Processes Control:

Second-Order Systems

A second-order system is one whose dynamic behavior is represented by the second-order differential equation of the type:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu(t) \qquad (1)$$

where ***y(t)*** and ***u(t)*** are respectively the system's output and input variables. Once again, it is customary to rearrange such an equation to a "standard form" in which the characteristic system parameters will be more obvious. In this case, the "standard form" is:

$$\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = Ku(t) \qquad (2)$$

and the newly introduced parameters are given (for $a_0 \neq 0$) by:

$$\tau^2 = a_2/a_0 \qquad 2\xi\tau = a_1/a_0 \qquad K = b/a_0$$

Assuming, as usual, that the model in Eq. (1) is in terms of deviation variables, Laplace transformation and subsequent rearrangement gives the transfer function model:

$$y(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} u(s)$$

so that the general transfer function for the second-order system is given by:

$$G(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} \qquad (3)$$

Note that whereas the transfer function of the first-order system has a first-order denominator polynomial, the transfer function for the second-order system has a second-order denominator polynomial (i.e., a quadratic).

The second-order system has three characteristic parameters:

1. ***K***, the steady-state gain,

2. $\boldsymbol{\xi}$, the damping coefficient, and

3. $\boldsymbol{\tau}$, the natural period (or the inverse natural frequency)

Physical Examples of Second-Order Systems

1. Two First-Order Systems in Series

Let us now recall the dynamic behavior of two first-order systems in series. We obtained the differential equation model for the non-interacting arrangement as the second-order differential equation in Eq. (7-Lecture 16). Comparing this now with the standard form in Eq. (2), we see that:

The system consisting of a noninteracting, series arrangement of two first-order systems (with time constants $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ and steady-state gains $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$) is a second-order system with the following parameters:

$$\boldsymbol{\tau^2 = \tau_1\tau_2 \Rightarrow \tau = \sqrt{\tau_1\tau_2}}$$

$$\boldsymbol{K = K_1\, K_2}$$

$$\boldsymbol{2\xi\tau = (\tau_1 + \tau_2)}$$

$$\boldsymbol{\xi = \frac{1}{2}\frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1\tau_2}}}$$

A comparison of the transfer function representation in Eq. (17-Lecture 16) with Eq. (3) shows that the interacting configuration gives rise to a second-order system in which $\boldsymbol{\tau}$ and $\boldsymbol{K}$ are as given above for the noninteracting system, but with $\boldsymbol{\xi}$ given by:

$$\boldsymbol{\xi = \frac{1}{2}\frac{(\tau_1 + \tau_2 + K\tau_1)}{\sqrt{\tau_1\tau_2}}}$$

2. The U-Tube Manometer

The U-tube manometer shown in Figure 1 is a device used for measuring pressure. The dynamic behavior of the liquid level in each leg of the manometer tube in response to pressure changes can be obtained by carrying out a force balance on this system. The resulting equation is:

$$\boldsymbol{\frac{d^2h}{dt^2} + \frac{6\mu}{R^2\rho}\frac{dh}{dt} + \frac{3}{2}\frac{g}{L}h = \frac{3}{4\rho L}\Delta P} \qquad (4)$$

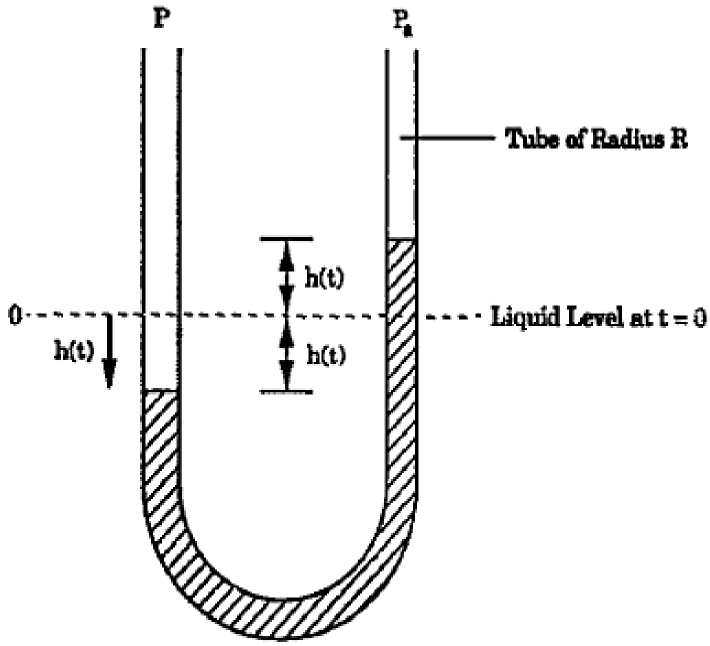

Fig. 1. The U-tube manometer

where $\boldsymbol{h}$ is the displacement of the liquid level from rest position, $\boldsymbol{L}$ is the total length of liquid in the manometer, $\boldsymbol{R}$ is the radius of the manometer tube; $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ are respectively the density and viscosity of the manometer liquid; $\Delta\boldsymbol{P}$ is the pressure difference across the tops of the two manometer legs; and **g** is the acceleration due to gravity. When Eq. (4) is arranged in the standard form, we observe that the dynamic behavior of this system is second order with:

$$K = \frac{1}{2\rho g}$$

$$\tau^2 = \frac{2}{3}\frac{L}{g} \quad \rightarrow \quad \tau = \sqrt{\frac{2}{3}\frac{L}{g}}$$

$$2\xi\tau = \frac{4\mu L}{\rho g R^2} \quad \rightarrow \quad \xi = \frac{\mu}{\rho R^2}\sqrt{\frac{6L}{g}}$$

Response of Second-Order Systems to Various Inputs

Step Response

The response of the second-order system to a step function of magnitude $\boldsymbol{A}$ may be obtained using Eq. (3). Since $\boldsymbol{u(s) = A/s}$ we have:

$$y(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}\ \frac{A}{s} \qquad (5)$$

We now choose to rearrange this expression to read:

$$y(s) = \frac{AK/\tau^2}{s(s - r_1)(s - r_2)} \qquad (6)$$

where r_1 and r_2 are the roots of the denominator quadratic (the transfer function poles). Ordinarily, one would invert Eq. (6), after partial fraction expansion, and obtain the general result:

$$y(t) = A_0 + A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (7)$$

where, as we now know, A_0 , A_1 , and A_2 are the usual constants obtained during partial fraction expansion. However, the values of the roots r_1 and r_2, obtained using the quadratic formula, are:

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau} \qquad (8)$$

By examining the quantity under the radical sign we may now observe that these roots can be real or complex, depending on the value of the parameter ξ. Observe further that the type of response obtained in Eq. (7) depends on the nature of these roots. In particular when $0 < \xi < 1$, Eq. (8) indicates complex conjugate roots, and we will expect the response under these circumstances to be different from that obtained when $\xi > 1$, and the roots are real and distinct. When $\xi = 1$, we have a pair of repeated roots, giving rise to yet another type of response which is expected to be different from the other two. The case $\xi < 0$ can occur only in special circumstances and signals process instability.

There are thus three different possibilities for the response in Eq. (7), depending on the nature of the roots r_1, r_2, that are dependent on the value of the parameter ξ. Let us now consider each case in turn.

H.W. 19.1. *Find Eq. 7 from Eq. 6.*

***CASE 1*: 0< ξ <1 (*r₁* and *r₂* Complex conjugates)**

It can be shown that in this case, Eq. (7) becomes, after some simplification:

$$y(t) = AK\left[1 - \frac{1}{\beta} e^{-\frac{\xi t}{\tau}} sin\left(\frac{\beta}{\tau}t + \phi\right)\right] \qquad (9)$$

Where:

$$\beta = \left|\xi^2 - 1\right|^{1/2}$$

$$\phi = tan^{-1}\left(\frac{\beta}{\xi}\right)$$

Note:

1. The time behavior of fuis response is that of a damped sinusoid with β/τ as the frequency of oscillation. The damping is provided by the exponential term $e^{-\frac{\xi t}{\tau}}$ which gets smaller in magnitude with time and eventually goes to zero as t → ∞.
2. The response ultimately settles, as t → ∞, to the value AK.

***CASE 2*: ξ = 1 (r_1 and r_2 Real and equal roots)**

In this case because we have a pair of repeated roots, we cannot use Eq. (7) directly. Laplace inversion of the appropriately modified version of Eq. (6):

$$y(s) = \frac{AK/\tau^2}{s(s-r)^2} \qquad (10)$$

gives the required response:

$$y(t) = AK\left[1 - \left(1 + \frac{t}{\tau}\right) e^{-\frac{t}{\tau}}\right] \qquad (11)$$

since $r = -1/\tau$. The time behavior indicated by this equation is an exponential approach to the ultimate value of ***AK***. Note that just as in *Case 1*, as t → ∞, ***y(t)*** → ***AK***.

***CASE 2*: ξ > 1 (r_1 and r_2 Real and distinct roots)**

With β as defined before, the roots are now given by:

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\beta}{\tau} \qquad (12)$$

and the response is given by:

$$y(t) = AK\left[1 - e^{-\frac{\xi t}{\tau}}\left(cosh\frac{\beta}{\tau}t + \frac{\xi}{\beta}sinh\frac{\beta}{\tau}t\right)\right] \qquad (13)$$

Where the hyperbolic functions sinh, cosh are defined as::

$$sinh\,\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

$$cosh\,\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

The indicated time response is another exponential approach to the ultimate value of AK. Although perhaps not immediately obvious from Eq. (13), it is true, however, that this particular exponential approach is somewhat slower than the one indicated in Eq. (11).

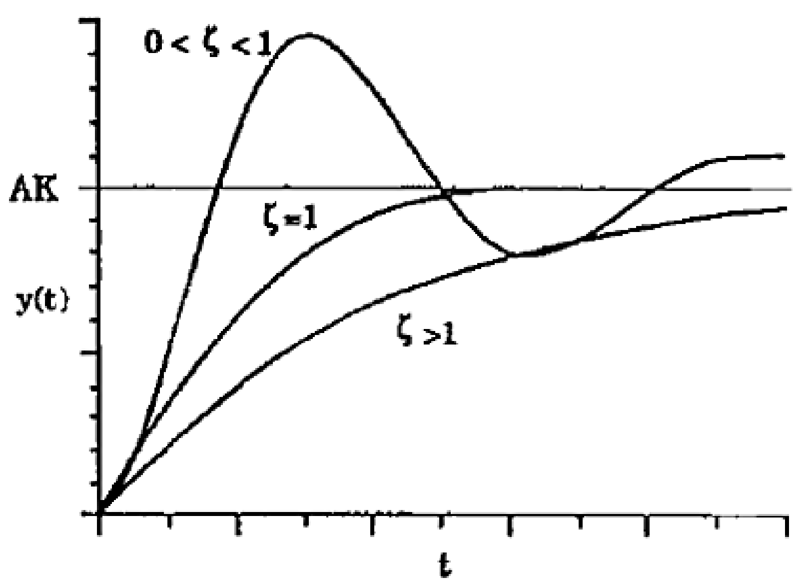

Fig. 2. Step responses of the second-order system

These three responses are sketched in Figure 2.

1. The Case 1 response (when 0< ξ <1) is oscillatory and is said to be under-damped.
2. The Case 2 response (when ξ = 1) is said to be critically damped. It offers the most rapid approach to the final value without oscillation.
3. The Case 3 response (when ξ > 1) is sluggish and is said to be over-damped.

We thus see that whether the second-order response is under-damped, over-damped, or critically damped is determined solely by one parameter ξ. This is why it is referred to as the *damping coefficient*.

Let us now return to the under-damped response in Eq. (9) and consider what happens when (is set equal to zero. This represents the situation in which there is no damping at all. The resulting response in this case is:

$$y(t) = AK\left[1 - sin\left(\frac{1}{\tau}t + \frac{\pi}{2}\right)\right] \qquad (14)$$

a pure, undamped sine wave, with frequency $1/\tau$. This is referred to as the natural frequency of oscillation ω_n. Observe that its reciprocal, the natural period of oscillation, is τ, establishing the reason for the name given to this parameter.

Having established that there are three categories of second-order systems (under-damped, critically damped, and over-damped).

Processes Control:

Second-Order Systems

A second-order system is one whose dynamic behavior is represented by the second-order differential equation of the type:

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu(t) \qquad (2.1)$$

where ***y(t)*** and ***u(t)*** are respectively the system's output and input variables. Once again, it is customary to rearrange such an equation to a "standard form" in which the characteristic system parameters will be more obvious. In this case, the "standard form" is:

$$\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = Ku(t) \qquad (2.2)$$

and the newly introduced parameters are given (for $a_0 \neq 0$) by:

$$\tau^2 = a_2/a_0 \qquad 2\xi\tau = a_1/a_0 \qquad K = b/a_0$$

Assuming, as usual, that the model in Eq. (2.1) is in terms of deviation variables, Laplace transformation and subsequent rearrangement gives the transfer function model:

$$y(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} u(s)$$

so that the general transfer function for the second-order system is given by:

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Note that whereas the transfer function of the first-order system has a first-order denominator polynomial, the transfer function for the second-order system has a second-order denominator polynomial (i.e., a quadratic).

The second-order system has three characteristic parameters:

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Physical Examples of Second-Order Systems

1. Two First-Order Systems in Series

Let us now recall the dynamic behavior of two first-order systems in series. We obtained the differential equation model for the non-interacting arrangement as the second-order differential equation in Eq. (7-Course 1-Lecture 16). Comparing this now with the standard form in Eq. (2.2), we see that:

The system consisting of a noninteracting, series arrangement of two first-order systems (with time constants $\boldsymbol{\tau_1}$and $\boldsymbol{\tau_2}$ and steady-state gains $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$) is a second-order system with the following parameters:

$$\boldsymbol{\tau^2 = \tau_1 \tau_2 \Rightarrow \tau = \sqrt{\tau_1 \tau_2}} \qquad \textbf{(2.4)}$$

$$\boldsymbol{K = K_1\, K_2}$$

$$\boldsymbol{2\xi\tau = (\tau_1 + \tau_2)}$$

$$\boldsymbol{\xi = \frac{1}{2} \frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1 \tau_2}}} \qquad (\textbf{2.5})$$

A comparison of the transfer function representation in Eq. (17-Course 1-Lecture 16) with Eq. (2.3) shows that the interacting configuration gives rise to a second-order system in which $\boldsymbol{\tau}$ and $\boldsymbol{K}$ are as given above for the noninteracting system, but with $\boldsymbol{\xi}$ given by:

$$\boldsymbol{\xi = \frac{1}{2} \frac{(\tau_1 + \tau_2 + K\tau_1)}{\sqrt{\tau_1 \tau_2}}}$$

2. The U-Tube Manometer

The U-tube manometer shown in Figure 2.1 is a device used for measuring pressure. The dynamic behavior of the liquid level in each leg of the manometer tube in response to pressure changes can be obtained by carrying out a force balance on this system. The resulting equation is:

$$\boldsymbol{\frac{d^2h}{dt^2} + \frac{6\mu}{R^2\rho}\frac{dh}{dt} + \frac{3}{2}\frac{g}{L}h = \frac{3}{4\rho L}\Delta P} \qquad (\textbf{2.6})$$

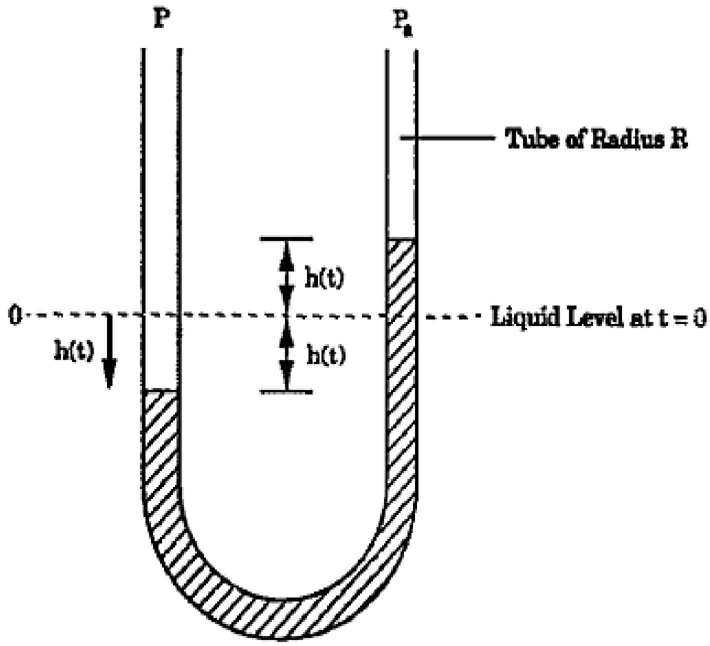


Fig. 2.1. The U-tube manometer

where $\boldsymbol{h}$ is the displacement of the liquid level from rest position, $\boldsymbol{L}$ is the total length of liquid in the manometer, $\boldsymbol{R}$ is the radius of the manometer tube; $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ are respectively the density and viscosity of the manometer liquid; $\Delta\boldsymbol{P}$ is the pressure difference across the tops of the two manometer legs; and **g** is the acceleration due to gravity. When Eq. (2.6) is arranged in the standard form, we observe that the dynamic behavior of this system is second order with:

$$K = \frac{1}{2\rho g}$$

$$\tau^2 = \frac{2}{3}\frac{L}{g} \quad \rightarrow \quad \tau = \sqrt{\frac{2}{3}\frac{L}{g}}$$

$$2\xi\tau = \frac{4\mu L}{\rho g R^2} \quad \rightarrow \quad \xi = \frac{\mu}{\rho R^2}\sqrt{\frac{6L}{g}}$$

Response of Second-Order Systems to Various Inputs

Step Response

The response of the second-order system to a step function of magnitude A may be obtained using Eq. (2.5). Since $u(s) = A/s$ we have:

$$y(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}\,\frac{A}{s} \qquad (2.7)$$

We now choose to rearrange this expression to read:

$$y(s) = \frac{AK/\tau^2}{s(s-r_1)(s-r_2)} \qquad (2.8)$$

where r_1 and r_2 are the roots of the denominator quadratic (the transfer function poles). Ordinarily, one would invert Eq. (2.8), after partial fraction expansion, and obtain the general result:

$$y(t) = A_0 + A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (2.9)$$

where, as we now know, A_0 , A_1 , and A_2 are the usual constants obtained during partial fraction expansion. However, the values of the roots r_1 and r_2, obtained using the quadratic formula, are:

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau} \qquad (2.10)$$

By examining the quantity under the radical sign we may now observe that these roots can be real or complex, depending on the value of the parameter ξ. Observe further that the type of response obtained in Eq. (2.9) depends on the nature of these roots. In particular when $0 < \xi < 1$, Eq. (2.10) indicates complex conjugate roots, and we will expect the response under these circumstances to be different from that obtained when $\xi > 1$, and the roots are real and distinct. When $\xi = 1$, we have a pair of repeated roots, giving rise to yet another type of response which is expected to be different from the other two. The case $\xi < 0$ can occur only in special circumstances and signals process instability.

There are thus three different possibilities for the response in Eq. (2.9), depending on the nature of the roots r_l, r_2, that are dependent on the value of the parameter ξ. Let us now consider each case in turn.

***CASE 1*: 0< ξ <1 (*r*₁ and *r*₂ Complex conjugates)**

It can be shown that in this case, Eq. (2.9) becomes, after some simplification:

$$y(t) = AK\left[1 - \frac{1}{\beta} e^{-\frac{\xi t}{\tau}} sin\left(\frac{\beta}{\tau} t + \phi\right)\right] \qquad (2.11)$$

Where:

$$\beta = \left|\xi^2 - 1\right|^{1/2}$$

$$\phi = tan^{-1}\left(\frac{\beta}{\xi}\right)$$

Note:

1. The time behavior of fuis response is that of a damped sinusoid with β/τ as the frequency of oscillation. The damping is provided by the exponential term $e^{-\frac{\xi t}{\tau}}$ which gets smaller in magnitude with time and eventually goes to zero as t → ∞.

2. The response ultimately settles, as t → ∞, to the value AK.

***CASE 2*: ξ = 1 (r_1 and r_2 Real and equal roots)**

In this case because we have a pair of repeated roots, we cannot use Eq. (2.9) directly. Laplace inversion of the appropriately modified version of Eq. (2.8):

$$y(s) = \frac{AK/\tau^2}{s(s-r)^2} \qquad (2.12)$$

gives the required response:

$$y(t) = AK\left[1 - \left(1 + \frac{t}{\tau}\right) e^{-\frac{t}{\tau}}\right] \qquad (2.13)$$

since $r = -1/\tau$. The time behavior indicated by this equation is an exponential approach to the ultimate value of ***AK***. Note that just as in *Case 1*, as t → ∞, ***y(t)*** → ***AK***.

***CASE 2*: $\xi > 1$ (r_1 and r_2 Real and distinct roots)**

With β as defined before, the roots are now given by:

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\beta}{\tau} \qquad (2.14)$$

and the response is given by:

$$y(t) = AK\left[1 - e^{-\frac{\xi t}{\tau}}\left(\cosh\frac{\beta}{\tau}t + \frac{\xi}{\beta}\sinh\frac{\beta}{\tau}t\right)\right] \qquad (2.15)$$

Where the hyperbolic functions sinh, cosh are defined as::

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

$$\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

The indicated time response is another exponential approach to the ultimate value of AK. Although perhaps not immediately obvious from Eq. (2.15), it is true, however, that this particular exponential approach is somewhat slower than the one indicated in Eq. (2.13).

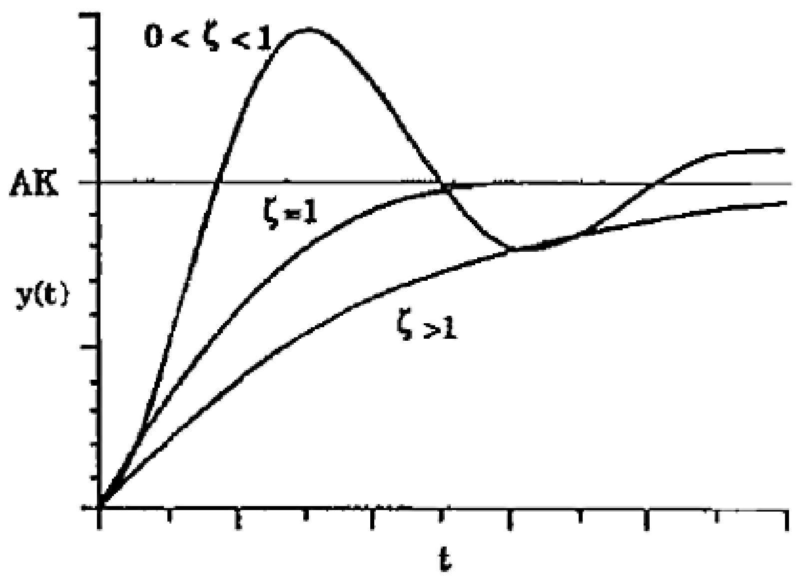


Fig. 2.2. Step responses of the second-order system

These three responses are sketched in Figure 2.2.

1. The Case 1 response (when 0< ξ <1) is oscillatory and is said to be under-damped.

2. The Case 2 response (when ξ = 1) is said to be critically damped. It offers the most rapid approach to the final value without oscillation.
3. The Case 3 response (when ξ > 1) is sluggish and is said to be over-damped.

We thus see that whether the second-order response is under-damped, over-damped, or critically damped is determined solely by one parameter ξ. This is why it is referred to as the *damping coefficient*.

Let us now return to the under-damped response in Eq. (2.11) and consider what happens when (is set equal to zero. This represents the situation in which there is no damping at all. The resulting response in this case is:

$$y(t) = AK\left[1 - sin\left(\frac{1}{\tau}t + \frac{\pi}{2}\right)\right] \qquad (2.16)$$

a pure, undamped sine wave, with frequency $1/\tau$. This is referred to as the natural frequency of oscillation ω_n. Observe that its reciprocal, the natural period of oscillation, is τ, establishing the reason for the name given to this parameter.
Having established that there are three categories of second-order systems (under-damped, critically damped, and over-damped).

Processes Control:

Second-Order Systems

Example 3.1. Having shown that two first-order systems in series (be they interacting or otherwise) constitute a second-order system, the following questions are to be answered.

1. What type of second-order system (under-damped, over-damped, or critically damped) is the noninteracting system?
2. Compare the damping characteristics of the interacting and the noninteracting configurations and hence determine the type of second-order system that describes the interacting system.

Ans.:

1. As earlier demonstrated, the damping characteristics of any second-order system are determined solely by the value of the parameter ξ. For the noninteracting arrangement, we had earlier shown that this parameter is given by:

$$\xi = \frac{1}{2}\frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1 \tau_2}} \qquad (2.5)$$

and our task is now to find out whether this quantity is greater than, less than, or equal to 1. First observe that Eq. (2.5) is a ratio of the arithmetic mean and the geometric mean of the two time constants τ_1 and τ_2. We may thus use the argument that the arithmetic mean of two positive quantities can never be smaller than their geometric mean to establish that $\xi \geq 1$.

Alternatively, let us assume that the converse is true: that is $\xi < 1$. This leads us to conclude from Eq. (2.5) that

$$\tau_1 + \tau_2 < 2\sqrt{\tau_1 \tau_2} \qquad (3.1)$$

Since the quantities involved in this inequality are all positive we may square both sides without altering the inequality; the result is:

$$\tau_1^2 + 2\tau_1 \tau_2 + \tau_2^2 < 4\tau_1 \tau_2$$

which simplifies to:

$$\tau_1^2 - 2\tau_1 \tau_2 + \tau_2^2 < 0$$

or

$$(\boldsymbol{\tau_1 - \tau_2})^2 < \mathbf{0} \qquad (\mathbf{3.2})$$

Observe, however, that regardless of the actual values of $\boldsymbol{\tau_1}$ or $\boldsymbol{\tau_2}$ a squared quantity can never be less than zero, thus proving false the initial assumption of $\xi < 1$. We therefore conclude that $\boldsymbol{\xi} \geq \mathbf{1}$. Note that when $\boldsymbol{\tau_1} = \boldsymbol{\tau_2}$ Eq. (2.5) indicates that $\boldsymbol{\xi} = \mathbf{1}$.

The conclusion is therefore that the noninteracting arrangement of two first-order systems in series is either critically damped (when the two time constants are identical) or over-damped; it can never be under-damped.

2. For the interacting case, the damping coefficient was earlier given as:

$$\boldsymbol{\xi} = \frac{\mathbf{1}}{\mathbf{2}} \frac{(\boldsymbol{\tau_1 + \tau_2 + K_2 \tau_1})}{\sqrt{\boldsymbol{\tau_1 \tau_2}}} \qquad (\mathbf{3.3})$$

A comparison of this expression with Eq. (2.5) shows that unless $\boldsymbol{K_2}$ is negative (which is not the case with the two physical interacting tanks) the damping coefficient for the interacting system takes on a value that is even greater than that for the noninteracting system.

Thus if the noninteracting system is over-damped, then the interacting system is even more so.

H.W. 3.1. *Prove that the interacting system can never be critically damped.*

Thus we conclude that the interacting system is also over-damped; exhibiting "heavier" damping characteristics than the noninteracting system. This implies the interacting system will show more sluggish response characteristics than the noninteracting system; this, of course, had already been indicated earlier.

One of the most important conclusions from this example is that two first-order systems in series can never exhibit under-damped characteristics. There are a number of other physical systems that do exhibit under-damped behavior. For example, the U-tube manometer will exhibit under-damped behavior for its most common configuration with water or mercury.

Under-damped behavior in chemical process systems often arises as a result of combining a feedback controller and the process itself. For this reason, the underdamped response takes on special significance in process control practice, and as a result, some special terminology (which will be defined) has been developed to characterize this behavior.

Characteristics of the Under-damped Response

Using Figure 3.1 as reference, the following are the terms used to characterize the under-damped response.

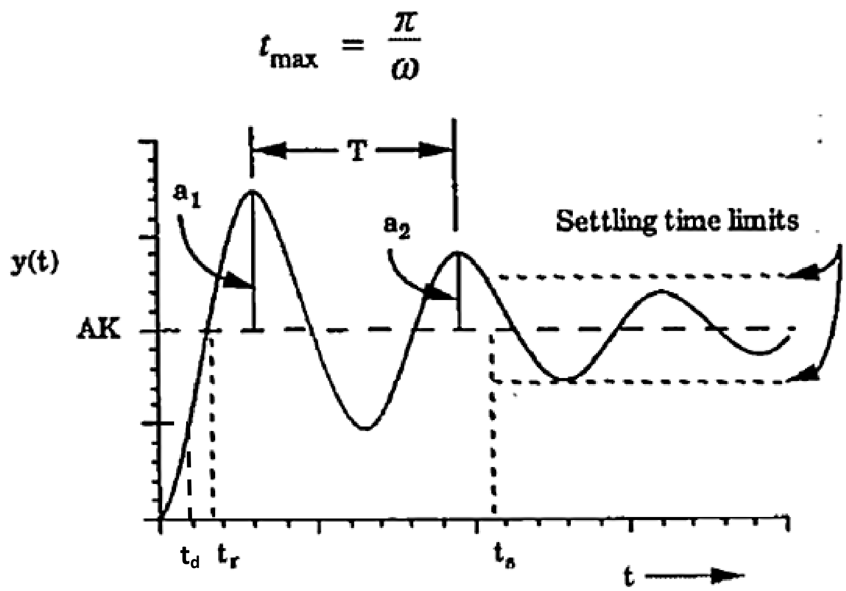


Fig. 3.1. Characteristics of the under-damped response

1. ***Delay Time***, t_d : Time to reach 50% of the ultimate value for the first time:

$$t_d = \tau(1 + 0.7\xi) = \frac{1 + 0.7\xi}{\omega_n} = \frac{1 + 0.7\xi}{\omega_d}\sqrt{1 - \xi^2} \qquad (3.4)$$

Where: $\tau = 1/\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \xi^2}$

2. ***Rise Time***, t_r : Time to reach the ultimate value for the first time; it can be shown that, for a second-order system:

$$t_r = \frac{\tau}{\beta}[\pi - \emptyset] \qquad (3.5)$$

where β and $\emptyset$ have been defined as:

$$\beta = \left|\xi^2 - 1\right|^{1/2}$$

$$\phi = tan^{-1}(\frac{\beta}{\xi})$$

ϕ must be in radian. ($rad = degree \times \pi/180$)

3. ***Overshoot***: The maximum amount by which the transient exceeds the ultimate value ***AK***; expressed as a fraction of this ultimate value, i.e., the ratio at a_1/AK in the diagram. For a second-order system, the maximum value attained by this response is given by:

$$y_{max} = AK\left[1 + exp\left(\frac{-\pi\xi}{\beta}\right)\right] \qquad (3.6)$$

so that the overshoot is now given by:

$$overshoot = \frac{a_1}{AK} = exp\left(\frac{-\pi\xi}{\beta}\right) \qquad (3.7)$$

The time to achieve this maximum value is:

$$t_{max} = \frac{\pi}{\omega_d} \qquad (3.8)$$

4. ***Period of Oscillation***: From Eq. (9-Course 1-Lecture 19) observe that the radian frequency of oscillation is:

$$\omega = \frac{\beta}{\tau}$$

and since the period (in times/ cycle) is given by $2\pi/\omega$, we therefore have:

$$T = \frac{2\pi}{\omega} = \frac{2\pi\tau}{\beta} \qquad (3.9)$$

5. ***Decay Ratio***: A measure of the rate at which oscillations are decaying, expressed as the ratio a_2/a_1 in the diagram:

$$Decay\ Ratio = \frac{a_2}{a_1} = exp\left(\frac{-2\pi\xi}{\beta}\right) \qquad (3.10)$$

H.W. 3.2*. Find the mathematical relation between decay ratio and the overshoot.*

6. ***Settling Time***, $\boldsymbol{t_s}$: A somewhat arbitrary quantity defined as the time for the process response to settle to within some small neighborhood of the ultimate value; usually taken to be within ± 5%.

For 2% tolerance:

$$\boldsymbol{t_s = \frac{4\tau}{\xi} = \frac{4}{\xi\omega_n} = \frac{4\sqrt{1-\xi^2}}{\xi\omega_d}} \qquad (\mathbf{3.11}\ \boldsymbol{a})$$

For 5% tolerance:

$$\boldsymbol{t_s = \frac{3\tau}{\xi} = \frac{3}{\xi\omega_n} = \frac{3\sqrt{1-\xi^2}}{\xi\omega_d}} \qquad (\mathbf{3.11}\ \boldsymbol{b})$$

7. ***Natural Period of Oscillation***: It is the time required to finish a complete cycle when the system is free of damping. The form of this parameter is:

$$\boldsymbol{T_N = 2\pi\tau} \qquad (\mathbf{3.12})$$

Example 3.2. A system having the following transfer function

$$G(s) = \frac{36}{s^2 + 5s + 36}$$

Check the damping status and characterize it.

Ans.:

$$G(s) = \frac{36/36}{(s^2/36) + (5s/36) + (36/36)}$$

$$G(s) = \frac{1}{0.027\,s^2 + 0.138\,s + 1}$$

$$AK = 1 \quad ;\ \tau^2 = 0.027\ \rightarrow\ \tau = 0.164$$

$$2\xi\tau = 0.138 \quad ;\ \xi = 0.42$$

Since the value of the damping factor is **0.42** so the system is under-damped system. Now we will characterize it.

1. ***Delay Time, t_d*** :

$$\omega_n = \frac{1}{\tau} = \frac{1}{0.164} = 6.086\ s^{-1}$$

$$\omega_d = \omega_n \sqrt{1-\xi^2} = 6.086\sqrt{1-0.42^2} = 5.523\ s^{-1}$$

$$t_d = \tau(1+0.7\xi) = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7\xi}{\omega_d}\sqrt{1-\xi^2} = 0.213\ s$$

2. ***Rise Time, t_r*** :

$$\beta = |\xi^2-1|^{1/2} = |0.42^2-1|^{1/2} = 0.908$$

$$\phi = tan^{-1}\left(\frac{\beta}{\xi}\right) = tan^{-1}\left(\frac{0.908}{0.42}\right) = 65.177$$

ϕ must be in radian.

$$rad = degree \times \frac{\pi}{180} = 65.177 \times \frac{\pi}{180} = 1.138$$

$$t_r = \frac{\tau}{\beta}[\pi - \emptyset] = \frac{0.164}{0.908}[\pi - 1.138] = 0.362\ s$$

3. ***Overshoot***:

$$overshoot = \frac{a_1}{AK} = exp\left(\frac{-\pi\xi}{\beta}\right) = exp\left(\frac{-\pi \times 0.42}{0.908}\right) = 0.233$$

$$a_1 = AK \times 0.233 = 1 \times 0.233 = 0.233$$

$$y_{max} = AK\left[1 + exp\left(\frac{-\pi\xi}{\beta}\right)\right] = 1\left[1 + exp\left(\frac{-\pi \times 0.42}{0.908}\right)\right] = 1.233$$

The time to achieve this maximum value is:

$$t_{max} = \frac{\pi}{\omega_d} = \frac{\pi}{5.523} = 0.569\ s$$

4. ***Period of Oscillation***:

$$T = \frac{2\pi\tau}{\beta} = \frac{2\pi \times 0.164}{0.908} = 1.135\frac{sec}{cycle}$$

5. ***Decay Ratio***:

$$\boldsymbol{Decay\ Ratio = \frac{a_2}{a_1} = exp\left(\frac{-2\pi\xi}{\beta}\right) = exp\left(\frac{-2\pi \times 0.42}{0.908}\right) = 0.054}$$

$$\boldsymbol{a_2 = 0.054 \times a_1 = 0.052 \times 0.233 = 0.012}$$

6. ***Settling Time***, t_s:

For 2% tolerance:

$$t_s = \frac{4\tau}{\xi} = \frac{4 \times 0.164}{0.42} = 1.562\ s$$

For 5% tolerance:

$$t_s = \frac{3\tau}{\xi} = \frac{3 \times 0.164}{0.42} = 1.171\ s$$

7. ***Natural Period of Oscillation***:

$$T_N = 2\pi\tau = 2\pi \times 0.164 = 1.030 \frac{sec}{cycle}$$

Processes Control:

Second-Order Systems

Let us now recall the dynamic behavior of two first-order systems in a noninteracting series with the following parameters:

$$\boldsymbol{\tau^2 = \tau_1 \tau_2 \; \Rrightarrow \; \tau = \sqrt{\tau_1 \tau_2}} \qquad (2.4)$$

$$\boldsymbol{\xi = \frac{1}{2} \frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1 \tau_2}}} \qquad (2.5)$$

Solving for $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ gives

$$\boldsymbol{\tau_1 = \frac{\tau}{\xi - \sqrt{\xi^2 - 1}}} \quad (\boldsymbol{\xi \geq 1}) \qquad (4.1)$$

$$\boldsymbol{\tau_2 = \frac{\tau}{\xi + \sqrt{\xi^2 - 1}}} \quad (\boldsymbol{\xi \geq 1}) \qquad (4.2)$$

H.W. 4.1. *Find equations (4.1) and (4.2) from equations (2.4) and (2.5)*

Example 4.1. An overdamped system consists of two first-order processes operating in series ($\boldsymbol{\tau_1}$ = 4, $\boldsymbol{\tau_2}$ = 1). Find the equivalent values of $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$ for this system.

Ans.:

From equations (2.4) and (2.5),

$$\boldsymbol{\tau = \sqrt{\tau_1 \tau_2} = \sqrt{(4)(1)} = 2}$$

$$\boldsymbol{\xi = \frac{1}{2} \frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1 \tau_2}} = \frac{1}{2} \frac{(4+1)}{\sqrt{(4)(1)}} = 1.25}$$

Equations (4.1) and (4.2) could be used to check on these results:

$$\boldsymbol{\tau_1 = \frac{\tau}{\xi - \sqrt{\xi^2 - 1}} = \frac{2}{1.25 - \sqrt{(1.25)^2 - 1}} = 4}$$

$$\tau_2 = \frac{\tau}{\xi + \sqrt{\xi^2 - 1}} = \frac{2}{1.25 + \sqrt{(1.25)^2 - 1}} = 1$$

Example 4.2. A step change from 15 to 31 psi in actual pressure results in the measured response from a pressure indicating element shown in Fig. 4.1.

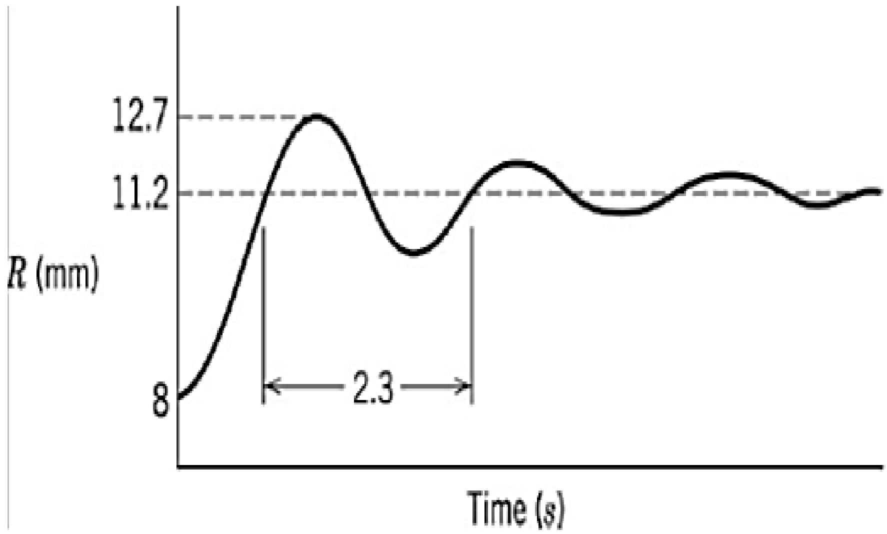


Fig. 4.1. An example of second order system

Assuming second-order dynamics, calculate all important parameters and write and approximate transfer function in the form

$$\frac{R'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where $\boldsymbol{R'}$ is the instrument output deviation (mm), $\boldsymbol{P'}$ is the actual pressure deviation (psi).

Ans.:

$$Gain = \frac{(11.2\ mm - 8\ mm)}{(31\ psi - 15\ psi)} = 0.2\ mm/psi$$

$$overshoot = \frac{a_1}{AK} = \frac{(12.7\ mm - 11.2\ mm)}{(11.2\ mm - 8\ mm)} = 0.47$$

$$overshoot = exp\left(\frac{-\pi\xi}{\beta}\right) = 0.47$$

Since

$$\boldsymbol{\beta = \left|\xi^2 - 1\right|^{1/2}}$$

From the above, $\boldsymbol{\xi = 0.234}$ so the system is underdamped.

$$\boldsymbol{\beta = \left|(0.234)^2 - 1\right|^{1/2} = 0.972}$$

$$\boldsymbol{T = \frac{2\pi}{\omega} = \frac{2\pi\tau}{\beta} = 2.3\ sec}$$

$$\boldsymbol{2.3 = \frac{2\pi\tau}{0.972}}$$

$$\boldsymbol{\tau = 0.356\ sec}$$

Other parameters could be calculated as follows:

Delay Time, t_d :

$$\boldsymbol{\omega_n = \frac{1}{\tau} = \frac{1}{0.356} = 2.809\ sec^{-1}}$$

$$\boldsymbol{\omega_d = \omega_n\sqrt{1-\xi^2} = 2.809\sqrt{1-0.234^2} = 2.731\ sec^{-1}}$$

$$\boldsymbol{t_d = \tau(1+0.7\xi) = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7\xi}{\omega_d}\sqrt{1-\xi^2} = 0.414\ sec}$$

Rise Time, t_r :

$$\boldsymbol{\phi = tan^{-1}\left(\frac{\beta}{\xi}\right) = tan^{-1}\left(\frac{0.972}{0.234}\right) = 76.464}$$

$\boldsymbol{\phi}$ must be in radian.

$$\boldsymbol{rad = degree \times \frac{\pi}{180} = 76.464 \times \frac{\pi}{180} = 1.334}$$

$$\boldsymbol{t_r = \frac{\tau}{\beta}[\pi - \emptyset] = \frac{0.356}{0.972}[\pi - 1.334] = 0.662\ s}$$

The time to achieve the maximum value (12.7 mm) is:

$$t_{max} = \frac{\pi}{\omega_d} = \frac{\pi}{2.731} = \boxed{1.15\ s}$$

Decay Ratio:

$$Decay\ Ratio = \frac{a_2}{a_1} = exp\left(\frac{-2\pi\xi}{\beta}\right) = exp\left(\frac{-2\pi \times 0.234}{0.972}\right) = \boxed{0.220}$$

$$a_2 = 0.220 \times a_1 = 0.220\ (12.7\ mm - 11.2\ mm) = \boxed{0.330}$$

Settling Time, t_s:

For 2% tolerance:

$$t_s = \frac{4\tau}{\xi} = \frac{4 \times 0.356}{0.234} = \boxed{6.085\ sec}$$

For 5% tolerance:

$$t_s = \frac{3\tau}{\xi} = \frac{3 \times 0.356}{0.234} = \boxed{4.564\ sec}$$

Natural Period of Oscillation:

$$T_N = 2\pi\tau = 2\pi \times 0.356 = \boxed{2.237\frac{sec}{cycle}}$$

The final transfer form is

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.127s^2 + 0.167s + 1}$$

H.W. 4.2. *A second-order control system having a radian frequency for the control system is 1.9 rad/min. The time constant is 0.5 min. The control system is subjected to a step change of the magnitude 2. Characterize the status of this system and calculate all parameters.*

Processes Control:

Second-Order Systems

Steady-state error analysis

The function of a feedback control system is to ensure that the closed-loop system has desirable dynamic and steady-state response characteristics. Ideally, we would like the closed-loop system to satisfy the following performance criteria:

1. The closed-loop system must be stable.

2. The effects of disturbances are minimized, providing good disturbance rejection.

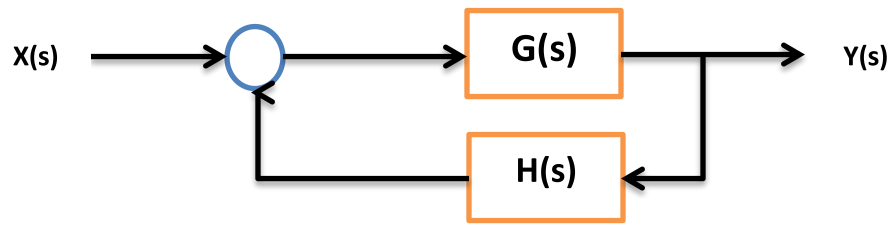
3. Rapid, smooth responses to set-point changes are obtained, that is, good set-point tracking.

4. Steady-state error (offset) is eliminated.

5. Excessive control action is avoided.

6. The control system is robust, that is, insensitive to changes in process conditions and to inaccuracies in the process model.

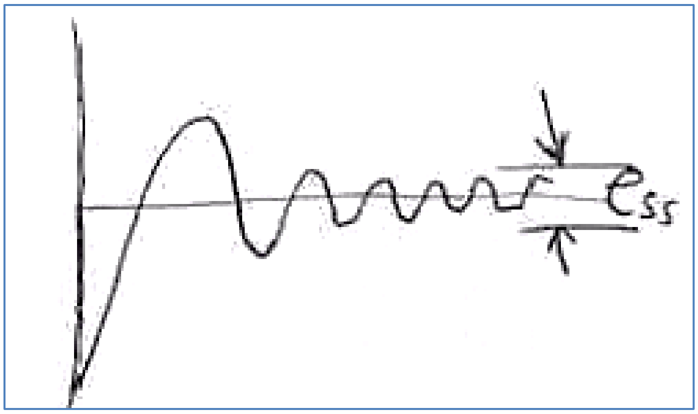
The steady-state error (or offset) occurs after a set-point change or a sustained disturbance. Consider the general figure for a negative feedback closed-loop control system shown as follows



$T(s) = G(s)/[1+G(s)\,H(s)]$

For such a system, the steady-state error (e_{ss}) could be expressed as follows

$$e_{ss} = \lim_{s\to 0} \frac{S\,X(s)}{1 + G(s)H(s)} \qquad (5.1)$$



(1) Steady-state error for step input

As known that the formula of step input is

$$X(t) = A \;\; ; \; X(s) = \frac{A}{s}$$

So

$$e_{ss} = \lim_{s\to 0} \frac{s\,X(s)}{1 + G(s)H(s)} = \lim_{s\to 0} \frac{s\,\frac{A}{s}}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{A}{1 + \lim_{s\to 0} G(s)H(s)} \qquad (5.2)$$

The term $[\lim_{s\to 0} G(s)H(s)]$ is constant and is called the "static position coefficient (or constant)", and it is denoted as K_p

$$K_p = \lim_{s\to 0} G(s)H(s) \qquad (5.3)$$

$$e_{ss} = \frac{A}{1 + K_p} \qquad (5.4)$$

(2) Steady-state error for ramp input

The formula of ramp input is

$$X(t) = At \ ; \ X(s) = \frac{A}{s^2}$$

So

$$e_{ss} = \lim_{s\to 0} \frac{s\,X(s)}{1 + G(s)H(s)} = \lim_{s\to 0} \frac{s\,\frac{A}{s^2}}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{A}{s + \lim_{s\to 0} s\,G(s)H(s)} \qquad (5.5)$$

The term $[\lim_{s\to 0} s\,G(s)H(s)]$ is constant and is called the “static velocity coefficient (or constant)”, and it is denoted as K_v

$$K_v = \lim_{s\to 0} s\,G(s)H(s) \qquad (5.6)$$

$$e_{ss} = \frac{A}{K_v} \qquad (5.7)$$

(3) Steady-state error for parabolic input

The formula of parabolic input is

$$X(t) = A\frac{t^2}{2} \ ; \ X(s) = \frac{A}{s^3}$$

So

$$e_{ss} = \lim_{s\to 0} \frac{s\,X(s)}{1 + G(s)H(s)} = \lim_{s\to 0} \frac{s\,\frac{A}{s^3}}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{A}{s^2 + \lim_{s\to 0} s^2\,G(s)H(s)} \qquad (5.8)$$

The term $[\lim_{s\to 0} s^2\, G(s)H(s)]$ is constant and is called the "static acceleration coefficient (or constant)", and it is denoted as K_a

$$K_a = \lim_{s\to 0} s^2\, G(s)H(s) \qquad (5.9)$$

$$e_{ss} = \frac{A}{K_a} \qquad (5.10)$$

Steady-state error for type-0, type-1, and type-2 systems

Types of the steady-state systems could be determined according to the open-loop transfer function [**G(s) H(s)**] which written as follows

$$G(s)H(s) = \frac{K(1+\tau_1 s)(1+\tau_2 s)\,\ldots\ldots..(1+\tau_p s)}{s^n(1+\tau_a s)(1+\tau_b s)\,\ldots\ldots.(1+\tau_q s)} \qquad (5.11)$$

The type of the system is (n) where

n = 0 ➔ Type-0

n = 1 ➔ Type-1

n = 2 ➔ Type-2

Example 5.1. Determine the types of the following systems:

(1) $G(s) = \frac{K(1+8s)}{s^2}$ **and** $H(s) = \frac{(1+2s)}{(s^2+9s+8)}$

(2) $G(s) = \frac{3}{(s^2+4s+5)}$ **and** $H(s) = (s+4)$

Ans.:

(1) $G(s)H(s) = \frac{K(1+8s)(1+2s)}{s^2(s^2+9s+8)}$ ➔ $Type-2$

(2) $G(s)H(s) = \frac{3(s+4)}{(s^2+4s+5)}$ ➔ $Type-0$

Steady-state error for type-0 system

For type-0 system where (n=0), the open-loop transfer function [**G(s) H(s)**] will be as follows:

$$G(s)H(s) = \frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots..(1+\tau_p s)}{(1+\tau_a s)(1+\tau_b s)\ldots\ldots.(1+\tau_q s)} \qquad (5.12)$$

(1) Steady-state error for (type-0) and step input

Since the steady-state error and the "static position coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{1+K_p}$$

And

$$K_p = \lim_{s\to 0} G(s)H(s)$$

So

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots..(1+\tau_p s)}{(1+\tau_a s)(1+\tau_b s)\ldots\ldots.(1+\tau_q s)}$$

$$K_p = \frac{K\,.\,1\,.\,1.\ldots.\,1}{1.\,1\,.\,1\ldots\ldots1} = K$$

$$e_{ss} = \frac{A}{1+K} \qquad (5.13)$$

(2) Steady-state error for (type-0) and ramp input

The steady-state error and the "static velocity coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{K_v}$$

And

$$K_v = \lim_{s \to 0} s\, G(s)H(s)$$

So

$$K_v = \lim_{s \to 0} s\, G(s)H(s) = \lim_{s \to 0} \frac{s\, K(1+\tau_1 s)(1+\tau_2 s) \ldots \ldots .. (1+\tau_p s)}{(1+\tau_a s)(1+\tau_b s) \ldots \ldots . (1+\tau_q s)}$$

$$K_v = \frac{0. K\,.1\,.1. \ldots .1}{1.1\,.1 \ldots \ldots 1} = 0$$

$$e_{ss} = \frac{A}{0} = \infty \qquad (5.14)$$

(3) Steady-state error for (type-0) and parabolic input

The steady-state error and the "static acceleration coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{K_a}$$

And

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s)$$

So

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = \lim_{s \to 0} \frac{s^2\, K(1+\tau_1 s)(1+\tau_2 s) \ldots \ldots .. (1+\tau_p s)}{(1+\tau_a s)(1+\tau_b s) \ldots \ldots . (1+\tau_q s)}$$

$$K_a = \frac{0. K\,.1\,.1. \ldots .1}{1.1\,.1 \ldots \ldots 1} = 0$$

$$e_{ss} = \frac{A}{0} = \infty \qquad (5.14)$$

Steady-state error for type-1 system

For type-1 system where (n=1), the open-loop transfer function [**G(s) H(s)**] will be as follows:

$$G(s)H(s) = \frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots(1+\tau_p s)}{s(1+\tau_a s)(1+\tau_b s)\ldots\ldots(1+\tau_q s)} \qquad (5.15)$$

(1) Steady-state error for (type-1) and step input

Since the steady-state error and the "static position coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{1+K_p}$$

And

$$K_p = \lim_{s\to 0} G(s)H(s)$$

So

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots(1+\tau_p s)}{s(1+\tau_a s)(1+\tau_b s)\ldots\ldots(1+\tau_q s)}$$

$$K_p = \frac{K\,.\,1\,.\,1.\ldots.\,1}{0.\,1\,.\,1\ldots\ldots1} = \infty$$

$$e_{ss} = \frac{A}{1+\infty} = 0 \qquad (5.16)$$

(2) Steady-state error for (type-1) and ramp input

The steady-state error and the "static velocity coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{K_v}$$

And

$$K_v = \lim_{s \to 0} s\, G(s)H(s)$$

So

$$K_v = \lim_{s \to 0} s\, G(s)H(s) = \lim_{s \to 0} \frac{s\, K(1+\tau_1 s)(1+\tau_2 s) \dots \dots ..(1+\tau_p s)}{s(1+\tau_a s)(1+\tau_b s) \dots \dots .(1+\tau_q s)}$$

$$K_v = \frac{K\,.\,1\,.\,1.\dots .\,1}{1.\,1\,.\,1 \dots \dots 1} = K$$

$$e_{ss} = \frac{A}{K} \qquad (5.17)$$

(3) Steady-state error for (type-1) and parabolic input

The steady-state error and the “static acceleration coefficient (or constant)” have been determined as

$$e_{ss} = \frac{A}{K_a}$$

And

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s)$$

So

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = \lim_{s \to 0} \frac{s^2\, K(1+\tau_1 s)(1+\tau_2 s) \dots \dots ..(1+\tau_p s)}{s(1+\tau_a s)(1+\tau_b s) \dots \dots .(1+\tau_q s)}$$

$$K_a = \frac{0.\,K\,.\,1\,.\,1.\dots .\,1}{1.\,1\,.\,1 \dots \dots 1} = 0$$

$$e_{ss} = \frac{A}{0} = \infty \qquad (5.18)$$

Steady-state error for type-2 system

For type-2 system where (n=2), the open-loop transfer function [**G(s) H(s)**] will be as follows:

$$G(s)H(s) = \frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots..(1+\tau_p s)}{s^2(1+\tau_a s)(1+\tau_b s)\ldots\ldots.(1+\tau_q s)} \qquad (5.19)$$

(1) Steady-state error for (type-2) and step input

Since the steady-state error and the “static position coefficient (or constant)” have been determined as

$$e_{ss} = \frac{A}{1+K_p}$$

And

$$K_p = \lim_{s\to 0} G(s)H(s)$$

So

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{K(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots..(1+\tau_p s)}{s^2(1+\tau_a s)(1+\tau_b s)\ldots\ldots.(1+\tau_q s)}$$

$$K_p = \frac{K\,.\,1\,.\,1.\ldots.\,1}{0.\,1\,.\,1\ldots\ldots 1} = \infty$$

$$e_{ss} = \frac{A}{1+\infty} = 0 \qquad (5.20)$$

(2) Steady-state error for (type-2) and ramp input

The steady-state error and the “static velocity coefficient (or constant)” have been determined as

$$e_{ss} = \frac{A}{K_v}$$

And

$$K_v = \lim_{s \to 0} s\, G(s)H(s)$$

So

$$K_v = \lim_{s \to 0} s\, G(s)H(s) = \lim_{s \to 0} \frac{s\, K(1 + \tau_1 s)(1 + \tau_2 s) \dots \dots .. (1 + \tau_p s)}{s^2(1 + \tau_a s)(1 + \tau_b s) \dots \dots . (1 + \tau_q s)}$$

$$K_v = \frac{K\,.\,1\,.\,1. \dots .\,1}{0.\,1\,.\,1 \dots \dots 1} = \infty$$

$$e_{ss} = \frac{A}{\infty} = 0 \qquad (5.21)$$

(3) Steady-state error for (type-2) and parabolic input

The steady-state error and the "static acceleration coefficient (or constant)" have been determined as

$$e_{ss} = \frac{A}{K_a}$$

And

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s)$$

So

$$K_a = \lim_{s \to 0} s^2\, G(s)H(s) = \lim_{s \to 0} \frac{s^2\, K(1 + \tau_1 s)(1 + \tau_2 s) \dots \dots .. (1 + \tau_p s)}{s^2(1 + \tau_a s)(1 + \tau_b s) \dots \dots . (1 + \tau_q s)}$$

$$K_a = \frac{K\,.\,1\,.\,1. \dots .\,1}{1.\,1\,.\,1 \dots \dots 1} = K$$

$$e_{ss} = \frac{A}{K} \qquad (5.22)$$

Processes Control:

Second-Order Systems

Advantages and disadvantages of static error coefficients

- This method provides the variation of error with time.
- It is applicable only for step, ramp, and parabolic forcing functions. It could not be applied for others of forcing function.
- The values of steady-state error of ***zero*** and ***infinity*** cannot give exact value of the error.
- This method could only be used for stable systems.

Dynamic error coefficients

This method could be used to overcome the disadvantages of static error coefficients. According to this method the (e_{ss}) could be expressed as follows:

$$e_{ss} = K_0 R(t) + K_1 R'(t) + K_2 R''(t) \qquad (6.1)$$

Where:

R(t) is the input forcing function in time (t) domain.

$$K_0 = \lim_{s \to 0} F_1(s) \qquad (6.2)$$

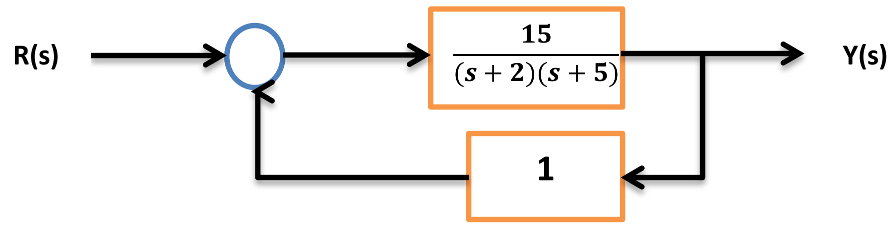
$$K_1 = \lim_{s \to 0} \frac{dF_1(s)}{ds} \qquad (6.3)$$

$$K_2 = \lim_{s \to 0} \frac{d^2 F_1(s)}{ds^2} \qquad (6.4)$$

Where

$$F_1(s) = \frac{1}{1 + G(s)H(s)} \qquad (6.5)$$

Example 6.1. Find the steady-state error for the input forcing function as [**3+8t+(5/2) t^2**] for the following system:



Ans.:

$$G(s) = \frac{15}{(s+2)(s+5)} \quad ; \quad H(s) = 1 \quad ; \quad R(t) = 3 + 8t + \frac{5}{2}t^2$$

$$e_{ss} = K_0 R(t) + K_1 R'(t) + K_2 R''(t) \qquad (6.1)$$

$$R'(t) = 8 + 5t$$

$$R''(t) = 5$$

$$F_1(s) = \frac{1}{1 + G(s)H(s)} \qquad (6.5)$$

$$F_1(s) = \frac{1}{1 + \frac{15}{(s+2)(s+5)} * 1} = \frac{1}{\frac{(s+2)(s+5)+15}{(s+2)(s+5)}}$$

$$F_1(s) = \frac{(s+2)(s+5)}{(s+2)(s+5)+15} = \frac{s^2+7s+10}{s^2+7s+25}$$

So,

$$F_1(s) = \frac{s^2+7s+10}{s^2+7s+25}$$

$$K_0 = \lim_{s \to 0} F_1(s) \qquad (6.2)$$

$$K_0 = \lim_{s \to 0} \frac{s^2+7s+10}{s^2+7s+25} = \frac{10}{25}$$

$$K_0 = 0.4$$

$$K_1 = \lim_{s\to 0} \frac{dF_1(s)}{ds} \qquad (6.3)$$

$$K_1 = \lim_{s\to 0} \frac{d}{ds}\left[\frac{s^2+7s+10}{s^2+7s+25}\right]$$

$$K_1 = \lim_{s\to 0} \left[\frac{(s^2+7s+25)(2s+7)-(s^2+7s+10)(2s+7)}{(s^2+7s+25)^2}\right]$$

$$K_1 = \lim_{s\to 0} \left[\frac{30s+105}{(s^2+7s+25)^2}\right] = \frac{105}{(25)^2}$$

$$K_1 = 0.168$$

$$K_2 = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} \qquad (6.4)$$

$$K_2 = \lim_{s\to 0} \frac{d^2}{ds^2}\left[\frac{s^2+7s+10}{s^2+7s+25}\right]$$

$$K_2 = \lim_{s\to 0} \left\{\frac{(s^2+7s+25)^2(30)-(30\,s+105)[2(s^2+7s+25)(2s+7)}{(s^2+7s+25)^4}\right\}$$

$$K_2 = \lim_{s\to 0} \left[\frac{30\,(25)^2-2(105)(25)(7)}{(25)^4}\right]$$

$$K_2 = -0.046$$

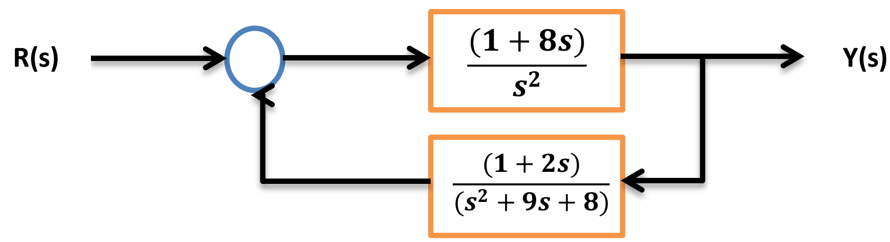
$$e_{ss} = K_0R(t) + K_1R'(t) + K_2R''(t)$$

$$e_{ss} = 0.4\left(3+8t+\frac{5}{2}t^2\right) + 0.168\,(8+5t) - 0.046\,(5)$$

$$e_{ss} = t^2 + 4.04\,t + 2.424$$

H.W. 6.1. *For the final equation obtained in Example 6.1, take the values of time as (0, 0.5, 1, 1.5, 2, 2.5, 3) and draw a graph related the (e_{ss}) with time and discuss the behavior of this graph obtained.*

Example 6.2. Find the steady-state error for the input forcing function as [**3+8t+(5/2) t**^{2}] for the following system:



Ans.:

$$G(s) = \frac{(1+8s)}{s^2} \;;\; H(s) = \frac{(1+2s)}{(s^2+9s+8)} \quad ; \quad R(t) = 3 + 8t + \frac{5}{2}t^2$$

$$e_{ss} = K_0 R(t) + K_1 R'(t) + K_2 R''(t) \qquad (6.1)$$

$$R'(t) = 8 + 5t$$

$$R''(t) = 5$$

$$F_1(s) = \frac{1}{1 + G(s)H(s)} \qquad (6.5)$$

$$F_1(s) = \frac{1}{1 + \frac{(1+8s)}{s^2}\frac{(1+2s)}{(s^2+9s+8)}}$$

$$F_1(s) = \frac{1}{\frac{s^2(s^2+9s+8)+(1+8s)(1+2s)}{s^2(s^2+9s+8)}}$$

$$F_1(s) = \frac{s^4+9s^3+8s^2}{s^4+9s^3+24s^2+10s+1}$$

$$K_0 = \lim_{s\to 0} F_1(s) \qquad (6.2)$$

$$K_0 = \lim_{s\to 0} \frac{s^4 + 9s^3 + 8s^2}{s^4 + 9s^3 + 24s^2 + 10s + 1} = \frac{0}{1}$$

$$K_0 = 0$$

$$K_1 = \lim_{s\to 0} \frac{dF_1(s)}{ds} \qquad (6.3)$$

$$K_1 = \lim_{s\to 0} \frac{d}{ds}\left[\frac{s^4 + 9s^3 + 8s^2}{s^4 + 9s^3 + 24s^2 + 10s + 1}\right]$$

$$K_1 = \frac{0}{1}$$

$$K_1 = 0$$

$$K_2 = \lim_{s\to 0} \frac{d^2F_1(s)}{ds^2} \qquad (6.4)$$

$$K_2 = \lim_{s\to 0} \frac{d^2}{ds^2}\left[\frac{s^4 + 9s^3 + 8s^2}{s^4 + 9s^3 + 24s^2 + 10s + 1}\right]$$

$$K_2 = 0$$

$$e_{ss} = K_0R(t) + K_1R'(t) + K_2R''(t)$$

$$e_{ss} = 0\left(3 + 8t + \frac{5}{2}t^2\right) + 0\,(8 + 5t) - 0\,(5)$$

$$e_{ss} = 0 \qquad \textit{what does that mean}?$$

H.W. 6.2. Find the steady-state error for the input forcing function as [**3+8t+(5/2) t**^{2}] for the following system

$G(s) = \frac{3}{(s^2+4s+5)}$ and $H(s) = (s+4)$

Processes Control:

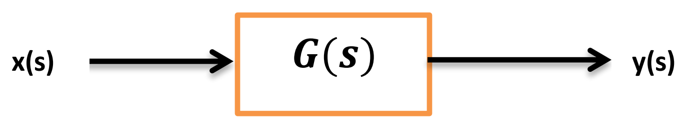
Block diagram development

Block diagram representations of control systems are developed by:

1. Identifying the individual elements of the control system (as was done in the previous course for the process mixing tank),
2. Identifying the input and output for each element,
3. Representing the individual input/ output transfer function relationship for each element in the block diagram,
4. Finally, combining the individual block diagrams for each element to obtain the overall block diagram.

Block diagrams containing the following elements:

1- ***Block*** : It is the schematic representation that relating the input and output for each unit in the studied process as follows:

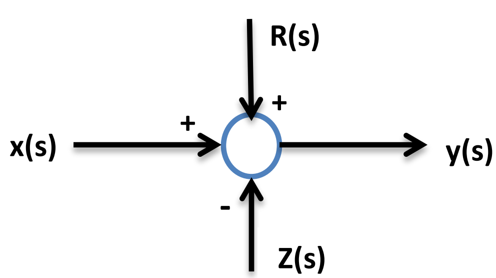


Where:

x(s) is the input (forcing function), y(s) is the output (response), and G(s) is the transfer function for a particular unit.

$$G(s) = \frac{y(s)}{x(s)}$$

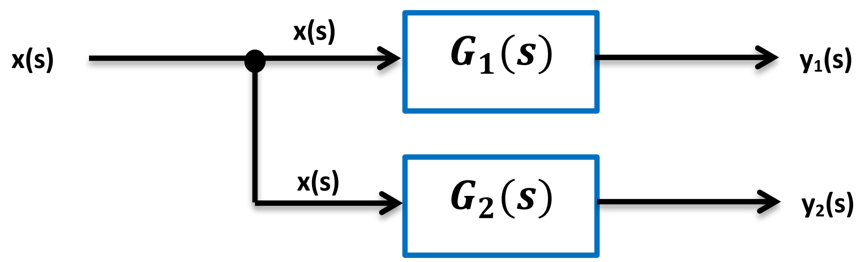
2- ***Summing point***: It is the representation of a point where two or more signals can be added or subtracted



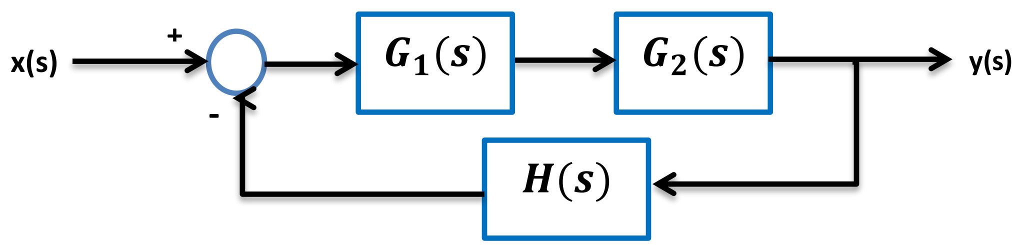
The output of any summing point is the algebraic sum of the entering input signals as follows:

$$y(s) = x(s) + R(s) - Z(s)$$

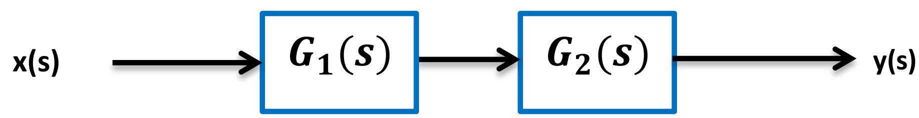
3- ***Take-off point***: It is the point at which the signal could be conducted to two or more blocks as input. This element will not alter the original signal as represented below:



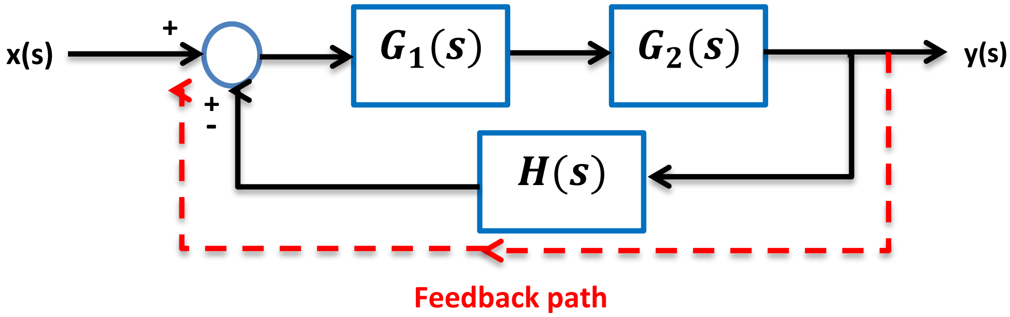
4- ***Forward path***: It is the direction of flow of signal from input to output. For the following closed-loop system:



The forward path is

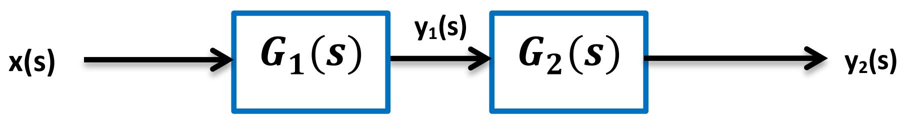


5- ***Feedback path***: It is the path of flow of signal from output to input such as the following closed-loop system:

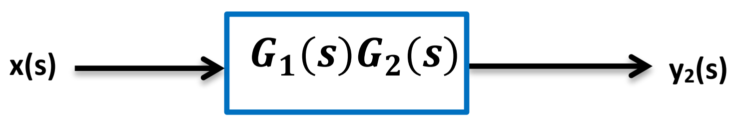


There are several techniques that could be used to reduce the items of the block diagram to obtain the final gain, i.e. the overall transfer function, of the studied process. These techniques are:

1- ***Blocks in series***: Blocks connected in cascaded could be reduced to a signal block, and the product of the individual transfer functions will be the new transfer function where the initial input and the final output will not alter. For example, the following cascaded blocks

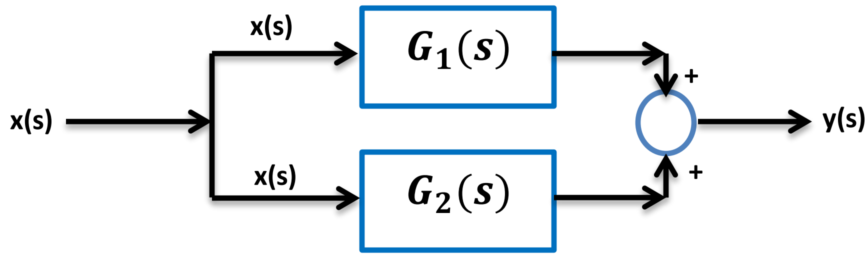


Will be reduced to

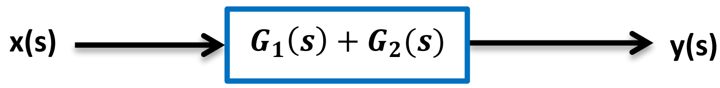


The condition of this procedure is that must be no summing points and/or take off points presented among blocks.

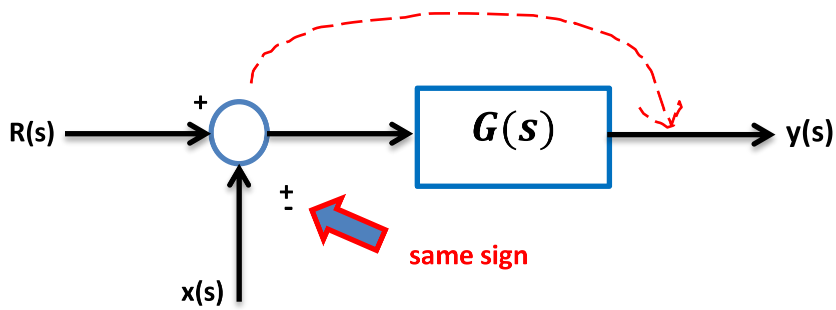
2- ***Blocks in parallel***: Blocks connected in parallel could be reduced to a signal block, and the new transfer function will be the sum of the individual transfer functions. The following example explain this technique.



This could be reduced to

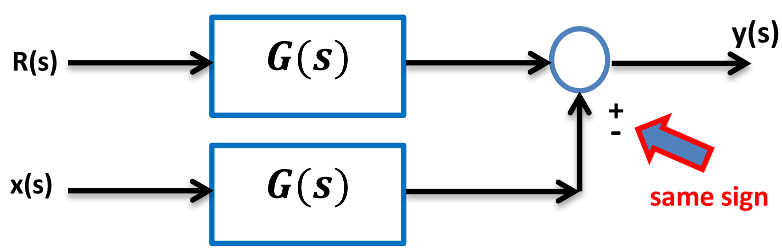


3- ***Moving a summing point after a block***: This technique could be performed to move a summing point from its location to a new location after the block such as follows



$$y(s) = [R(s) \pm x(s)]G(s)$$

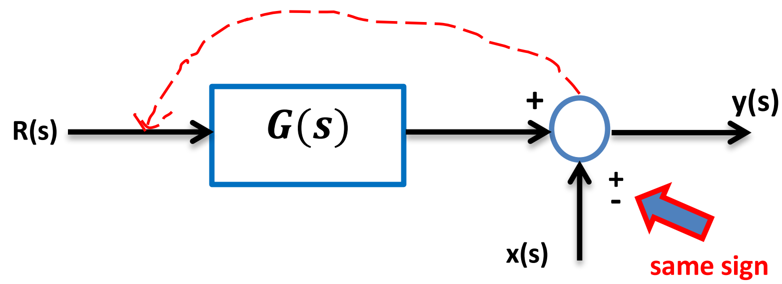
It will be after moving the summing point after the block



$$y(s) = [R(s) \pm x(s)]G(s)$$

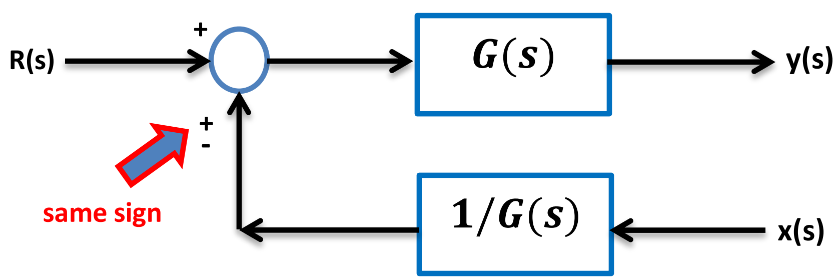
As observed, each of side-streams should be multiplied by the transfer function presented in the main path that was passed front by the summing point.

4- ***Moving a summing point before a block***: This technique could be used to move a summing point from its location to a new location before the block as shown below



$$y(s) = R(s)G(s) \pm x(s)$$

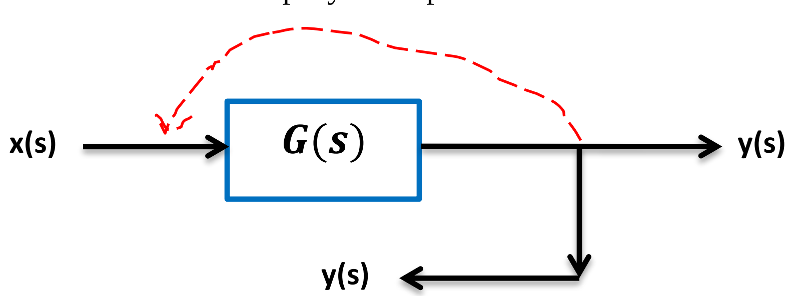
It will be after moving the summing point before the block



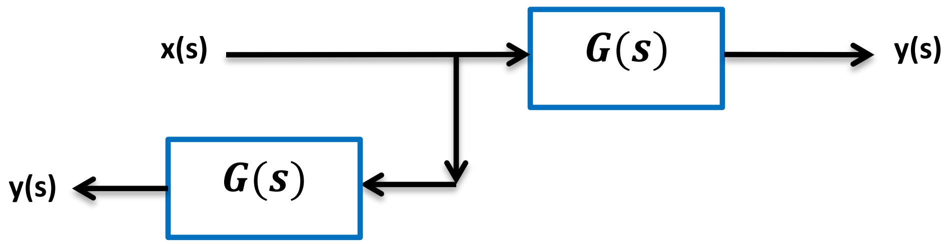
$$y(s) = \left[R(s) \pm \frac{1}{G(s)}x(s)\right]G(s)$$

As noted that each of side-streams should be multiplied by the reciprocal of the transfer function presented in the main path that was passed back by the summing point.

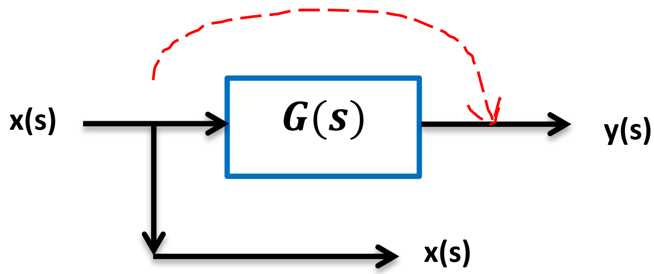
5- ***Moving a tack off point before a block***: This technique could be used to move a take-off point a head of a block to simplify some problems.



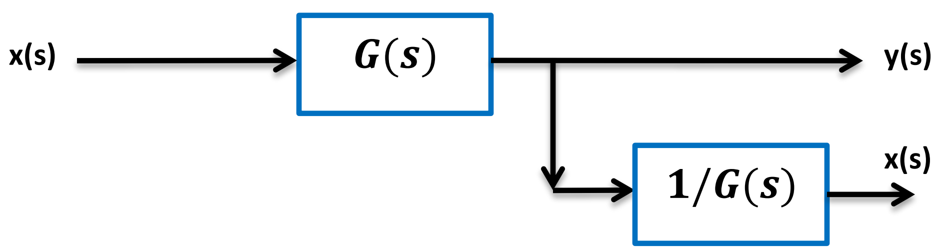
It will be after moving the take-off point before the block as follows:



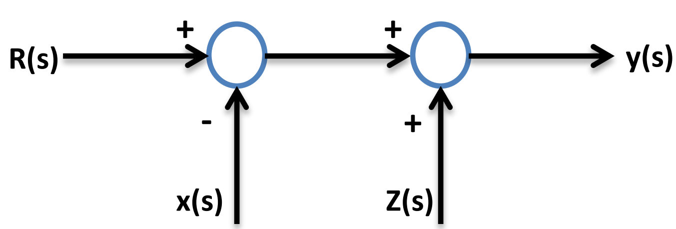
6- ***Moving a tack off point after a block***: This technique could be used to move a take-off point a head of a block to simplify some problems.



It will be after moving the take-off point after the block as follows:

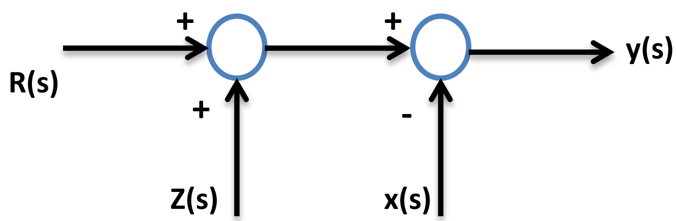


7- ***Interchanging of two summing points***: This technique provides the ability of interchanging the location of two summing points as shown below.



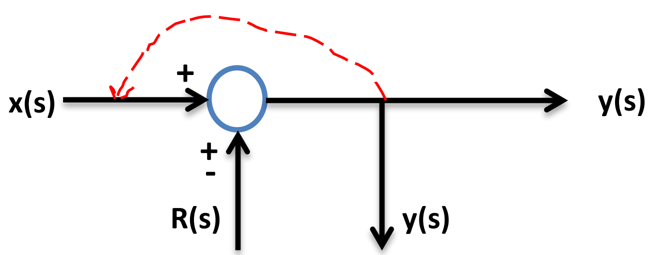
$$\boldsymbol{y(s) = R(s) - x(s) + Z(s)}$$

After the interchanging process, it will be as follows:

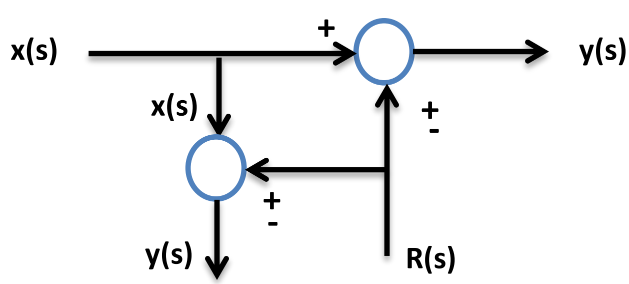


$$\boldsymbol{y(s) = R(s) + Z(s) - x(s)}$$

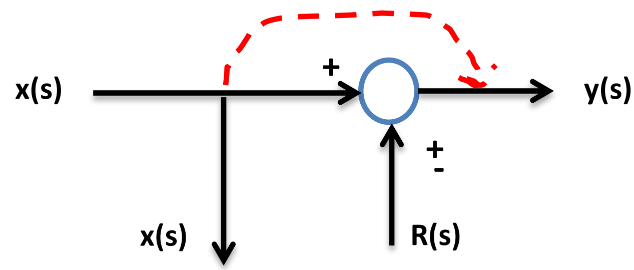
8- ***Moving a take-off point before a summing point***: There are some steps to applied this technique which are: (a) Move the take-off point, (b) Add a summing point to the moved side-stream, and (c) Algebraically, add the new stream with the side-stream entering the original summing point without changing any sign.



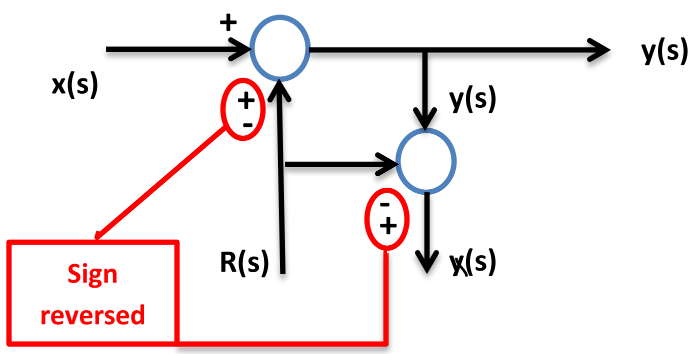
$$\boldsymbol{y(s) = x(s) \pm R(s)}$$



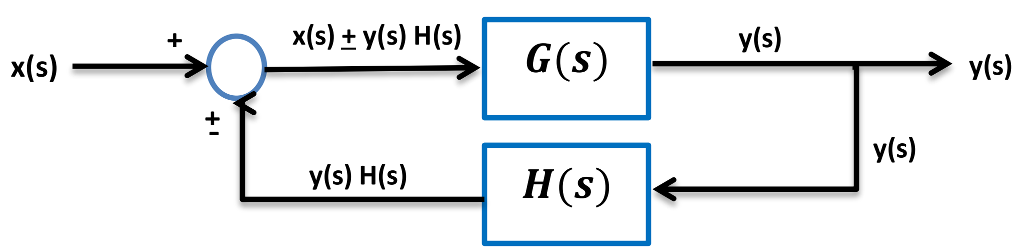
9- ***Moving a take-off point after a summing point***: This technique is used when the movement of a take-off point after a summing point is required as explained below:



$$y(s) = x(s) \pm R(s)$$



10- Eliminating a feedback loop: There two kinds of feedback closed-loop system, negative and positive. Both of them could be eliminated to only one block having the new transfer function as revealed below.



$$y(s) = [x(s) \pm y(s)H(s)]G(s)$$

$$y(s) = x(s)G(s) \pm y(s)H(s)G(s)$$

$$y(s) \mp y(s)H(s)G(s) = x(s)G(s)$$

$$y(s)[1 \mp H(s)G(s)] = x(s)G(s)$$

$$\frac{y(s)}{x(s)} = \frac{G(s)}{1 \mp H(s)G(s)}$$

So, the final transfer function ***T(s)*** or ***G(s)*** is

$$T(s) = \frac{G(s)}{1 \mp H(s)G(s)}$$

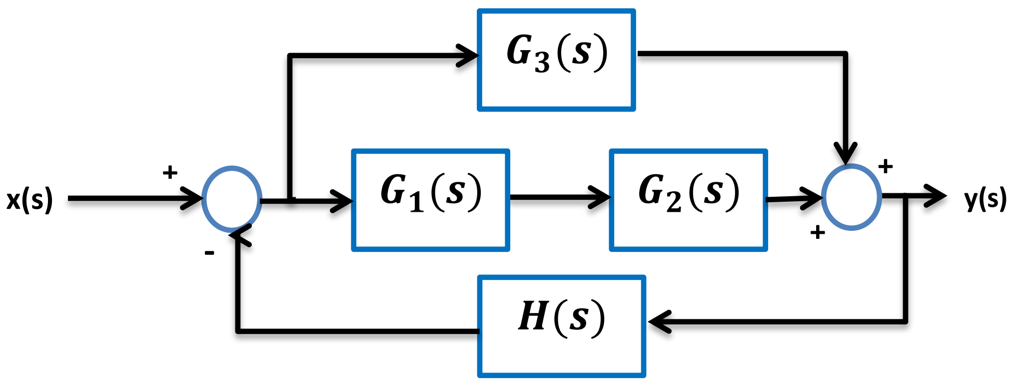
Follow the following steps for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

1- Check for the blocks connected in series and simplify.
2- Check for the blocks connected in parallel and simplify.
3- Check for the blocks connected in feedback loop and simplify.
4- If there is difficulty with take-off point while simplifying, shift it towards right.
5- If there is difficulty with summing point while simplifying, shift it towards left.
6- Repeat the above steps till you get the simplified form, i.e., single block.

Processes Control:

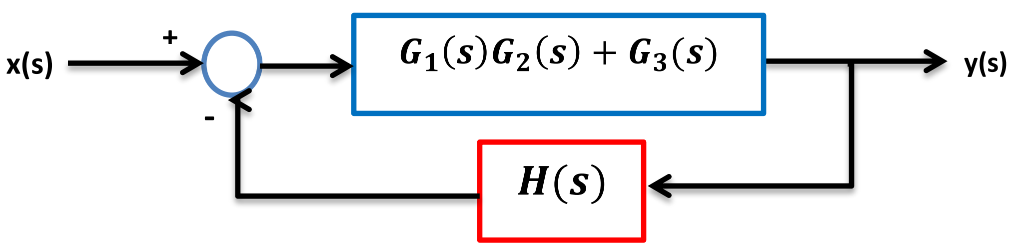
Block diagram development

Example 8.1. Find the overall transfer function for the following process:

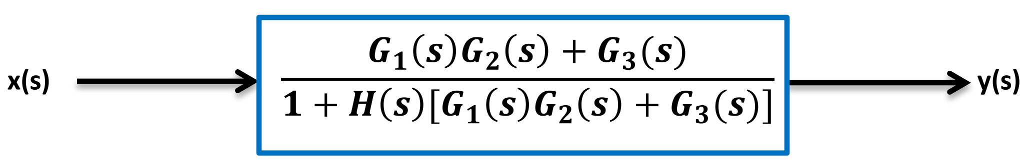


Ans.:

At first, reduce the cascade blocks [**G_1(s) G_2(s)**] to one block then reduce it with the parallel block [**G_3(s)**]. The new form of this process will be



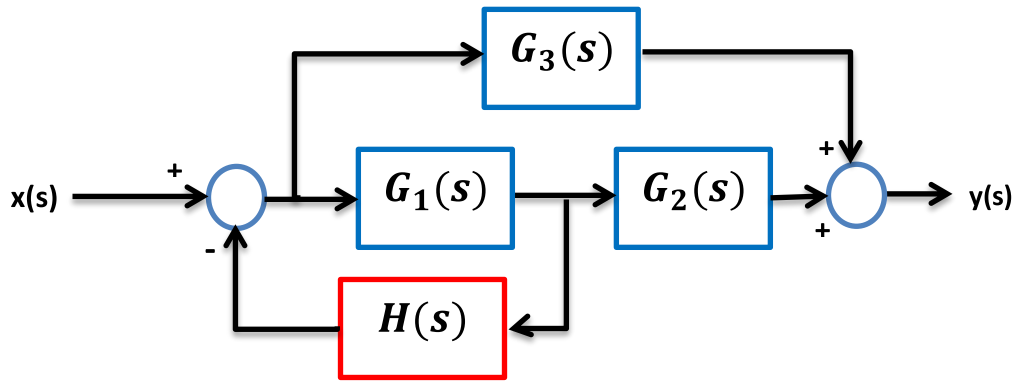
Now, reduce the negative feedback loop to be the final form of the process as follow:



So, the overall transfer function is

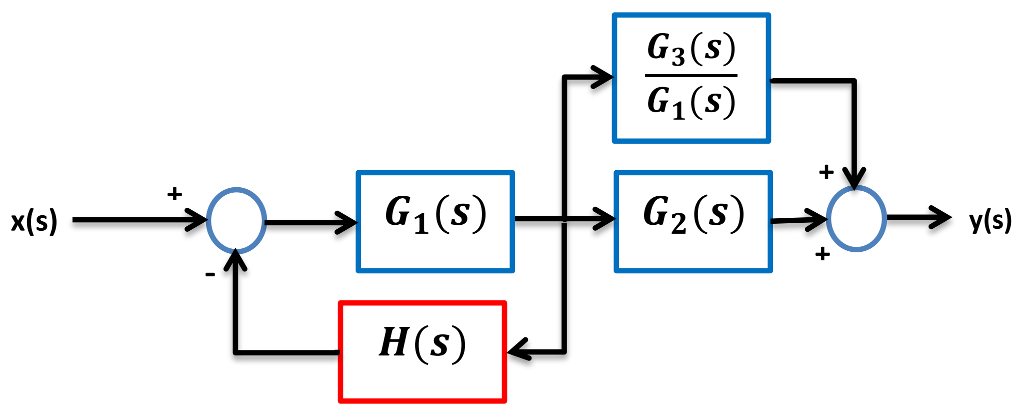
$$T(s) = \frac{G_1(s)G_2(s) + G_3(s)}{1 + H(s)[G_1(s)G_2(s) + G_3(s)]}$$

Example 8.2. Reduce the following block diagram and find the overall transfer function of the process.

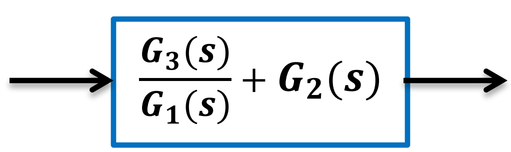


Ans.:

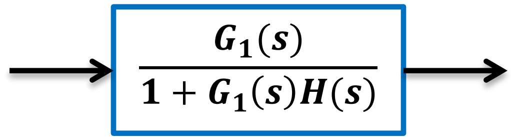
Move the take-off point of the block [**G_3(s)**] to the same location of the take-off of the feedback block [**H(s)**].



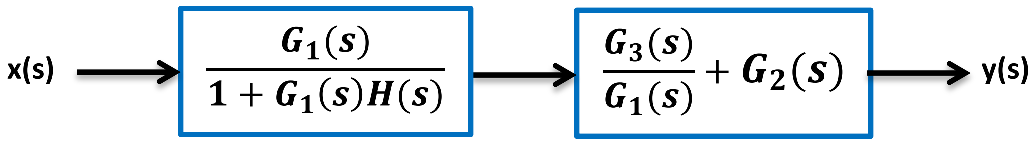
Now, simplify the parallel blocks to be



Then, reduce the negative feedback loop to be as follows



Finally, merge two blocks obtained to find the overall transfer function

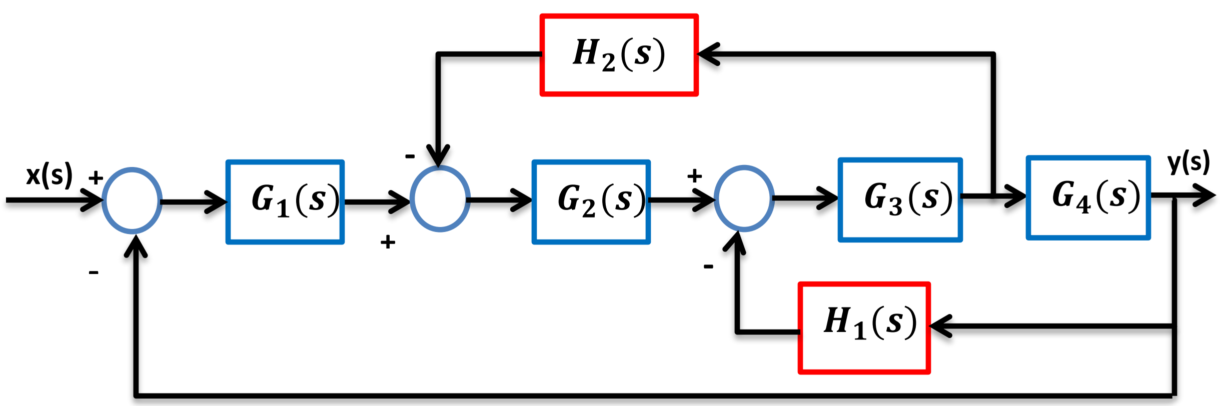


$$T(s)=\frac{G_1(s)}{1+G_1(s)H(s)}\times\frac{G_3(s)}{G_1(s)}+G_2(s)$$

$$T(s)=\frac{G_1(s)}{1+G_1(s)H(s)}\times\frac{G_3(s)+G_1(s)G_2(s)}{G_1(s)}$$

$$T(s)=\frac{G_1(s)G_2(s)+G_3(s)}{1+G_1(s)H(s)}$$

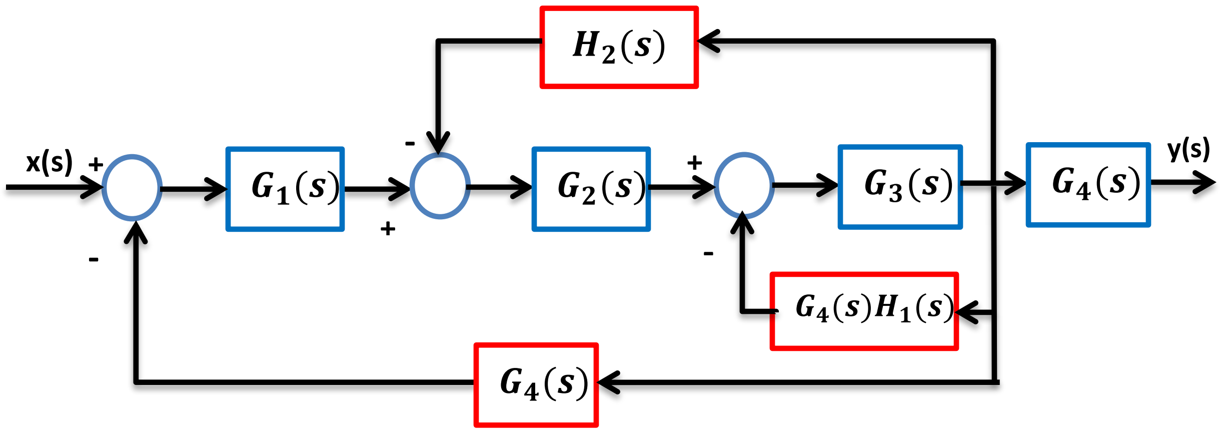
Example 8.3. Simplify the following process to find the overall transfer function.



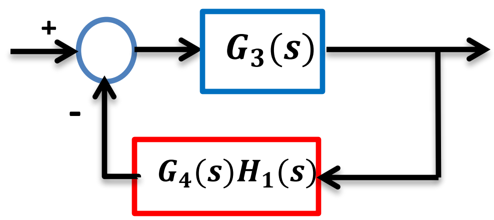
Ans.:

There are several suggestions to solve this diagram. Let following the next suggestion:

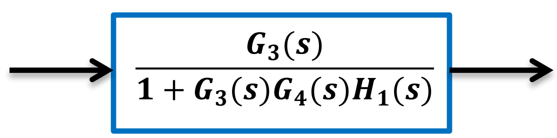
At first, move the last take-off point to be ahead of the block [**G_4(s)**] regarding the change of the feedback streams of the wire and the block [**H_1(s)**] as follows:



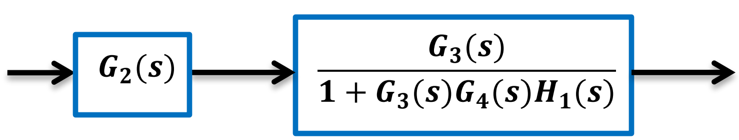
Now, simplify the following negative feedback loop



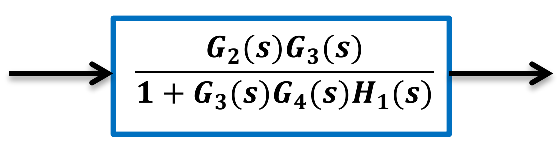
To be as follows:



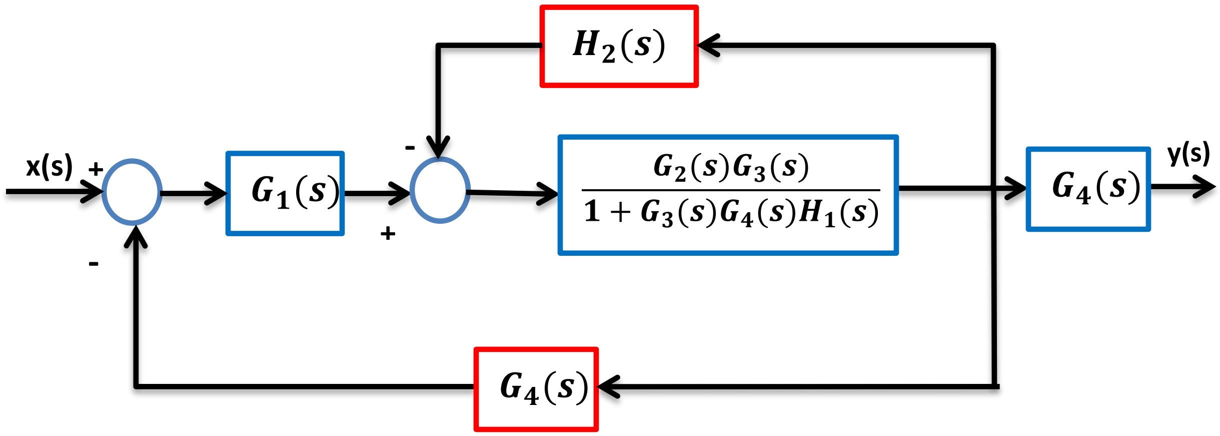
Then



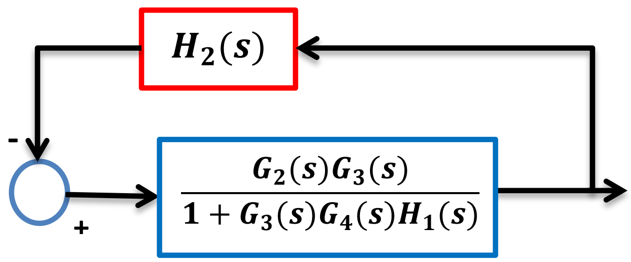
Will be as follows:



Until now, the new form will be



The negative feedback of [$H_2(s)$]

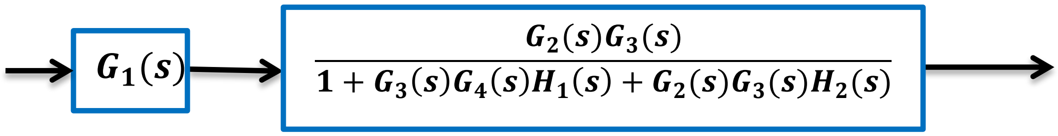


Will be as shown below:

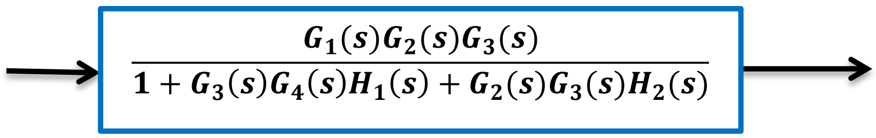
$$\frac{\dfrac{G_2(s)G_3(s)}{1+G_3(s)G_4(s)H_1(s)}}{1+\dfrac{G_2(s)G_3(s)}{1+G_3(s)G_4(s)H_1(s)}\times H_2(s)}$$

$$\frac{G_2(s)G_3(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)}$$

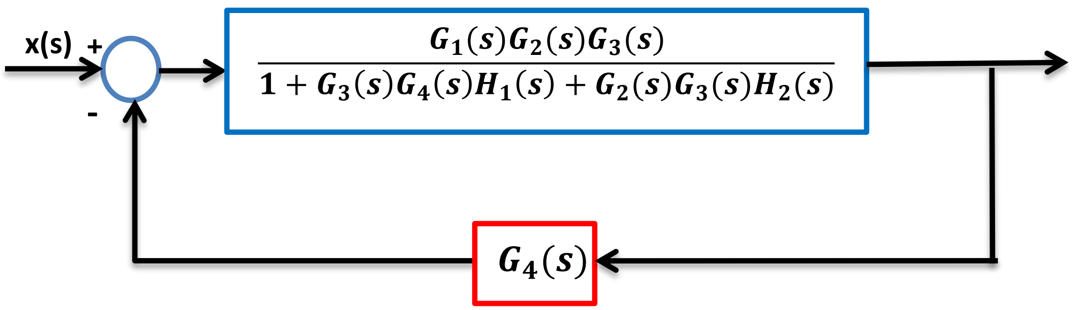
Then, multiply the last block by the block [$G_1(s)$] because they are connected in series



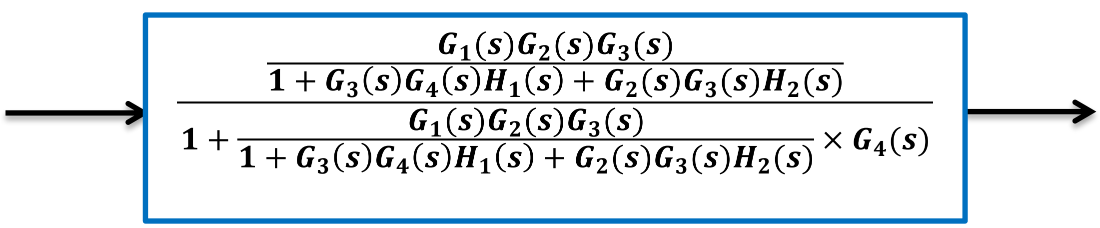
It will be:

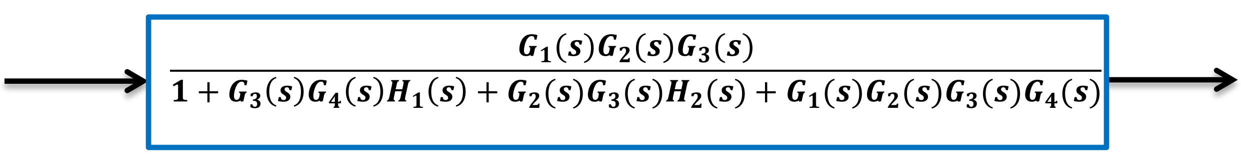


The negative feedback of [G_4(s)]



Will be as follows:





Consequently, multiply the final block by the block [**G_4(s)**] which are a cascade blocks

$$\longrightarrow \boxed{\frac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_1(s)G_2(s)G_3(s)G_4(s)}} \longrightarrow \boxed{G_4(s)} \longrightarrow$$

Will be reduced to be:

$$\longrightarrow \boxed{\frac{G_1(s)\ G_2(s)\ G_3(s)\ G_4(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_1(s)G_2(s)G_3(s)G_4(s)}} \longrightarrow$$

This block is the final form of the studied process and the overall transfer function is:

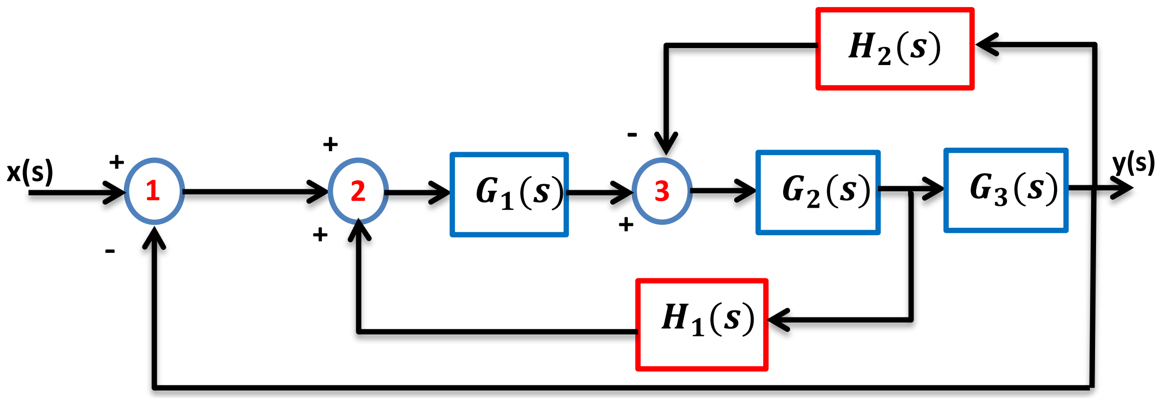
$$T(s)=\frac{G_1(s)\ G_2(s)\ G_3(s)\ G_4(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_1(s)G_2(s)G_3(s)G_4(s)}$$

H.W. 8.1. *Suggest another solution for the block diagram of Example 8.3 obtaining the same result of the overall transfer function.*

Processes Control:

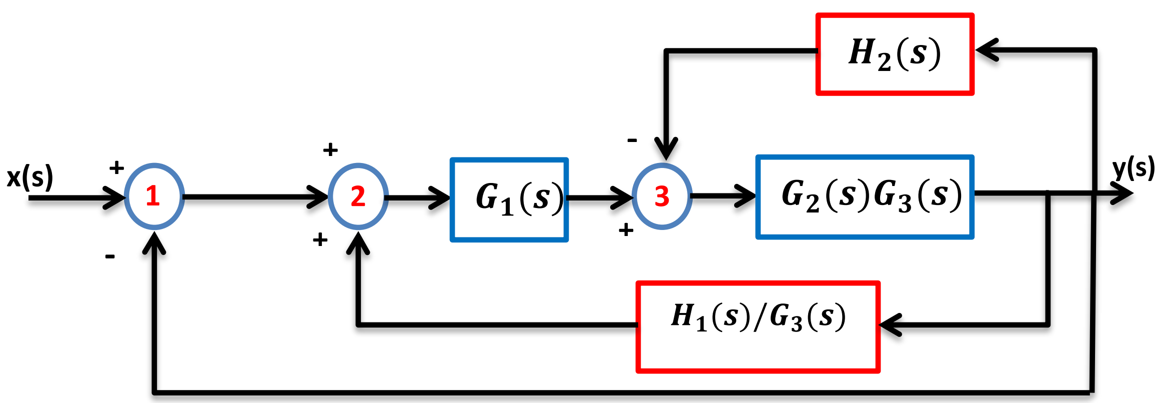
Block diagram development

Example 9.1. Reduce the following diagram and find the overall transfer function:

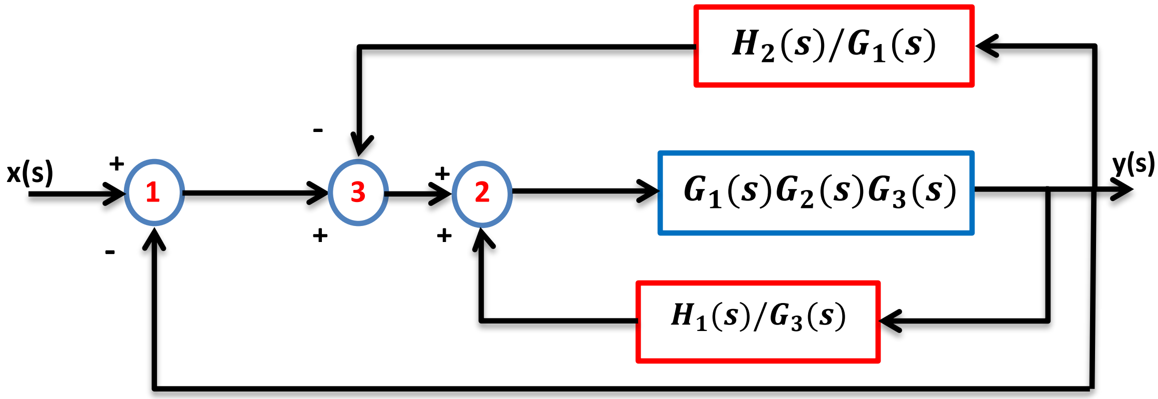


Ans.:

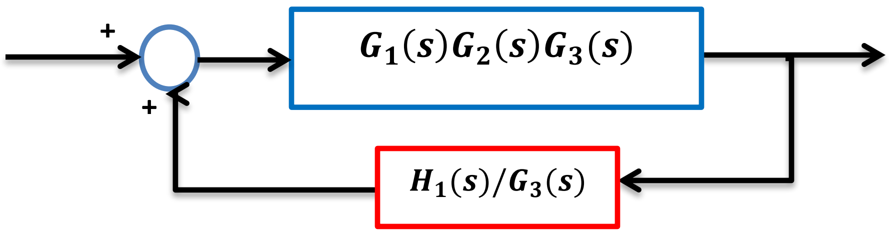
At first, move the take-off of the positive feedback of [$\mathbf{H_1(s)}$] to be after the block of [$\mathbf{G_3(s)}$] and divided the positive feedback of [$\mathbf{H_1(s)}$] by the block of [$\mathbf{G_3(s)}$]. Then multiply the cascade blocks of [$\mathbf{G_2(s)}$ and $\mathbf{G_3(s)}$] to be one block.



Now, move the summing point No. 3 ahead of the summing point No. 2 and simplify the cascade blocks of [$\mathbf{G_1(s)}$] and [$\mathbf{G_2(s)G_3(s)}$] as follows:



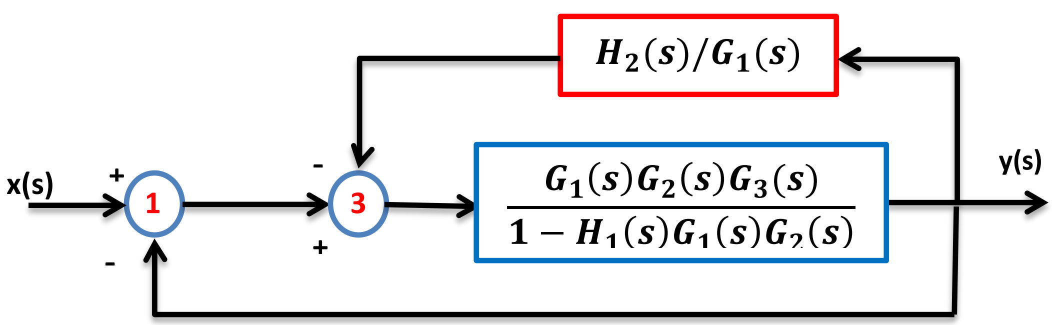
Simplify the following positive feedback



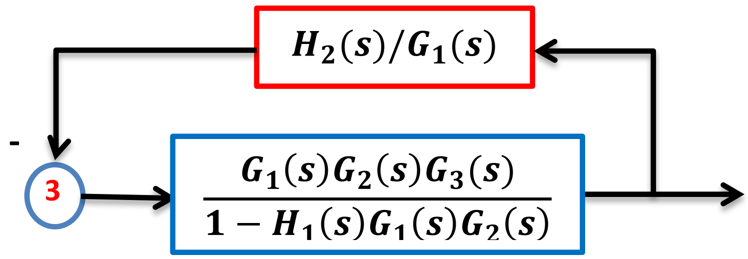
It will be as follow:

$$\frac{G_1(s)G_2(s)G_3(s)}{1-\frac{H_1(s)}{G_3(s)}[G_1(s)G_2(s)G_3(s)]}$$

$$\frac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)}$$



The negative feedback could be reduced to be as follows:

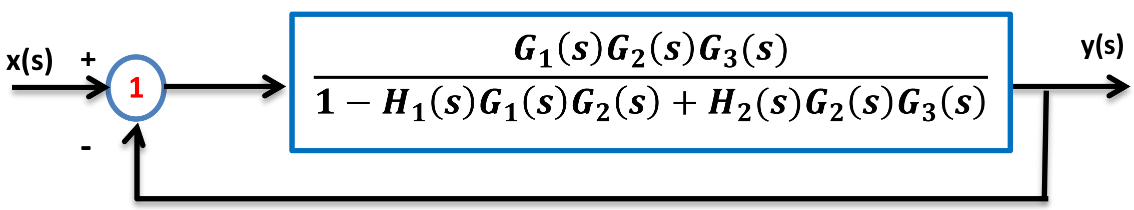


It will be after simplification as:

$$\frac{\dfrac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)}}{1+\dfrac{H_2(s)}{G_1(s)}\left[\dfrac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)}\right]}$$

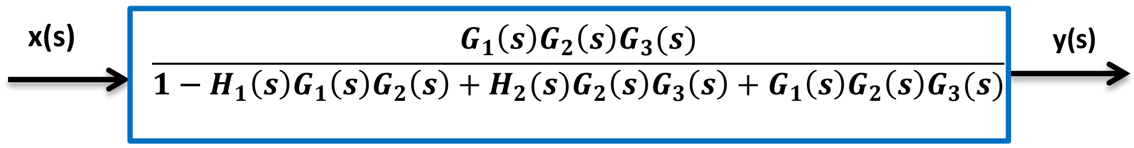
$$\frac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)+H_2(s)G_2(s)G_3(s)}$$

The new form of the block diagram is



The final form of this process after simplifying the unity-negative feedback is

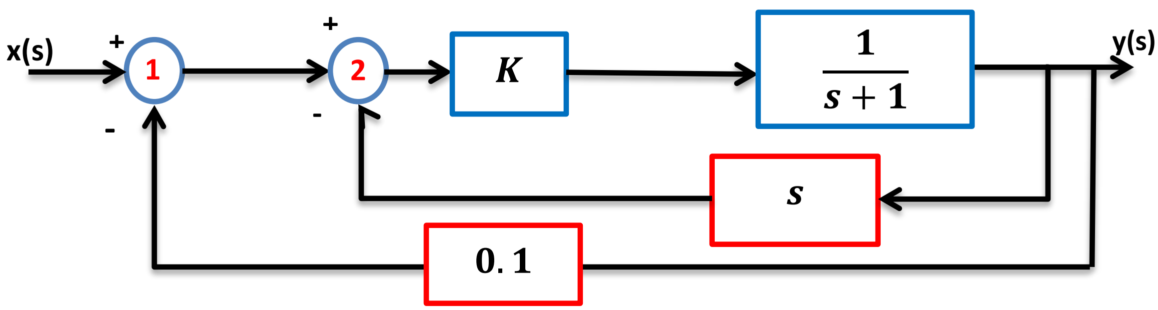
$$x(s) \rightarrow \frac{\dfrac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)+H_2(s)G_2(s)G_3(s)}}{1+\dfrac{G_1(s)G_2(s)G_3(s)}{1-H_1(s)G_1(s)G_2(s)+H_2(s)G_2(s)G_3(s)}} \rightarrow y(s)$$



The overall transfer function of this process will be:

$$T(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 - H_1(s)G_1(s)G_2(s) + H_2(s)G_2(s)G_3(s) + G_1(s)G_2(s)G_3(s)}$$

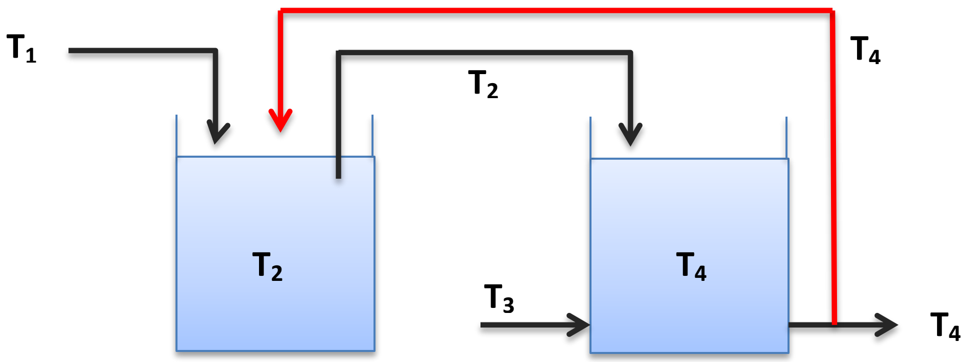
Example 9.2. Reduce the following diagram and find the overall transfer function.



Processes Control:

Block diagram development

Example 10.1. Two tanks are connected in series with recycle as shown in the following figure assuming that the flow rates entering the tanks are constants and are set such that accumulation occurs in neither tank. However, both T_1 and T_3 can vary, ultimately, leading to a change in the temperature of the fluid exiting the tank T_4.



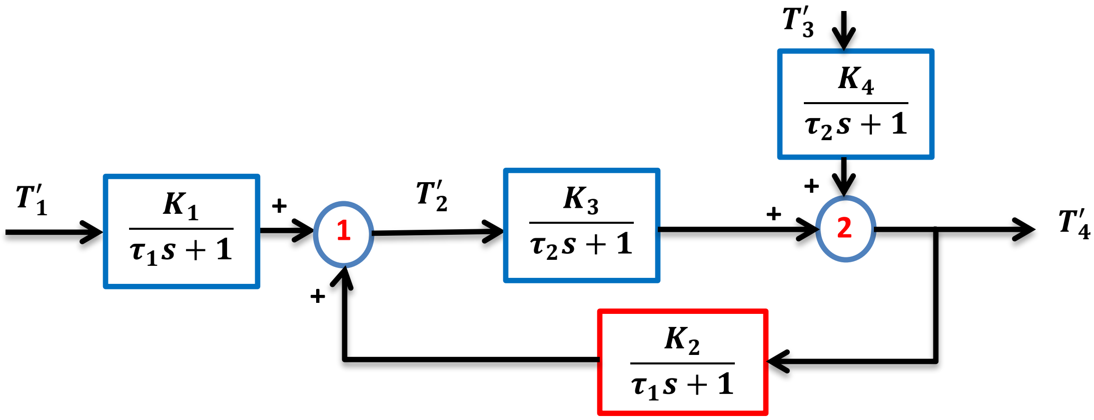
When an energy balance is completed on each tank, the relations shown below are developed. Turn these equations into block diagram and use the diagram to find T'_4/T'_1 assuming that T_3 is zero.

$$T'_2 = \frac{K_1}{\tau_1 s + 1} T'_1 + \frac{K_2}{\tau_1 s + 1} T'_4 \qquad \textit{left tank}$$

$$T'_4 = \frac{K_3}{\tau_2 s + 1} T'_2 + \frac{K_4}{\tau_2 s + 1} T'_3 \qquad \textit{right tank}$$

***Ans*.:**

The block diagram of this process will be developed depending on the mathematical relations between the deviation values of the temperatures entire the two tanks and the individual first order effect of other temperature in each tank. The form of the block diagram will be as follows:



$$T'_4 = \frac{K_3}{\tau_2 s+1} T'_2 + \frac{K_4}{\tau_2 s+1} T'_3$$

$$T'_4 = \frac{K_3}{\tau_2 s+1}\left[\frac{K_1}{\tau_1 s+1} T'_1 + \frac{K_2}{\tau_1 s+1} T'_4\right] + \frac{K_4}{\tau_2 s+1} T'_3$$

According to the assumption give that T_3 equals zero, so this equation will be as explained below:

$$T'_4 = \frac{K_3}{\tau_2 s+1}\left[\frac{K_1}{\tau_1 s+1} T'_1 + \frac{K_2}{\tau_1 s+1} T'_4\right]$$

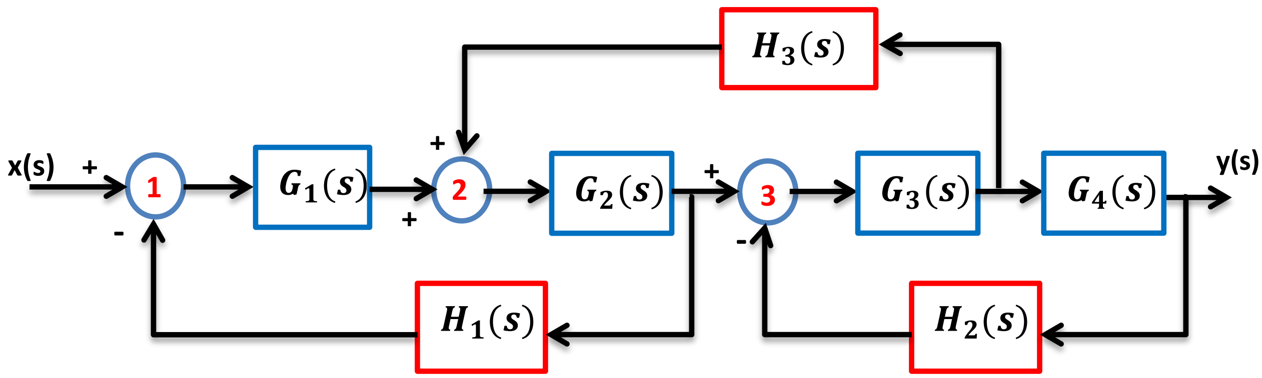
$$T'_4 = \frac{K_3}{\tau_2 s+1}\frac{K_1}{\tau_1 s+1} T'_1 + \frac{K_3}{\tau_2 s+1}\frac{K_2}{\tau_1 s+1} T'_4$$

$$T'_4\left[1 - \frac{K_3}{\tau_2 s+1}\frac{K_2}{\tau_1 s+1}\right] = \frac{K_3}{\tau_2 s+1}\frac{K_1}{\tau_1 s+1} T'_1$$

$$\frac{T'_4}{T'_1} = \frac{\frac{K_1 K_3}{(\tau_1 s+1)(\tau_2 s+1)}}{1 - \frac{K_2 K_3}{(\tau_1 s+1)(\tau_2 s+1)}}$$

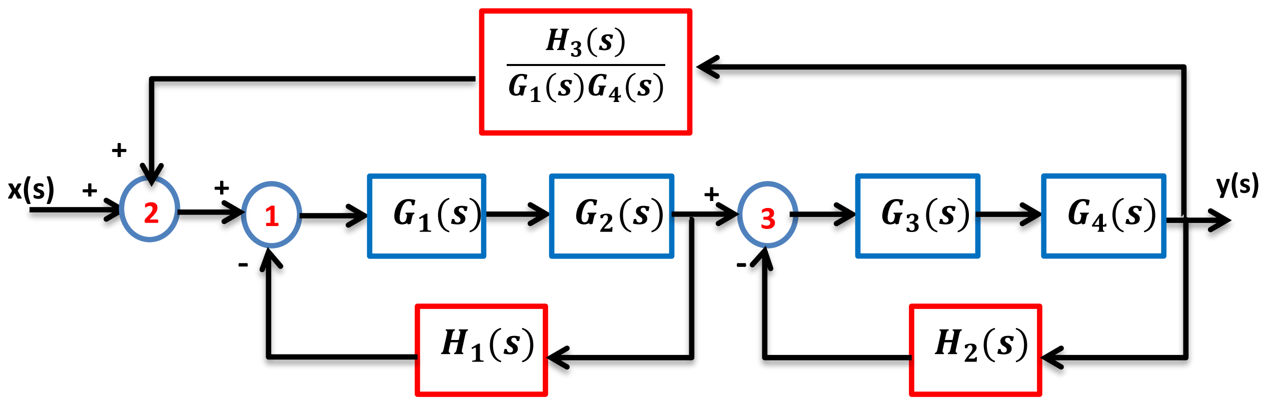
$$\frac{T'_4}{T'_1} = \frac{K_1 K_3}{(\tau_1 s+1)(\tau_2 s+1) - K_2 K_3}$$

Example 10.2. Simplify the block diagram shown below and obtain the overall closed-loop function.

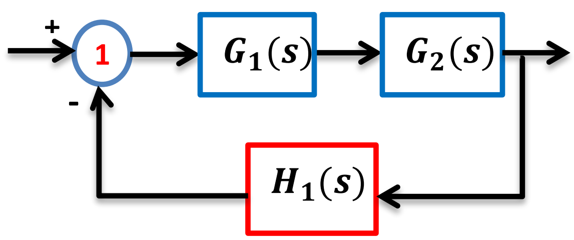


***Ans.*:**

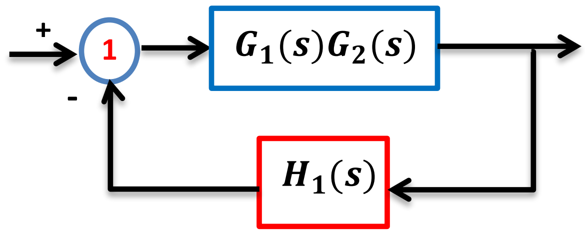
At first, move the summing point No. 2 ahead of the summing point No. 1 of the positive feedback of [$H_3(s)$] as well as its take-off point by the blocks [$G_1(s)$] and [$G_4(s)$].

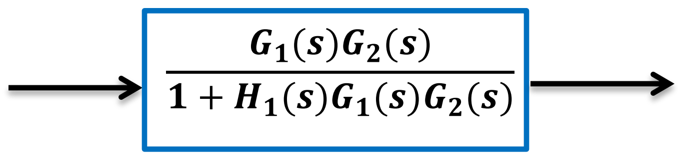


Now , we have two negative feedback loops that could be reduced as explained below individually.

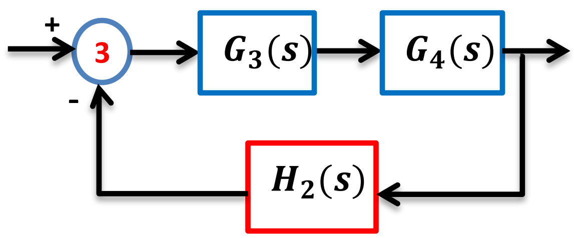


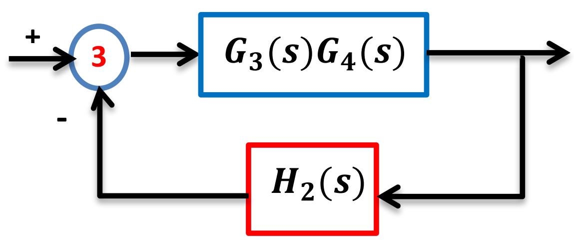
Loop No. 1

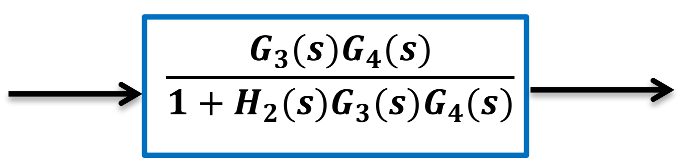




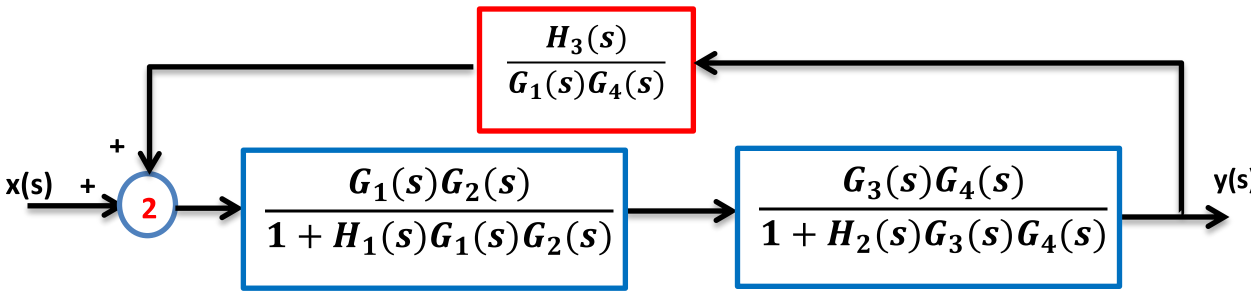
Loop No. 2



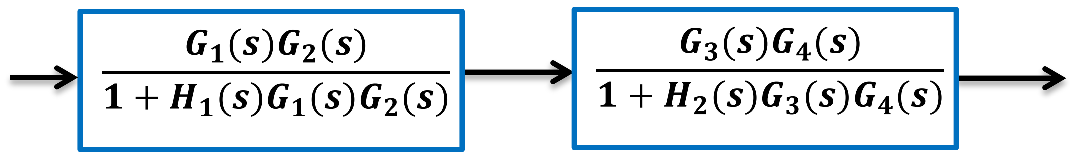




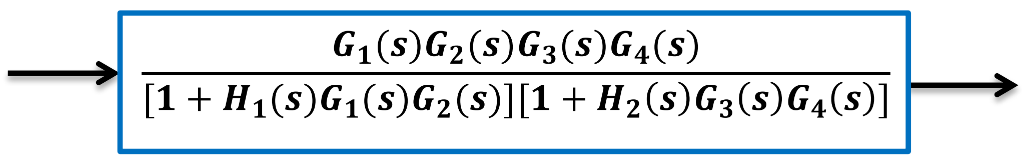
The new form of the block diagram will be as follows:

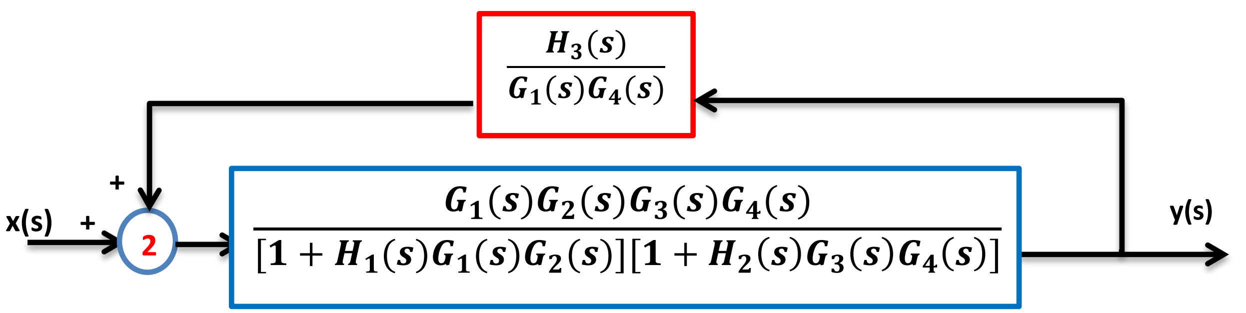


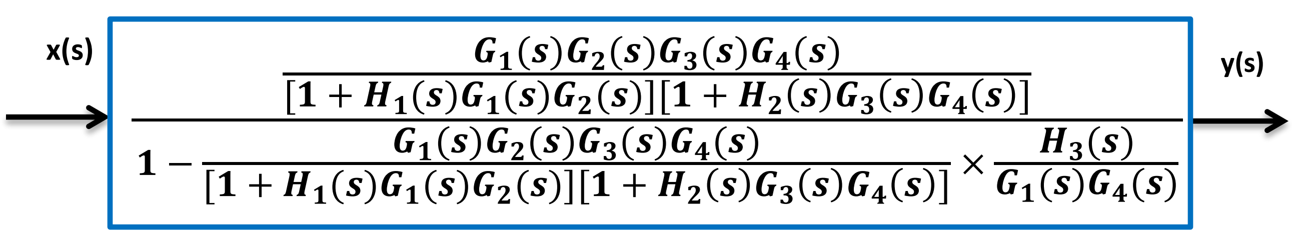
The cascade blocks should be reduced



To be as follows:







The final block diagram of the process will be

$$x(s) \longrightarrow \boxed{\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{[1+H_1(s)G_1(s)G_2(s)][1+H_2(s)G_3(s)G_4(s)]-H_3(s)G_2(s)G_3(s)}} \longrightarrow y(s)$$

The overall transfer function of the closed-loop is

$$T(s) = \frac{G_1(s)G_2(s)G_3(s)G_4(s)}{[1+H_1(s)G_1(s)G_2(s)][1+H_2(s)G_3(s)G_4(s)]-H_3(s)G_2(s)G_3(s)}$$

Processes Control:

Signal flow graphs

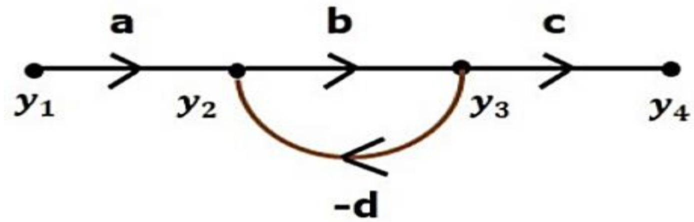
Signal flow graph is a graphical representation of algebraic equations. The basic concepts related signal flow graph and also learn how to draw signal flow graphs will be discussed.

Basic elements of signal flow graph

Nodes and branches are the basic elements of signal flow graph.

- **Node** is a point which represents either a variable or a signal. There are three types of nodes which are the input node, output node and mixed node.
- **Branch**: It is a direct line segment joining two nodes. It has both gain and direction. The arrow on the branch refers to the direction of the signal flow.
- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.
- **Path**: It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.
- **Forward Path**: The path that exists from the input node to the output node is known as forward path.
- **Forward Path Gain**: It is obtained by calculating the product of all branch gains of the forward path.
- **Loop**: The path that starts from one node and ends at the same node is known as loop. Hence, it is a closed path.
- **Loop Gain**: It is obtained by calculating the product of all branch gains of a loop.
- **Non-touching Loops**: These are the loops, which should not have any common node.

Example 11.1. Explain the following signal flow graph.



Ans.:

- The nodes present in this signal flow graph are $\mathbf{y_1}$, $\mathbf{y_2}$, $\mathbf{y_3}$, and $\mathbf{y_4}$.
- $\mathbf{y_1}$ and $\mathbf{y_4}$ are the input node and output node respectively.
- $\mathbf{y_2}$ and $\mathbf{y_3}$ are mixed nodes.

Let us construct a signal flow graph by considering the following algebraic equations

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

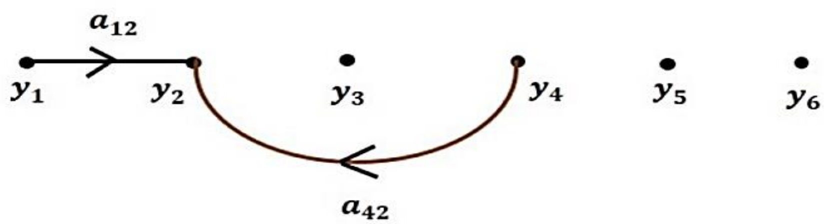
$$y_4 = a_{34}y_3$$

$$y_5 = a_{45}y_4 + a_{35}y_3$$

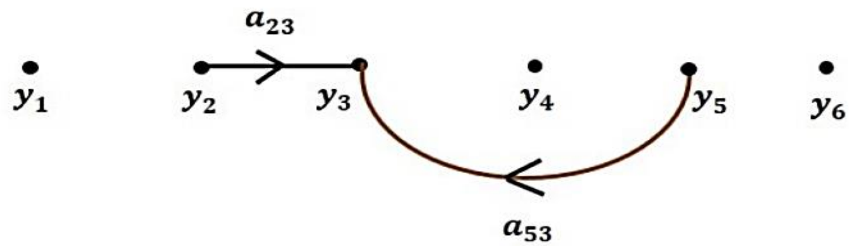
$$y_6 = a_{56}y_5$$

There will be six nodes ($\mathbf{y_1}$, $\mathbf{y_2}$, $\mathbf{y_3}$, $\mathbf{y_4}$, $\mathbf{y_5}$ and $\mathbf{y_6}$) and eight branches in this signal flow graph. The gains of the branches are ($\mathbf{a_{12}}$, $\mathbf{a_{23}}$, $\mathbf{a_{34}}$, $\mathbf{a_{45}}$, $\mathbf{a_{56}}$, $\mathbf{a_{42}}$, $\mathbf{a_{53}}$ and $\mathbf{a_{35}}$). To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below:

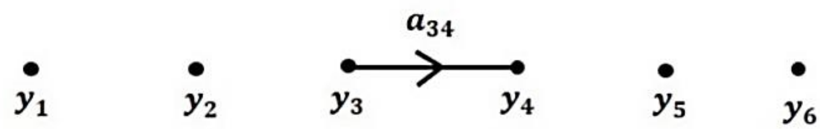
Step 1: Signal flow graph for ($\mathbf{y_2 = a_{12}\, y_1 + a_{42}\, y_4}$) is shown in the following figure:



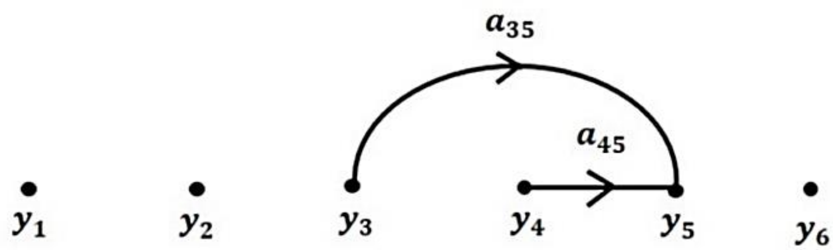
Step 2: Signal flow graph for (**$y_3 = a_{23}\, y_1 + a_{53}\, y_5$**) is shown in the following figure:



Step 3: Signal flow graph for (**$y_4 = a_{34}\, y_3$**) is shown in the following figure:

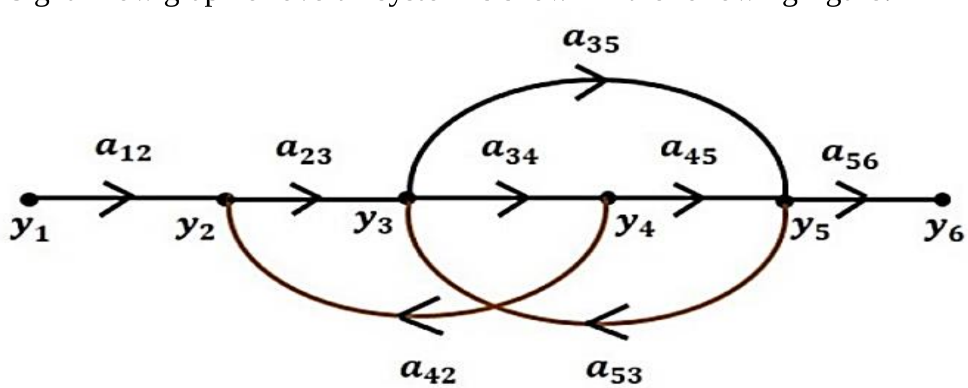


Step 4: Signal flow graph for (**$y_5 = a_{45}\, y_4 + a_{35}\, y_3$**) is shown in the following figure:



Step 5: Signal flow graph for (**$y_6 = a_{56}\, y_5$**) is shown in the following figure:

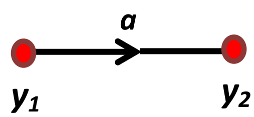
Step 6: Signal flow graph of overall system is shown in the following figure:



Rules of signal flow graph

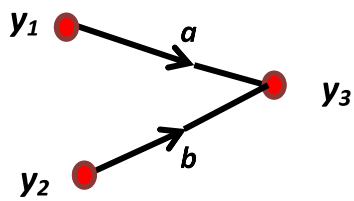
There are several rules that could be used to find the overall transfer function using the technique of signal flow graph. They are as follows:

1-



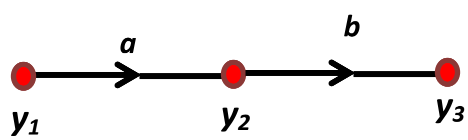
$$y_2 = a\, y_1$$

2-

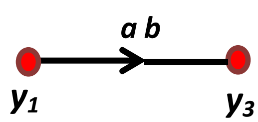


$$y_3 = a\, y_1 + b\, y_2$$

3-

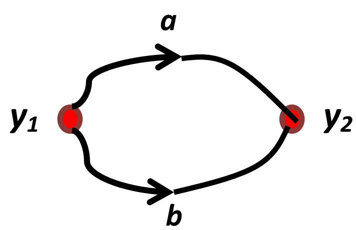


$y_2 = a\, y_1$ $\quad\quad$ $y_3 = b\, y_2$



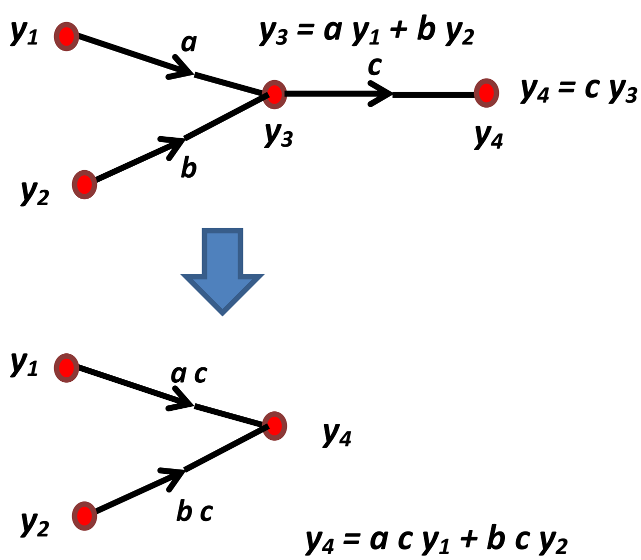
$$y_3 = (a\, b)\, y_1$$

4-

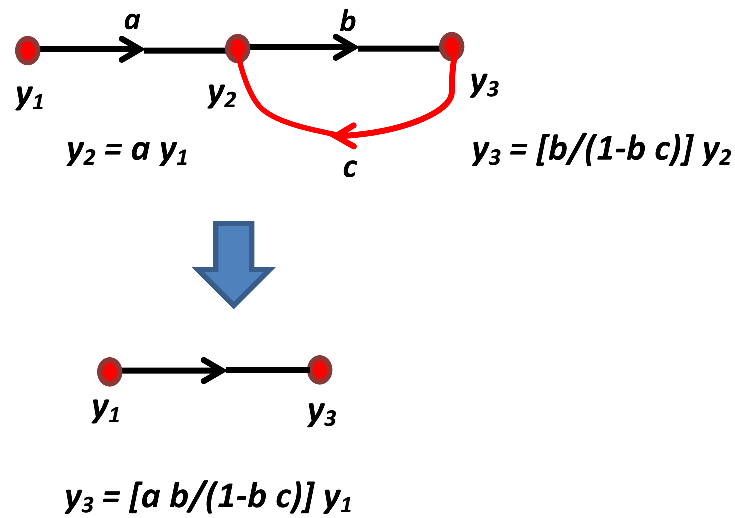


$$y_2 = (a + b)\, y_1$$

5-



6-



Mason's gain formula

Let us now discuss the Mason’s Gain Formula. Suppose there are ‘N’ forward paths in a signal flow graph. The gain between the input and the output nodes of a signal flow graph is nothing but the transfer function of the system. It can be calculated by using Mason’s gain formula. Mason’s gain formula is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \qquad (11.1)$$

Where,

C(s) is the output node

R(s) is the input node

T(s) is the transfer function or gain between and

$\mathbf{P_i}$ is the $\boldsymbol{i}^{th}$ forward path gain

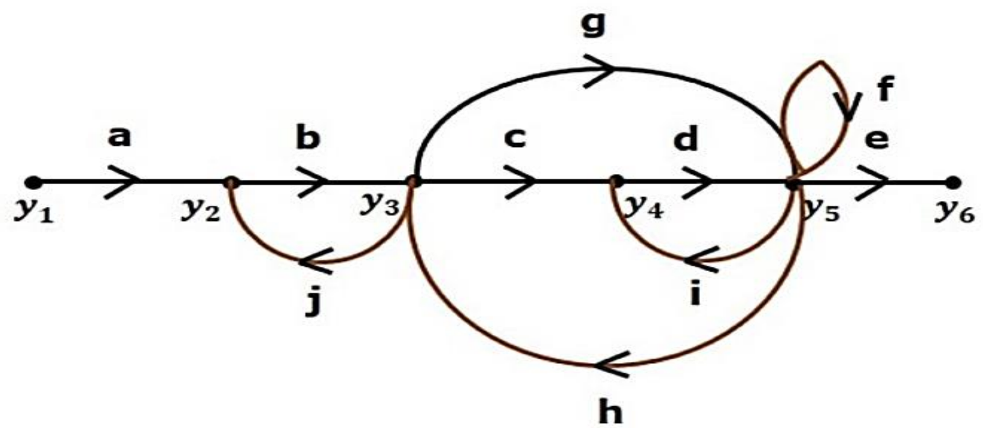
$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

$\mathbf{\Delta_i}$ is obtained from $\mathbf{\Delta}$ by removing the loops which are touching the $\boldsymbol{i}^{th}$ forward path.

Consider the following signal flow graph in order to understand the basic terminology involved here.



Path

Examples – $y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$ and $y_5 \rightarrow y_3 \rightarrow y_2$

Forward Path

Examples – $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$

Forward Path Gain

Examples – $abcde$ is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and abge is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Loop

Examples – $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_3 \rightarrow y_5 \rightarrow y_3$

Loop gain

Examples – b_j is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$ and g_h is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$.

Non-touching Loops

Examples – The loops, $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_4 \rightarrow y_5 \rightarrow y_4$ are non-touching.

Let us consider the same signal flow graph for finding transfer function

- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$
- First forward path gain, $p_1 = abcde$.
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

- Second forward path gain, $p_2 = abge$.
- Number of individual loops, L = 5.
- Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$

  $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.
- Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.
- Number of two non-touching loops = 2.
- First non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_4 \rightarrow y_5 \rightarrow y_4$.
- Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta_1 = 1$.

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, N = 2 in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{2} P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

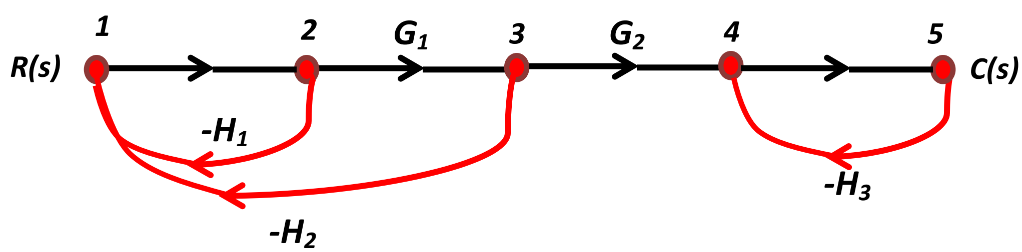
Therefore, the transfer function is -

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Processes Control:

Signal flow graphs

Example 12.1. Find the overall transfer function from the following signal flow graph of a process.



Ans.:

Number of forward paths, **N=1**

$$(\mathbf{P_1: 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5})$$

The Forward Path Gain:

$$(\mathbf{P_1 = G_1\, G_2})$$

Number of individual loops, **L=3** which are:

$$(\mathbf{P_{11}: 1 \rightarrow 2 \rightarrow 1})\ ;\ (\mathbf{P_{12}: 1 \rightarrow 2 \rightarrow 3 \rightarrow 1})\ ;\ (\mathbf{P_{13}: 4 \rightarrow 5 \rightarrow 4})$$

Loop gains are:

$$\mathbf{P_{11} = -\, H_1}$$

$$\mathbf{P_{12} = -\, G_1\, H_2}$$

$$\mathbf{P_{13} = -\, H_3}$$

Non-touching Loops are ***zero*** because all loops are touching the only forward-path.

Gain product of first non-touching loops pair,

$$\mathbf{P_{11}\,P_{13} = (-\,H_1)\,(-H_3) = H_1\,H_3}$$

Gain product of second non-touching loops pair,

$$\mathbf{P_{12}\,P_{13} = (-\,G_1\,H_2)\,(-H_3) = G_1\,H_2\,H_3}$$

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\mathbf{\Delta = 1 - (-\,H_1 - G_1\,H_2 - H_3) + (H_1\,H_3 + G_1\,H_2\,H_3)}$$

$$\mathbf{\Delta = 1 + H_1 + G_1\,H_2 + H_3 + H_1\,H_3 + G_1\,H_2\,H_3}$$

There is no loop which is non-touching to the forward path.

So,

$$\mathbf{\Delta_1 = 1 - 0 = 1}$$

Substitute in Mason's gain formula

$$\boldsymbol{T(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta}} \qquad (\mathbf{11.1})$$

$$\boldsymbol{T(s) = \frac{C(s)}{R(s)} = \frac{(G_1 G_2) \times 1}{1 + H_1 + G_1 H_2 + H_3 + H_1 H_3 + G_1 H_2 H_3}}$$

So the overall transfer function of this process is:

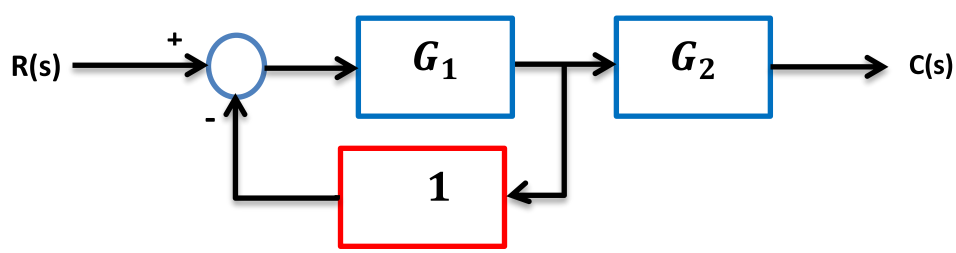
$$\boldsymbol{T(s) = \frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + H_1 + G_1 H_2 + H_3 + H_1 H_3 + G_1 H_2 H_3}}$$

Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

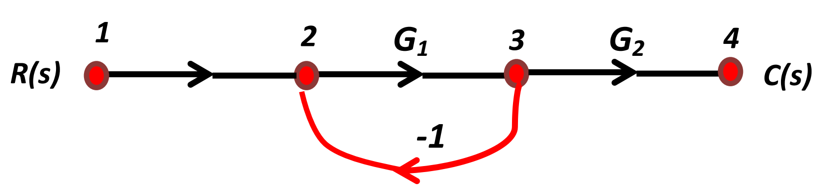
- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Represent the blocks of block diagram as **branches** in signal flow graph.
- Represent the transfer functions inside the blocks of block diagram as **gains** the of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. For example, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Example 12.2. Convert the following block diagram into its equivalent signal flow graph then find the overall transfer function.



Ans.:

Follow the general steps to convert this block diagram into its equivalent signal flow graph, it will be as follows:



Number of forward paths, **N=1**

(P_1: 1→2→3→4)

The Forward Path Gain:

($P_1 = G_1\ G_2$)

Number of individual loops, **L=1** which is:

(P_{11}: 2→3→2)

Loop gain is:

$P_{11} = -\ G_1$

Non-touching Loops are ***zero*** because all loops are touching the only forward-path.

Non-touching loop pairs are ***zero*** because there is only one loop.

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (-\ G_1)$$

$$\Delta = 1 + G_1$$

There is no loop which is non-touching to the forward path. So,

$$\Delta_1 = 1 - 0 = 1$$

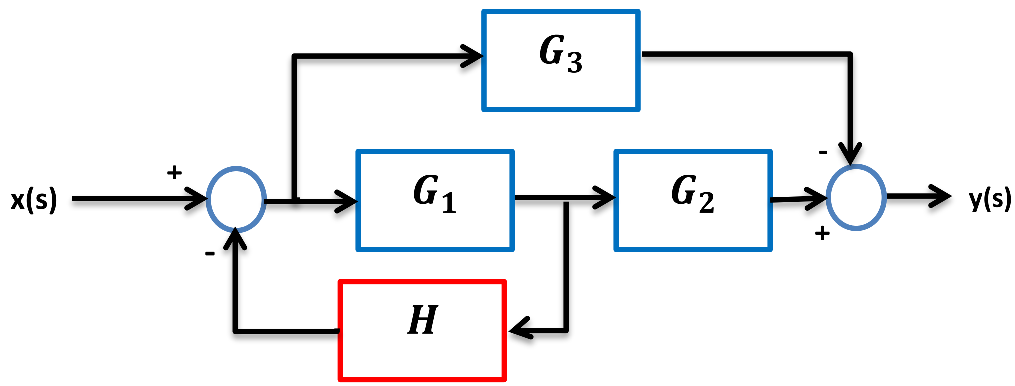
Substitute in Mason's gain formula

$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} = \frac{(G_1 G_2) \times 1}{1 + G_1}$$

So the overall transfer function of this process is:

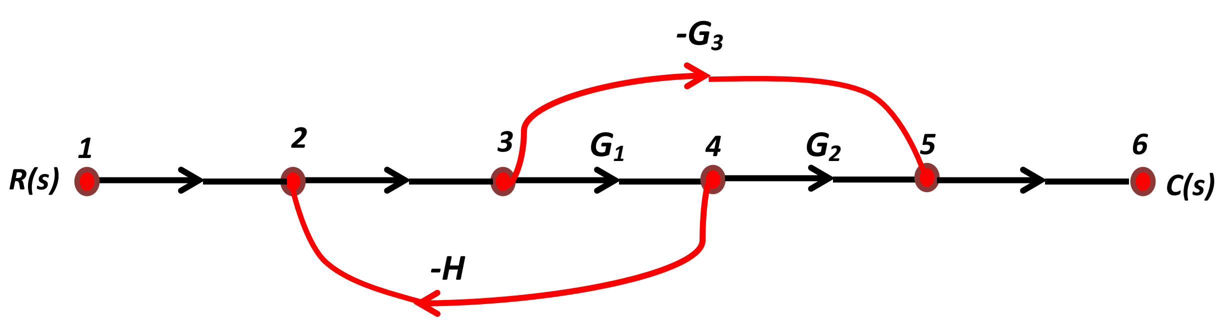
$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1G_2}{1+G_1}$$

Example 12.3. Convert the following block diagram into its equivalent signal flow graph and find the overall transfer function.



Ans.:

Follow the general steps to convert this block diagram into its equivalent signal flow graph, it will be as follows:



Ans.:

Number of forward paths, **N=2**

(P$_1$: 1→2→3→4→5→6)

(P$_2$: 1→2→3→5→6)

The gain for first forward path is:

$$(P_1 = G_1\, G_2)$$

The gain for second forward path is:

$$(P_2 = -\, G_3)$$

Number of individual loops, **L=1** which is:

$$(P_{11}: 2 \rightarrow 3 \rightarrow 4 \rightarrow 2)$$

Loop gain is:

$$P_{11} = -\, G_1\, H$$

Non-touching Loops are ***zero*** because all loops are touching the only forward-path.

Non-touching loop pairs are ***zero*** because there is only one loop.

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (-\, G_1\, H)$$

$$\Delta = 1 + G_1\, H$$

There is no loop which is non-touching to the first forward path. So,

$$\Delta_1 = 1 - 0 = 1$$

Also, there is no loop which is non-touching to the second forward path. So,

$$\Delta_2 = 1 - 0 = 1$$

Substitute in Mason's gain formula

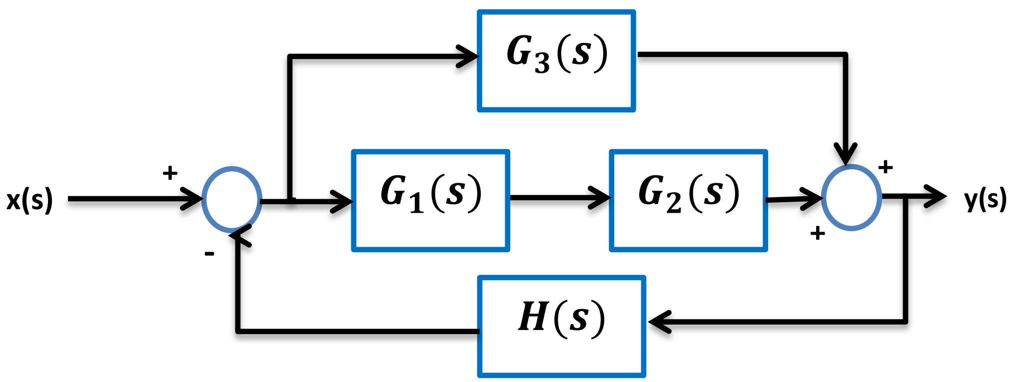
$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \qquad (11.1)$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{(G_1 G_2) \times 1 + (-G_3) \times 1}{1 + G_1 H}$$

So the overall transfer function of this process is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1 G_2 - G_3}{1 + G_1 H}$$

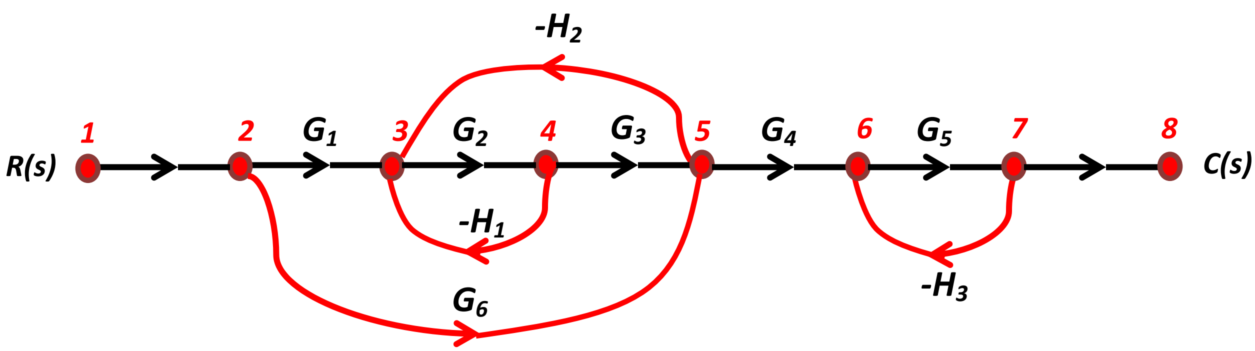
H.W. 12.1. Convert the following block diagram into its equivalent signal flow graph and find the overall transfer function



Processes Control:

Signal flow graphs

Example 13.1. Find the overall transfer function from the following signal flow graph of a process.



Ans.:

Number of forward paths, **N=2**

$$(\mathbf{P_1: 1\rightarrow2\rightarrow3\rightarrow4\rightarrow5\rightarrow6\rightarrow7\rightarrow8})$$

$$(\mathbf{P_2: 1\rightarrow2\rightarrow5\rightarrow6\rightarrow7\rightarrow8})$$

The gain for first forward path is:

$$(\mathbf{P_1= G_1\ G_2\ G_3\ G_4\ G_5})$$

The gain for second forward path is:

$$(\mathbf{P_2= G_4\ G_5\ G_6})$$

Number of individual loops, **L=3** which are:

$$(\mathbf{P_{11}: 3\rightarrow4\rightarrow3})\ ;\ (\mathbf{P_{12}: 3\rightarrow4\rightarrow5\rightarrow3})\ ;\ (\mathbf{P_{13}: 6\rightarrow7\rightarrow6})$$

Loop gains are:

$$\mathbf{P_{11}= -\ G_2\ H_1}$$

$$\mathbf{P_{12} = - G_2 G_3 H_2}$$

$$\mathbf{P_{13} = - G_5 H_3}$$

Gain product of first non-touching loops pair,

$$\mathbf{P_{11} P_{13} = (- G_2 H_1) (- G_5 H_3) = G_2 G_5 H_1 H_3}$$

Gain product of second non-touching loops pair,

$$\mathbf{P_{12} P_{13} = (- G_2 G_3 H_2) (- G_5 H_3) = G_2 G_3 G_5 H_2 H_3}$$

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$

$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$

$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\mathbf{\Delta = 1 - (- G_2 H_1 - G_2 G_3 H_2 - G_5 H_3) + (G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3)}$$

$$\mathbf{\Delta = 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

There is no loop which is non-touching to the first forward path (P_1).

So,

$$\mathbf{\Delta_1 = 1 - 0 = 1}$$

But, there is one loop non-touching to the second forward path (P_2) which is the (P_{11}).

So,

$$\mathbf{\Delta_2 = 1 - (- G_2 H_1) = 1 + G_2 H_1}$$

Substitute in Mason's gain formula

$$\boldsymbol{T(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta}} \qquad \mathbf{(11.1)}$$

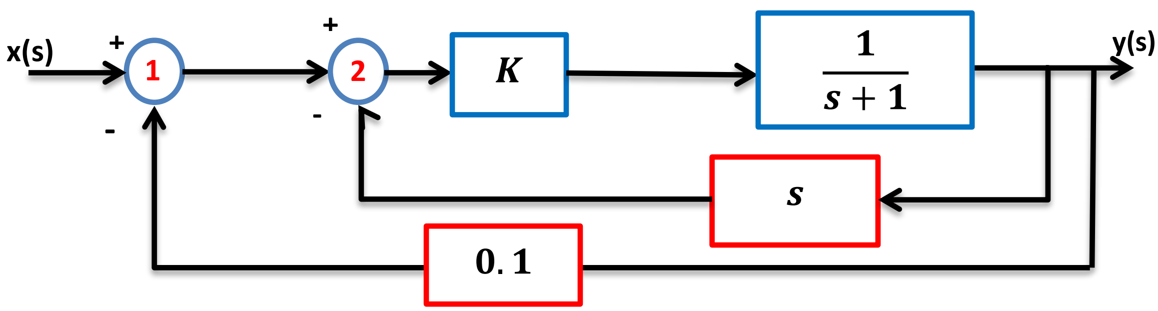
$$T(s) = \frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1\ G_2\ G_3\ G_4\ G_5\ + (G_4\ G_5\ G_6)(1\ + G_2H_1)}{1 +\ G_2H_1 +\ G_2G_3H_2 +\ G_5H_3 +\ G_2G_5H_1H_3 +\ G_2G_3G_5H_2H_3}$$

So the overall transfer function of this process is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1\ G_2\ G_3\ G_4\ G_5\ + G_2G_4\ G_5\ G_6H_1 + G_4\ G_5\ G_6}{1 +\ G_2H_1 +\ G_2G_3H_2 +\ G_5H_3 +\ G_2G_5H_1H_3 +\ G_2G_3G_5H_2H_3}$$

Example 13.2. Convert the following block diagram into its equivalent signal flow graph then find the overall transfer function.



Processes Control:

Stability

Most industrial processes are stable without feedback control. Thus, they are said to be open-loop stable, or self-regulating. An open-loop stable process will return to the original steady state after a transient disturbance occurs. By contrast, there are a few processes, such as exothermic chemical reactors, that can be open-loop unstable. These processes are extremely difficult to operate without feedback control. The ***stability*** criteria could be defined as an unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise, it is said to be unstable. So, it could be recognized as an important consequence of feedback control is that it can cause oscillatory responses. If the oscillation has a small amplitude and damps out quickly, then the control system performance is generally considered to be satisfactory. However, under certain circumstances, the oscillations may be undamped or even have an amplitude that increases with time until a physical limit is reached, such as a control valve being fully open or completely shut. In these situations, the closed-loop system is said to be ***unstable***, i.e. a system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.

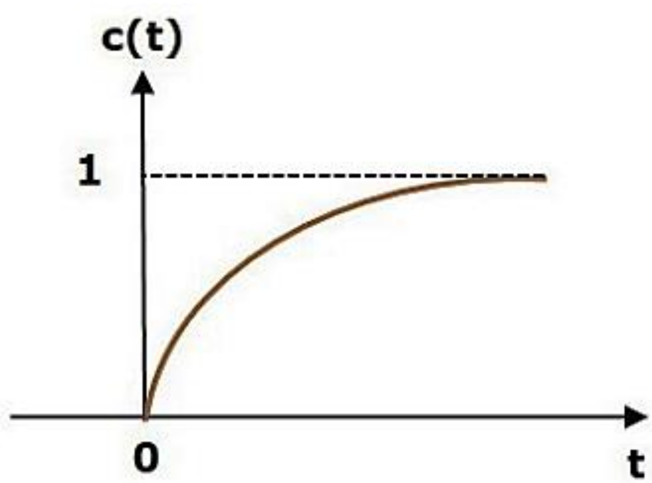


Fig. 14.1. The response of first order control system for unit step input

For example, the response of first order control system for unit step input (Fig. 14.1), this response has the values between **0** and **1**. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of t including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded. Other figure (Fig. 14.2) shows the difference between the stable and unstable systems as follows

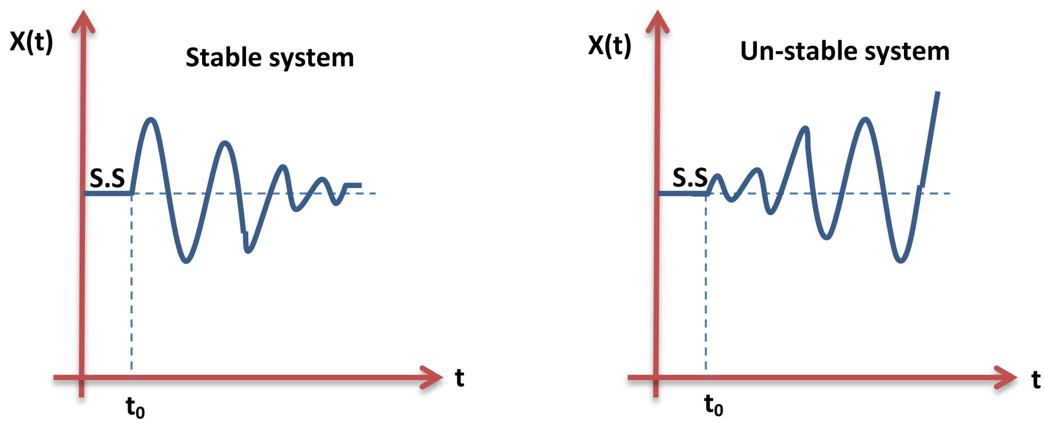


Fig. 14.2. The difference between the stable and unstable systems

Types of systems based on stability

Systems could be classified into several types based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely stable system

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of **'s' plane**. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the **'s' plane**.

Conditionally stable system

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

Marginally stable system or relatively stable system

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis. In other words, this type of system is a quantitative measure of how fast the transient vanish in the system.

Characteristic equation

The characteristic equation of a closed-loop system plays a decisive role in determining system stability. Consider a system having the following characteristic equation, i.e. the denominator of the overall transfer function:

$$\boldsymbol{q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0 = 0} \qquad (\mathbf{14.1})$$

The following conditions should be provided in the characteristic equation for the system to be expected as a stable system:

- All powers of (s) must be presented in descending order or no missing term should be presented.
- All the coefficients presented in the characteristic equation must have the same sign.

Several methods have been performed to evaluate the stability of any system without requiring calculation of the roots of the characteristic equation.

Routh-Hurwitz stability criterion

This technique is applied to analyze the coefficients of the characteristic equation, we can determine whether the closed-loop system is stable. criterion. It can be applied only to systems whose characteristic equations are polynomials in (***s***). It depends on converting the characteristic equation into an array called "***Routh's array***"

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts. Considering the characteristic equation presented in Eq. 14.1, there should not be any term missing in the $\mathbf{n}^{\mathbf{th}}$ order characteristic equation. This means that the $\mathbf{n}^{\mathbf{th}}$ order characteristic equation should not have any coefficient that is of zero value.

It depends on converting the characteristic equation into an array called "***Routh's array***"

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method

If all the roots of the characteristic equation exist to the left half of the '***s***' plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the '***s***' plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable. Follow this procedure for forming the Routh array as the following table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of s^n and continue up to the coefficient of s^0.
- Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of row s^0 is a_n. Here, a_n is the coefficient of s^0 in the characteristic polynomial.

Note − If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

The following table shows the Routh array of the nth order characteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6	$\cdots$	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$	$\cdots$
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	$b_3 = \frac{a_1 a_6 - a_7 a_0}{a_1}$	$\cdots$	$\cdots$	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2 = \frac{b_1 a_5 5 - b_3 a_1}{b_1}$	$\vdots$			

$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

Example 14.1. Evaluate the stability of the control system having characteristic equation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Ans.:

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability. All the coefficients of the characteristic polynomial are positive. So, the control system satisfies the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3\times3)-(2\times1)}{3} = \frac{7}{3}$	$\frac{(3\times1)-(0\times1)}{3} = \frac{3}{3} = 1$	
s^1	$\frac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}} = \frac{5}{7}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability. All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

H.W. 14.1. Check the stability of a system having the following characteristic equation.

$$\boldsymbol{s^3 + 5s^2 + 10s + 3 = 0}$$

Processes Control:

Stability

Special Cases of Routh Array

We may come across two types of situations, while forming the Routh table. It is difficult to complete the Routh table from these two situations.

The two special cases are:

- The first element of any row of the Routh array is zero.
- All the elements of any row of the Routh array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

First Element of any row of the Routh array is zero

If any row of the Routh array contains only the first element as *zero* and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh table. Now, find the number of sign changes in the first column of the Routh table by substituting ϵ tends to zero.

Example 15.1. Let us find the stability of the control system having characteristic equation,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Ans.:

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability. All the coefficients of the characteristic polynomial, $s^4 + 2s^3 + s^2 + 2s + 1$ are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	1	1
s^3	$\cancel{2}$ 1	$\cancel{2}$ 1	
s^2	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(0\times1)}{1} = 1$	
s^1			
s^0			

The row s^3 elements have 2 as the common factor. So, all these elements are divided by 2.

Special case (i) − Only the first element of row s^2 is zero. So, replace it by ϵ and continue the process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon\times1)-(1\times1)}{\epsilon} = \frac{\epsilon-1}{\epsilon}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability. As ϵ tends to zero, the Routh table becomes like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

All the Elements of any row of the Routh array are zero

In this case, follow these two steps:

- Write the auxilary equation, A(s) of the row, which is just above the row of zeros.
- Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example 15.2. Let us find the stability of the control system having characteristic equation,

$$\boldsymbol{s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0}$$

Ans.:

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability. All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^5	1	1	1
s^4	$\cancel{3}$ 1	$\cancel{3}$ 1	$\cancel{3}$ 1
s^3	$\frac{(1\times1)-(1\times1)}{1}=0$	$\frac{(1\times1)-(1\times1)}{1}=0$	
s^2			
s^1			
s^0			

The row $\boldsymbol{s^4}$ elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) − All the elements of row $\boldsymbol{s^3}$ are zero. So, write the auxiliary equation, A(s) of the row $\boldsymbol{s^4}$.

$$\boldsymbol{A(s) = s^4 + s^2 + 1}$$

Differentiate the above equation with respect to $\boldsymbol{s}$.

$$\boldsymbol{\frac{dA(s)}{ds} = 4s^3 + 2s}$$

Place these coefficients in row s^3.

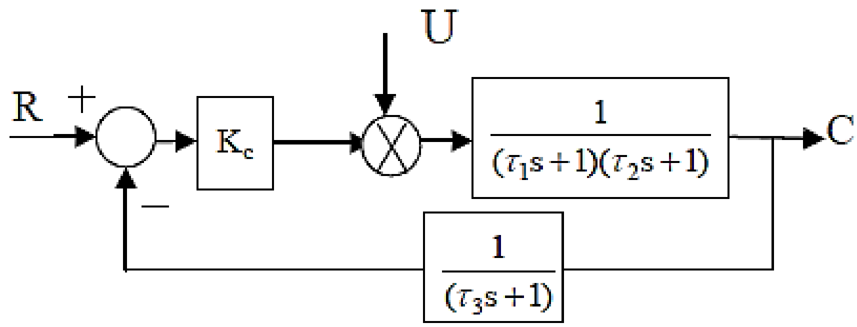
s^5	1	1	1
s^4	1	1	1
s^3	$\cancel{4}\ 2$	$\cancel{2}\ 1$	
s^2	$\frac{(2\times1)-(1\times1)}{2}=0.5$	$\frac{(2\times1)-(0\times1)}{2}=1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5}=\frac{-1.5}{0.5}=-3$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability. There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in on left half of the ‘s’ plane or on the right half of the ‘s’ plane or on an imaginary axis. So, we can’t find the nature of the control system. To overcome this limitation, there is a technique known as the root locus. We will discuss this technique later.

Example 15.3. Determine K_C for a stable system

$$\text{If}\ \tau_1 = 1,\ \tau_2 = \frac{1}{2},\ \tau_3 = \frac{1}{3}$$



Ans.:

The characteristic equation is

$$1+K_c\frac{1}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}=0$$

$$(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)+K_c=0$$

$$(\frac{1}{2}s^2+\frac{3}{2}s+1)(\frac{1}{3}s+1)+K_c=0$$

$$\frac{s^3}{6}+\frac{s^2}{2}+\frac{s}{3}+\frac{s^2}{2}+\frac{3s}{2}+1+K_c=0$$

$$\frac{1}{6}s^3+s^2+\frac{11}{6}s+1+K_c=0$$

Row		
1	1/6	11/6
2	1	$1+K_c$
3	$\frac{10-K_c}{6}$	0
4	$1+K_c$	

Since $K_c>0$

$\therefore$ The system will be stable

If $10-K_c>0$

$K_c<10$

Therfore K_c must within the range $0<K_c<10$

Processes Control:

Types of controllers

In order to keep the response of any system at the desired value, the output of this response should be measured and compared with the desired value to estimate how far this value deviated from the desired value. The technique of the controller is to receive the error, analyze it, and convert it to actuating signal C(t) which will handle the control action. Basically, three types of controllers exits as follows:

1- Proportional controller (P-only).

Its actuating output is proportional to the error:

$$\boldsymbol{C(t) = C_s + K_c\, e(t)} \qquad (17.1)$$

Where, $\boldsymbol{C_s}$ is Bias signal, and $\boldsymbol{K_c}$ is the gain of the proportional controller. When the ***e(t)*** equals zero the actuating signal equals the steady state value.

The P-only controller is described by the value of its proportional gain ($\boldsymbol{K_c}$) or equivalently by its proportional band (PB):

$$\boldsymbol{PB = \frac{100}{K_c}} \qquad (17.2)$$

So,

$$\boldsymbol{(G_c)_{P-only} = \frac{C'(s)}{e(s)} = K_c} \qquad (17.3)$$

The sensitivity of this controller to ***e(t)*** is directly proportional with the increase of the gain ($\boldsymbol{K_c}$).

An example, for error type change:

$$C(s) = C_s + K_c\, e(s)$$

$$C'(s) = K_c\, e(s)$$

$$C'(s) = K_c \times \frac{1}{s}$$

Taking Laplace invers

$$C'(t) = K_c$$

$$C(t) = C_s + K_c$$

2- Proportional – Integral controller (PI).

This controller is known also as "proportional-plus-reset controller". Its actuating signal is related to the error by the following equation:

$$C(t) = C_s + K_c\, e(t) + \frac{K_c}{\tau_I}\int e(t)dt \qquad (17.4)$$

Where, $\tau_I$is the integral time or reset time.

$$C'(t) = K_c\, e(t) + \frac{K_c}{\tau_I}\int e(t)dt \qquad (17.5)$$

$$(G_c)_{PI} = \frac{C'(s)}{e(s)} = K_c(1 + \frac{1}{\tau_I s}) \qquad (17.6)$$

Note that:

- The integral action eliminates the offset.
- First-order system in addition to PI-controller obey as a second-order system.
- The larger the value of $\boldsymbol{K_c}$, the smaller the offset.

3- Proportional – Integral – derivative controller (PID).

Sometimes, it called as "Proportional-plus-reset-plus-rate controller". Its actuating signal is related to the error by the following equation:

$$C(t) = C_s + K_c\, e(t) + \frac{K_c}{\tau_I}\int e(t)dt + K_c\tau_D\frac{de}{dt} \qquad (17.7)$$

Where, $\tau_D$is the derivative time or reset time.

$$C'(t) = K_c\, e(t) + \frac{K_c}{\tau_I}\int e(t)dt + K_c\tau_D\frac{de}{dt} \qquad (17.8)$$

If the error is constant, the derivative action will be omitted.

$$(G_c)_{PID} = \frac{C'(s)}{e(s)} = K_c(1 + \frac{1}{\tau_I s} + \tau_D s) \qquad (17.9)$$

The choice of controller parameters (K_c, τ_I, τ_D)depends basically on the nature of the process model. The adjustment of controller parameters to attain acceptable control action is called "controller tuning". The Zieglar-Nichols tuning technique goes through the following steps:

- Bring the system to the desired operational level, i.e. design condition.
- Using P-only controller with feed-back closed loop, introduce sinusoidal change with low amplitude ($\frac{K_p}{\sqrt{1+\tau^2\omega^2}}$) and varying frequencies until the system oscillates continuously. The frequency of continuous oscillation is the crossover frequency (ω_{co}).
- Let (M) be the amplitude ratio (AR) where (AR=$\sqrt{R^2+I^2}$); the ultimate gain (K_u=1/M), the ultimate period of sustained cycling ($\boldsymbol{P_u}$=$\boldsymbol{2\pi/\omega_{co}}$ $[min/cycle]$).
- Depending on the values of $\boldsymbol{K_u}$ and $\boldsymbol{P_u}$, Zieglar-Nichols recommended the following setting for feedback controller:

Type of controllers	K_c	$\boldsymbol{\tau_I}$ $(\boldsymbol{min})$	$\boldsymbol{\tau_D}$ $(\boldsymbol{min})$
P-only	$K_u/2$	---	---
PI	$K_u/2.2$	$P_u/1.2$	---
PID	$K_u/1.7$	$P_u/2$	$P_u/8$

Example 17.1. A control system has been constructed to control a temperature value. The ultimate gain was (K_u=0.4) and the ultimate period of sustained oscillation was (P_u=2). Find the controller parameters for P, PI, and PID controllers using Zieglar-Nichols method.

Ans.:

Type of controllers	K_c	$\boldsymbol{\tau_I}$ $(\boldsymbol{min})$	$\boldsymbol{\tau_D}$ $(\boldsymbol{min})$
P-only	$K_u/2=0.4/2=0.2$	---	---

PI	$K_u/2.2=0.4/2.2=0.182$	$P_u/1.2=2/1.2=1.67$	---
PID	$K_u/1.7=0.4/1.7=0.24$	$P_u/2=2/2=1$	$P_u/8=2/8=0.25$

Another type of controller is known as ON-Off controller. It is a simple inexpensive feedback controller that could be performed in heating systems, refrigerators, lab. Furnaces, etc.

$$C(t) = \begin{cases} C_{max} & if\ e \geq 0 \quad y_{sp} > y_m \quad fully\ opening\ valve \\ C_{min} & if\ e < 0 \quad y_{sp} < y_m \quad fully\ closed\ valve \end{cases}$$

Some examples of systems used this controller are as follows:

- Digital computer: $C_{max} = 100\%$; $C_{min} = 0\%$
- Current – based electronic controller: $C_{max} = 20\ mA$; $C_{min} = 4\ mA$
- Pneumatic controller: $C_{max} = 15\ psig$; $C_{min} = 3\ psig$

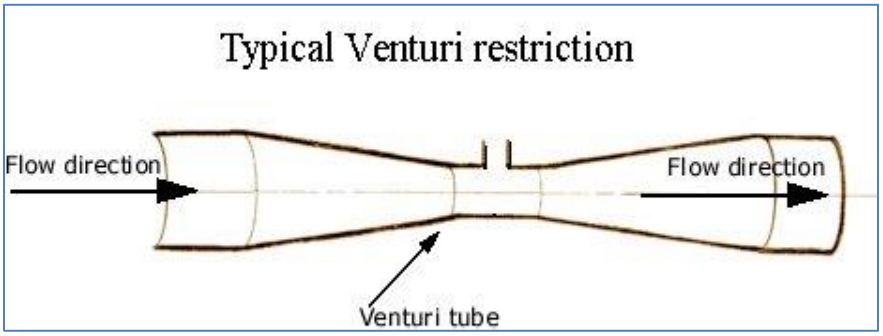
Note that:

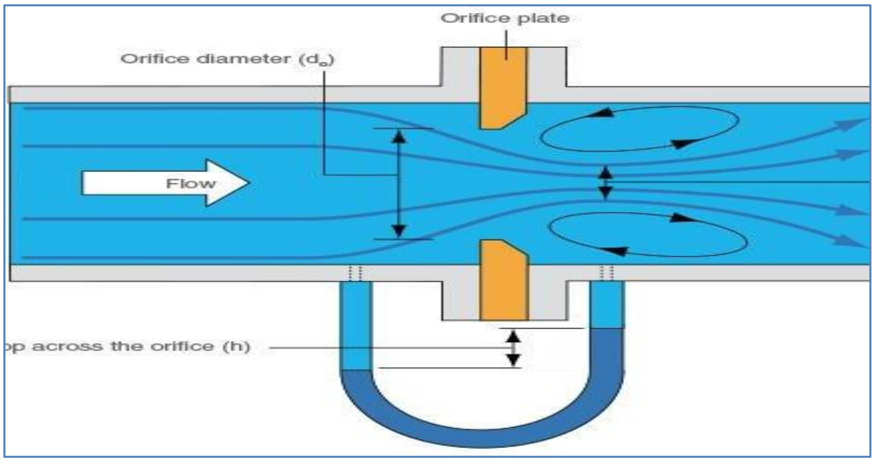
- If the gain (K_c) is very large for P-only controller, the control system behaves as ON-Off controller.
- The controllers P-only, PI, and PID are more efficient than the ON-off controller.

Measuring devices (sensors)

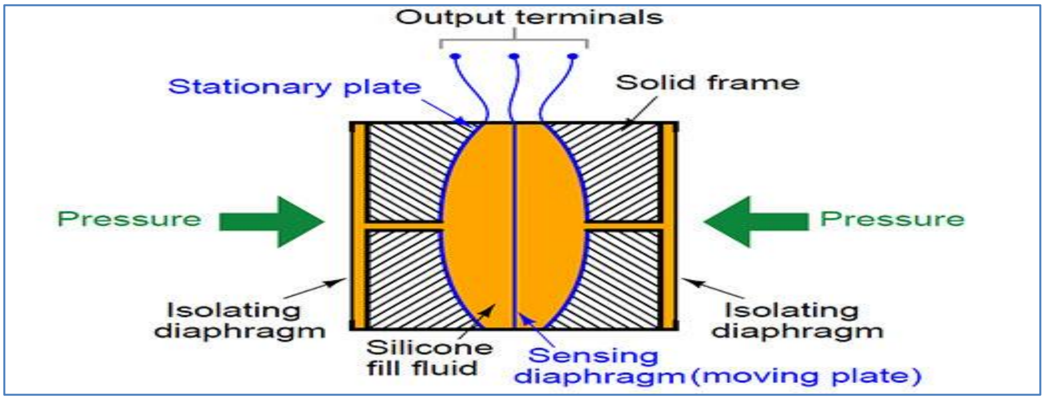
The effective operation of any feed-back control systems depends upon good measurement of the controller output and correct transmission of the controller. These sensors could be categorized as follows:

- ***Flow sensors***: These sensors are widely used in the industrial practice where they are measuring the pressure gradient developed across a constriction. By using Bernoulli equation, the flow rate could be computed. These sensors can be used for both gases and liquids. The orifice plate and Venturi tube are typical examples of sensors based on the above principle. Orifice plate is more popular due to its simplicity and low cost. The Venturi tube is more expensive but it is more accurate.



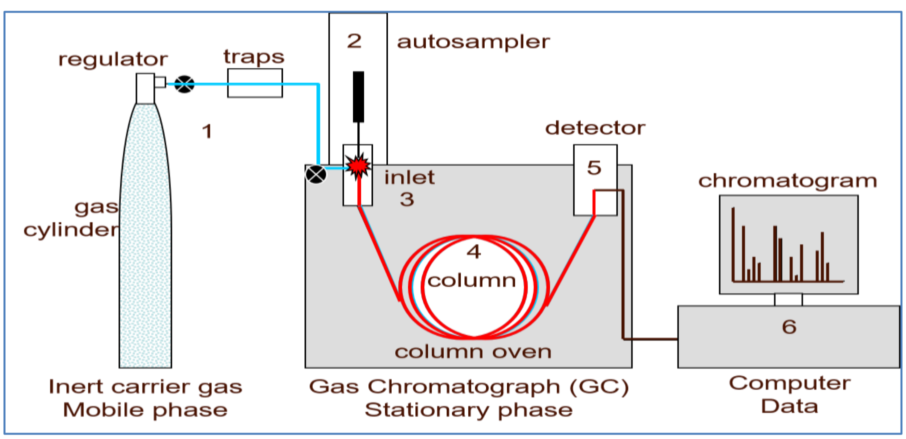


- ***Pressure sensors***: They are used to measure the pressure of a process or the pressure difference which is performed to manipulated a liquid level or a flow rate (orifice plate, Venturi tube). The available capacitance differential pressure transducer has become very popular.



- ***Temperature sensors***: The most common are thermocouples, resistance bulb thermometers, and thermistors. All provide the measurements in term of electrical signals.

- ***Composition analyzers***: Typical examples of such sensors are gas-liquid chromatographs and various types of spectroscopic analyzers. They are used to measure the composition of liquids and gases in terms of one or two key components or in terms of all components presented in a process stream.



The following details conclude the most important measuring devices:

Temperature:

1- Thermocouple Resistance.
2- Temperature detector (RTD).
3- Filled-system thermometer.
4- Bimetal thermometer.
5- Pyrometer (a. Total radiation ; b. Photoelectric; c. Ratio).

Flow:

1- Orifice.
2- Venturi.
3- Rotameter.
4- Turbine.
5- Vortex-shedding.

Pressure:

1- Liquid column.
2- Elastic element (a. Bourdon tube; b. Bellows; c. Diaphragm).
3- Strain gauges.
4- Piezoresistive transducers.

Level:

1- Float (a. Activated ; b. Chain gauge, lever; c. Magnetically coupled.
2- Head devices (Bubble tube).
3- Electrical (conductivity).

Composition:

1- Gas-liquid chromatography (GLC).
2- Mass spectrometry (MS).
3- Magnetic resonance analysis (MRA).
4- Infrared (IR) spectroscopy.