من ∞ كشرط اساسي. اما اذا كان اكبر اس موجود في البسط يكون الناتج ∞.

1

محاضرة ٧

$$\lim_{x \to \infty} \left(1 + \cos\frac{1}{x}\right)$$

$$= 1 + \cos\frac{1}{\infty}$$

$$= 1 + \cos 0$$

$$= 1 + 1 = 2$$

$$\lim_{x \to 0} \frac{\sin 3x}{x}$$

$$\lim_{x \to 0} \frac{\sin 3x}{x} * \frac{3}{3}$$ نضرب الدالة بـ ٣ ونقسمها ٣

$$\lim_{x \to 0} 3 * \cancel{\frac{\sin 3x}{3x}} = 3$$

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$$

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} \frac{3x}{3x} * \frac{5x}{5x}$$

$$\lim_{x \to 0} \cancel{\frac{\sin 3x}{3x}} \cancel{\frac{5x}{\sin 5x}} * \frac{3\cancel{x}}{5\cancel{x}} = \frac{3}{5}$$

Note:-

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{x}{\tan x} = 1$$

$$\lim_{y \to 0} \frac{\tan 2y}{3y}$$

$$\lim_{y \to 0} \frac{\tan 2y}{3y} * \frac{2}{2}$$

$$\lim_{y \to 0} \frac{\tan 2y}{2y} \cdot \frac{2}{3} = \frac{2}{3} \frac{\tan 2y}{2} = \frac{2}{3}$$

$$\lim_{x \to 0} \frac{3x}{\tan 2x}$$

$$\lim_{x \to 0} \frac{3x}{\tan 2x} \cdot \frac{2}{2}$$

$$\lim_{x \to 0} \frac{3}{2} \cancel{\frac{2x}{\tan 2x}} = \frac{3}{2}$$

$$\lim_{x \to 0} \frac{\sin 2x}{2x^2+x}$$

$$\lim_{x \to 0} \frac{\sin 2x}{x(2x+1)} * \frac{2}{2}$$

$$\lim_{x \to 0} \frac{2 \cdot \sin 2x}{2x(2x+1)}$$

$$\lim_{x \to 0} \cancel{\frac{\sin 2x}{2x}}^{1} \cdot \frac{2}{2x+1}$$

$$\lim_{x \to 0} \frac{2}{0+1} = 2$$

$$\lim_{x \to 0} \frac{\tan^2 \sqrt{x}}{x}$$

$$= \lim_{x \to 0} \frac{(\tan \sqrt{x})^2}{x}$$

$$= \lim_{x \to 0} \frac{\tan \cancel{\sqrt{x}} \cdot \tan \cancel{\sqrt{x}}}{\cancel{\sqrt{x}} \quad \cancel{\sqrt{x}}}$$

$$= 1 * 1 = 1$$

$$\lim_{x \to 0} (x \cdot \sec x \cdot \csc x)$$

$$\lim_{x \to 0} \frac{x}{\sin x} \cdot \frac{1}{\cos x}$$

$$= 1 * \frac{1}{1} = 1$$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin\theta}{\cos^2\theta}$$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin\theta}{1 - \sin^2\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\cancel{(1 - \sin\theta)}}{\cancel{(1 - \sin\theta)}(1 + \sin\theta)}$$

$$= \frac{1}{1 + \sin\frac{\pi}{2}} = \frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x \to \infty} \frac{x(x+1)(2x+1)}{x^3}$$

$$\lim_{x \to \infty} \frac{x[2x^2+3x+1]}{x^3}$$

$$\lim_{x \to \infty} \frac{2x^3+3x^2+x}{x^3}$$

$$\lim_{x \to \infty} \frac{\frac{2x^3}{x^3}+\frac{3x^2}{x^3}+\frac{x}{x^3}}{\frac{x^3}{x^3}}$$

$$\lim_{x \to \infty} \frac{2+\frac{3}{\infty}+\frac{1}{\infty}}{1} = 2$$

②

$$\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}}$$

$$\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}} * \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$$

$$\lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)}$$

$$\lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{x}$$

$$\lim_{x \to 0} (1+\sqrt{1-x}) = 1+\sqrt{1-0} = 2$$

③

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$$

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} * \frac{1 + \sin x}{1 + \sin x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin^2 x}{\cos x (1 + \sin x)}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{\cos x (1 + \sin x)}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{1 + \sin x} = 0$$

4

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} * \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$\lim_{h \to 0} \frac{\cancel{4}+h-\cancel{4}}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$= \frac{1}{\sqrt{4+0}+2} = \frac{1}{2+2} = \frac{1}{4}$$

$$\lim_{x \to 2} \frac{\cos \frac{\pi}{x}}{x-2}$$

Let $x-2=y$ when $x \to 2$

$x = y+2$ when $y \to 0$

$$\lim_{y \to 0} \frac{\cos \frac{\pi}{y+2}}{y+2-2}$$

$$\lim_{y \to 0} \frac{\cos \frac{\pi}{y+2}}{y}$$

$$\lim_{y \to 0} \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{y+2}\right)}{y}$$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

حولنا من (cos الزاوية) لـ

$\sin(90 -$ الزاوية$)$

لأننا نبحث دائماً عن $1 = \frac{\sin\theta}{\theta}$

$$\lim_{y \to 0} \frac{\sin\left(\frac{\pi(y+2) - 2\pi}{2(y+2)}\right)}{y}$$

$$\lim_{y \to 0} \frac{\sin \frac{\pi y + 2\pi - 2\pi}{2(y+2)}}{y}$$

$$\lim_{y \to 0} \frac{\sin \frac{\pi y}{2(y+2)}}{y} * \frac{\frac{\pi}{2(y+2)}}{\frac{\pi}{2(y+2)}} \to 1$$

$$\lim_{y \to 0} \frac{\pi}{2(y+2)} = \frac{\pi}{2(0+2)} = \frac{\pi}{4}$$

(1)

$$\lim_{x \to 3} \frac{\cos \frac{3\pi}{2x}}{x-3}$$

Let $y = x-3$ $\quad x \to 3 \quad \therefore y \to 0$

$x = y+3$

$$\lim_{y \to 0} \frac{\cos \frac{3\pi}{2(y+3)}}{y+3-3} = \lim_{y \to 0} \frac{\cos \frac{3\pi}{2y+3}}{y}$$

$\sin(\frac{\pi}{2} - \theta) = \cos\theta$

$$\lim_{y \to 0} \frac{\sin(\frac{\pi}{2} - \frac{3\pi}{2y+3})}{y} = \lim_{y \to 0} \frac{\sin \frac{\pi(y+3) - 3\pi}{2(y+3)}}{y}$$

$$\lim_{y \to 0} \frac{\sin \frac{\pi y}{2(y+3)}}{y} * \frac{\frac{\pi}{2(y+3)}}{\frac{\pi}{2(y+3)}}$$

$$\lim_{y \to 0} \frac{\sin \frac{\pi y}{2(y+3)}}{\frac{\pi y}{2(y+3)}} * \frac{\pi}{2(y+3)} = \lim_{y \to 0} \frac{\pi}{2(y+3)}$$

$$= \frac{\pi}{2(0+3)} = \frac{\pi}{6}$$

②

$$\lim_{x \to \pi} \frac{\sin x}{x - \pi}$$

sol let $y = x - \pi$ $\quad x \to \pi \quad y \to 0$

$x = y + \pi$

$$\lim_{y \to 0} \frac{\sin(y+\pi)}{y + \pi - \pi} = \lim_{y \to 0} \frac{\sin(y+\pi)}{y}$$

Note $\sin(\pi - \theta) = \sin\theta$

$\sin(\pi + \theta) = -\sin\theta$

$$\lim_{y \to 0} \frac{-\sin y}{y} = -1$$

$$\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}}$$

sol

$$\lim_{x \to 0} \frac{x}{1-\sqrt{1-x}} * \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$$

$$\lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)} = \lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{1-1+x}$$

$$\lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{x}$$

$$\lim_{x \to 0} (1+\sqrt{1-x}) = 1+\sqrt{1-0} = 2$$

4

$$\lim_{x \to 0} \frac{\sqrt{x^2+9-3}}{x^2}$$

$$\lim_{x \to 0} \frac{\sqrt{x^2+9-3}}{x^2} * \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$$

$$\lim_{x \to 0} \frac{\left(\sqrt{x^2+9}\right)^2 - 3^2}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$\lim_{x \to 0} \frac{x^2+9-9}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$\lim_{x \to 0} \frac{x^2}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$\lim_{x \to 0} \frac{1}{\left(\sqrt{x^2+9}+3\right)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Hopital's Rule

وهي قاعده او اسلوب لحل مسائل الغاية لكن لها شروط معينة وهي

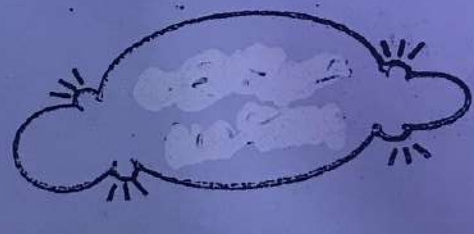
١). لا تستخدم ألا عندما يطلب استخدامها في السؤال.

٢) لا تستخدم ألا عندما يكون ناتج الغاية بعد التعويض هو $\frac{0}{0}$ أو $\frac{\infty}{\infty}$

* وتتضمن هذه الطريقة باشتقاق البسط والمقام كل منهما بغض النظر عن الآخر.

<u>Ex</u>: Use Hopitals Rule.

1). $\lim\limits_{x \to 0} \frac{\sin x}{x^2} = \frac{0}{0}$



$= \lim\limits_{x \to 0} \frac{\sin x}{x \cdot \cancel{x}}$

$= \lim\limits_{x \to 0} \frac{1}{x} = \frac{1}{0} = \infty$

by hopitals Rule :-

$\lim\limits_{x \to 0} \frac{\cos x}{2x} = \frac{1}{2 \times 0} = \infty$

$$= \frac{0}{0}$$

2). $$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

$$\lim_{x \to 0} \frac{0 - (-\sin x)}{1 + 2x}$$

$$\lim \frac{\sin x}{1 + 2x} = \frac{0}{1+0} = 0$$

3). $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\sin x}{6x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$