Processes Control:

Introduction:

In recent years the performance requirements for process plants have become increasingly difficult to satisfy. Stronger competition, tougher environmental and safety regulations, and rapidly changing economic conditions have been key factors. Consequently, product quality specifications have been tightened and increased emphasis has been placed on more profitable plant operation. A further complication is that modern plants have become more difficult to operate because of the trend toward complex and highly integrated processes. Thus, it is difficult to prevent disturbances from propagating from one unit to other interconnected units. In view of the increased emphasis placed on safe, efficient plant operation, it is only natural that the subject of process control has become increasingly important in recent years. Without computer-based process control systems, it would be impossible to operate modern plants safely and profitably while satisfying product quality and environmental requirements. Thus, it is important for chemical engineers to have an understanding of both the theory and practice of process control.

The two main subjects of this issue are ***process dynamics*** and ***process control***. The term *process dynamics* refers to unsteady-state (or transient) process behavior. By contrast, most of the chemical engineering curricula emphasize steady-state and equilibrium conditions in such courses as material and energy balances, thermodynamics, and transport phenomena. But the topic of process dynamics is also very important. Transient operation occurs during important situations such as start-ups and shutdowns, unusual process disturbances, and planned transitions from one product grade to another. The primary objective of process control is to maintain a process at the desired operating conditions, safely and economically, while satisfying environmental and product quality requirements.

The subject of process control is concerned with how to achieve these goals. In large-scale, integrated processing plants such as oil refineries or ethylene plants, thousands of process variables such as compositions, temperatures, and pressures are measured and must be controlled. Fortunately, thousands of process variables (mainly flow rates) can usually be manipulated for this purpose. Feedback control systems compare measurements with their desired values and then adjust the manipulated variables accordingly. Feedback control is a fundamental concept that is absolutely critical for both biological and manmade systems. Without feedback control, it would be very difficult, if not impossible, to keep complicated systems at the desired conditions.

Representative Process Control Problems:

The foundation of process control is process understanding. Thus, we begin this section with a basic question: what is a process? For our purposes, a brief definition is appropriate:

Process: The conversion of feed materials to products using chemical and physical operations, i.e. In the chemical process industry, the primary objective is to combine chemical processing units, such as chemical reactors, distillation columns, extractors, evaporators, heat exchangers, etc., integrated in a rational fashion into a chemical process in order to transform raw materials and input energy into finished products which mean that any single process unit, or combinations of processing nits, used for the conversion of raw materials (through any combination of chemical, physical, mechanical, or thermal changes) into finished products, is a chemical process. In practice, the term process tends to be used for both the processing operation and the processing equipment. There are three broad categories of processes: continuous, batch, and semibatch. Next, we consider representative processes and briefly summarize key control issues.

1. Continuous Processes

Four continuous processes are shown schematically in Fig. 1:

(a) ***Tubular heat exchanger.*** A process fluid on the tube side is cooled by cooling water on the shell side. Typically, the exit temperature of the process fluid is controlled by manipulating the cooling water flow rate. Variations in the inlet temperatures and the process fluid flow rate affect the heat exchanger operation. Consequently, these variables are considered to be disturbance variables.

(b) ***Continuous stirred-tank reactor (CSTR).*** If the reaction is highly exothermic, it is necessary to control the reactor temperature by manipulating the flow rate of coolant in a jacket or cooling coil. The feed conditions (composition, flow rate, and temperature) can be manipulated variables or disturbance variables.

(c) ***Thermal cracking furnace.*** Crude oil is broken down ("cracked") into a number of lighter petroleum fractions by the heat transferred from a burning fuel/air mixture. The furnace temperature and amount of excess air in the flue gas can be controlled by manipulating the fuel flow rate and the fuel/air ratio. The crude oil composition and the heating quality of the fuel are common disturbance variables.

(d) ***Kidney dialysis unit.*** This medical equipment is used to remove waste products from the blood of human patients whose own kidneys are failing or have failed. The blood flow rate is maintained by a pump, and "ambient conditions," such as temperature in the unit, are controlled by adjusting a flow rate. The dialysis is continued long enough to reduce waste concentrations to acceptable levels.

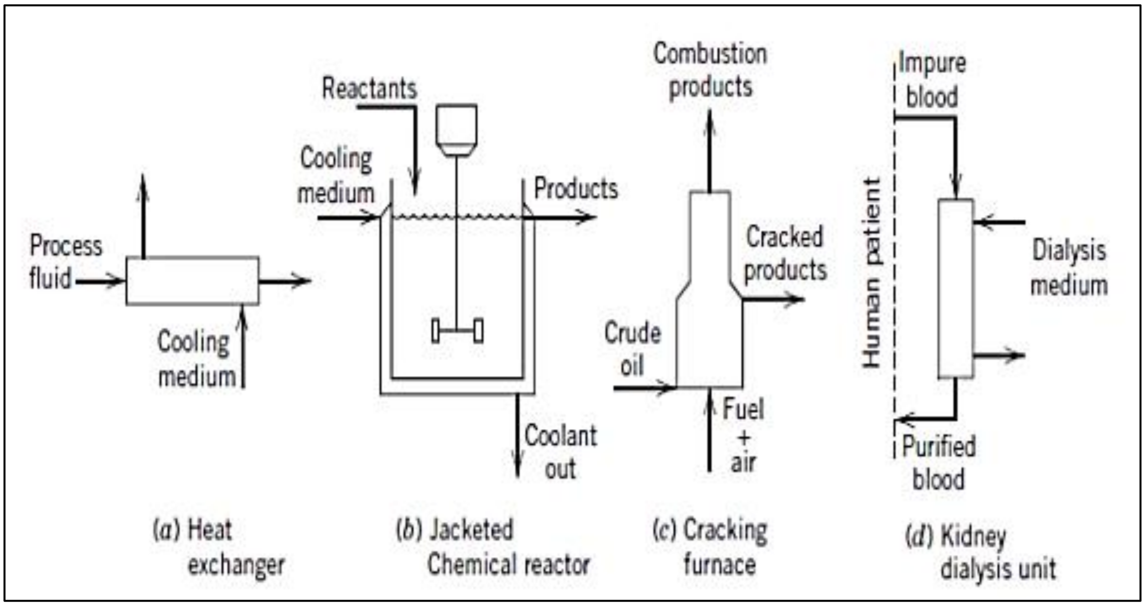


Fig. 1. Some typical continuous processes

2. Batch and Semibatch Processes

Batch and semibatch processes are used in many process industries, including microelectronics, pharmaceuticals, specialty chemicals, and fermentation. Batch and semibatch processes provide needed flexibility for multiproduct plants, especially when products change frequently and production quantities are small. Figure 2 shows four representative batch and semibatch processes:

(e) Jacketed batch reactor: In a batch reactor, an initial charge (e.g., reactants and catalyst) is placed in the reactor, agitated, and brought to the desired starting conditions. For exothermic reactions, cooling jackets are used to keep the reactor temperature at or near the desired set point. Typically, the reactor temperature is regulated by adjusting the coolant flow rate. The endpoint composition of the batch can be controlled by adjusting the temperature set point and/or the cycle time, the time period for reactor operation. At the end of the batch, the reactor contents are removed and either stored or transferred to another process unit such as a separation process.

(f) Semibatch bioreactor: For a semibatch reactor, one of the two alternative operations is used: (i) a reactant is gradually added as the batch proceeds or (ii) a product stream is withdrawn during the reaction. The first configuration can be used to reduce the side reactions while the second configuration allows the reaction equilibrium to be changed by withdrawing one of the products. For bioreactors, the first type of semibatch operation is referred to as a fed-batch operation; it is shown in Fig. 2(f). In order to better regulate the growth of the desired microorganisms, a nutrient is slowly added in a predetermined manner.

(g) Semibatch digester in a pulp mill: Both continuous and semibatch digesters are used in paper manufacturing to break down wood chips in order to extract the cellulosic fibers. The end point of the chemical reaction is indicated by the kappa number, a measure of lignin content. It is controlled to a desired value by adjusting the digester temperature, pressure, and/or cycle time.

(h) Plasma etcher in semiconductor processing: A single wafer containing hundreds of printed circuits is subjected to a mixture of etching gases under conditions suitable to establish and maintain a plasma (a high voltage applied at high temperature and extremely low pressure). The unwanted material on a layer of a microelectronics circuit is selectively removed by chemical reactions. The temperature, pressure, and flow rates of etching gases to the reactor are controlled by adjusting electrical heaters and control valves.

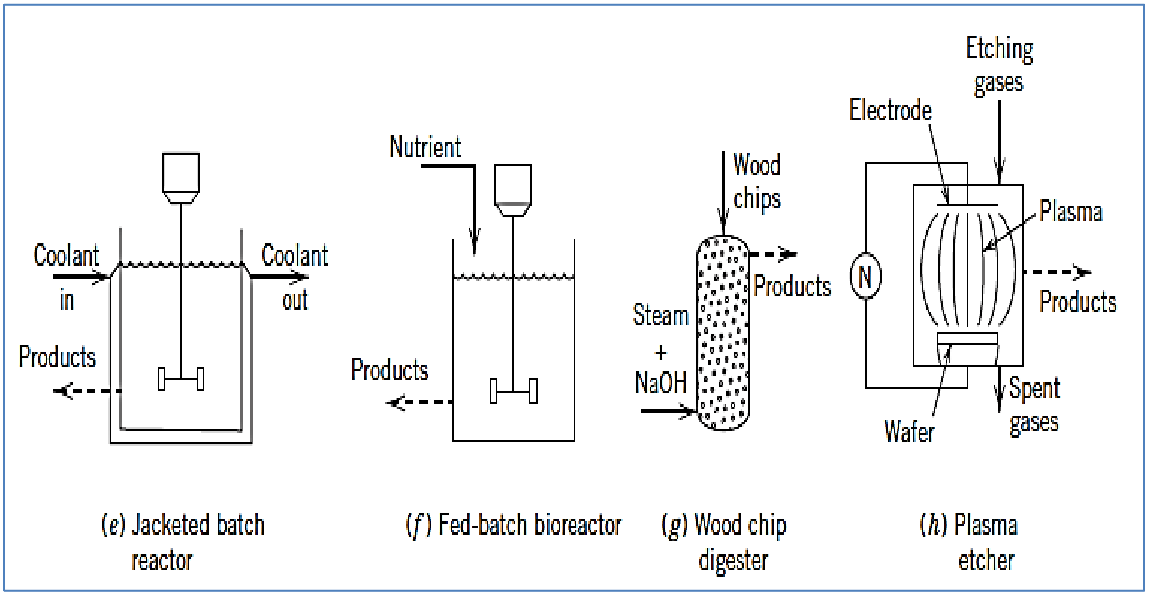


Fig. 2. Some typical processes whose operation is noncontinuous

The basic principles guiding the operation of the processing units of a chemical process are based on the following broad objectives:

1. **It is desirable to operate the processing units safely**: This mean that no unit should be operated at, or near, conditions considered to be potentially dangerous either to the health of the operators or to the life of the equipment. The safety of the immediate, as well as the remote, environment also comes into consideration here. Process operating conditions that may lead to the violation of environmental regulations must be avoided.

2. **Specified production rates must be maintained**: The amount of product output required of a plant at any point in time is usually dictated by market requirements. Thus, production rate specifications must be met and maintained, as much as possible.

3. **Product quality specifications must be maintained**: Products not meeting the required quality specifications must either be discarded as waste, or, where possible, reprocessed at extra cost. The need for economic uti1ization of resources therefore provides the motivation for striving to satisfy product quality specifications.

For example, For the process shown in Fig.3, some operating constraints mandated by safety would be that the furnace tubes should not exceed their metallurgical temperature limit and the fractionation unit should not exceed its pressure rating.

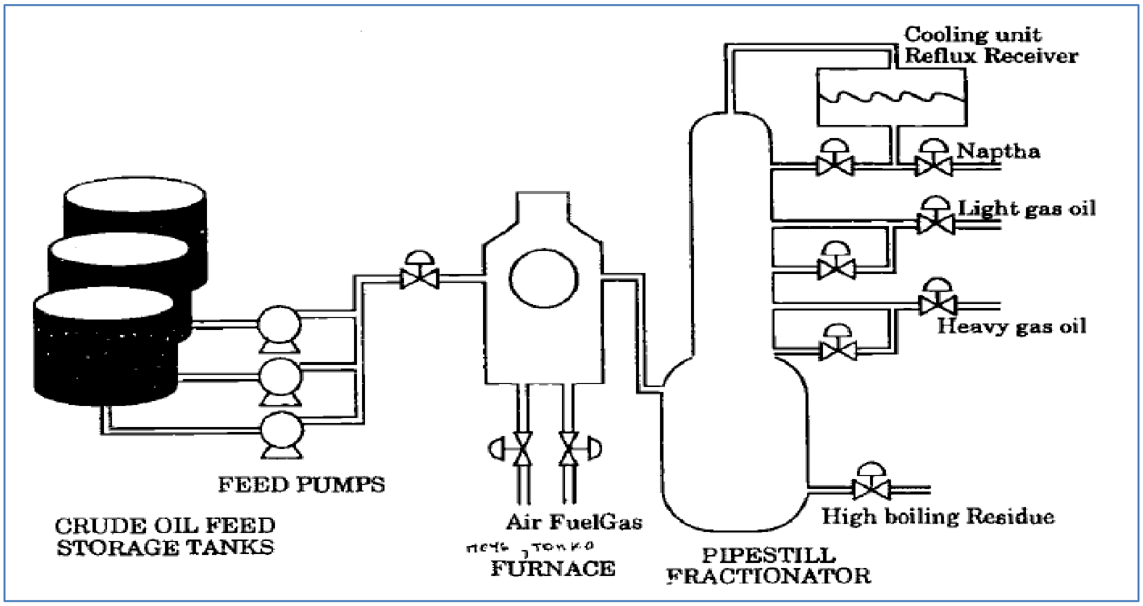


Fig. 3. The upstream end of an oily refinery

The issues of maintaining *production rates* and *product quality* are linked for this process. The products available from crude oil are determined by their boiling points. Thus a lighter crude oil feed could Produce more naptha and light gas oil, while a heavier crude oil would produce more heavy gas oil and high boiling residue. Hence, the production rate possible for each of the products depends on the particular crude oil being fractionated and the quality specifications (usually a maximum boiling point for each fraction above the bottom). Thus by shifting the maximum boiling point upwards for a product such as naptha or gas oil, one could produce more of it, but it would have a lower quality (i.e., more high-boiling materials).

Now, chemical processes are, by nature, *dynamic*, by which we mean that their variables are always changing *with time.* It is clear, therefore, that to achieve the above noted objectives, there is the need to *monitor*, and be able to induce change in, those key process variables that are related to *safety, production rates,* and *product quality.*

This dual task of:

1. Monitoring certain process condition indicator variables, and,

2. Inducing changes in the appropriate process variables in order to improve process conditions.

is the job of the *control system*. To achieve good designs for these control systems one must embark on the study of a new field, defined as follows:

Process Dynamics and Control is that aspect of chemical engineering concerned with the analysis, design, and implementation of control systems that facilitate the achievement of specified objectives of process safety, production rates, and product quality.

H.W. 1.1*: Give examples about auto and manual control systems found in your daily life.*

Processes Control:

Introductory concepts:

An industrial perspective of a typical process control problem:

The next phase of our presentation of introductory concepts involves the definition of certain terms that are routinely used in connection with various components of a chemical process, and an introduction to the concept of a process control system. The process unit under consideration is the furnace used to preheat the crude oil feed material to the fractionator. A more detailed schematic diagram is shown in Fig. 1. Such units are typically found in refineries and petrochemical plants.

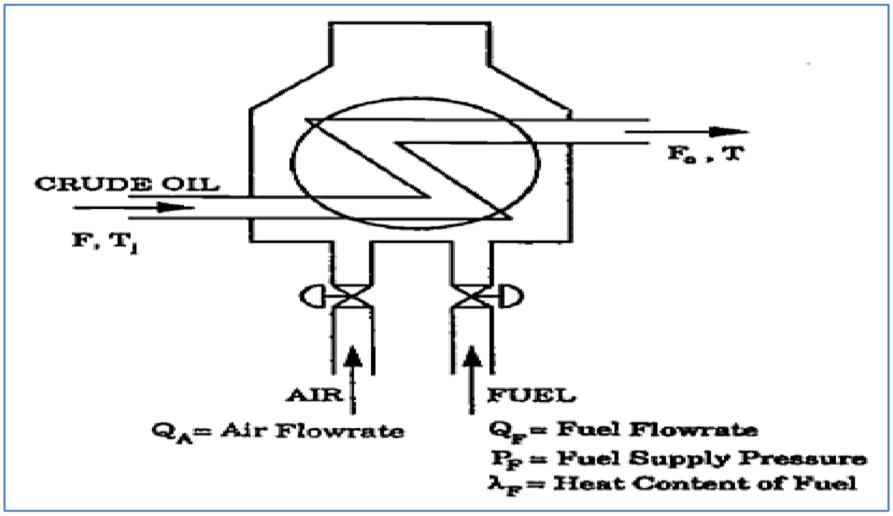


Fig. 1. Crude oil preheater furnace

The crude oil flowrate **F** and temperature $\mathbf{T_i}$; at the inlet of the furnace tend to fluctuate substantially. The flowrate and temperature of the crude oil at the outlet of the furnace are, respectively, $\mathbf{F_0}$ and **T**. It is desired to deliver the crude oil feed to the fractionator at a constant temperature **T***, regardless of the conditions at the furnace inlet. For plant safety reasons, and because of metallurgical limits, it is mandatory that the furnace tube temperature not exceed the value $\mathbf{T_m}$· The heat content of the heating fuel, as well as the fuel supply pressure, are also known to vary because of disturbances in the fuel gas coming from a different processing unit in the refinery complex. The furnace control problem may be summarized as follows:

Deliver crude oil feed to the fractionator at a constant *temperature* ***T****, *and flowrate* $\mathbf{F_0}$, *regardless of all the factors potentially capable of causing the furnace outlet temperature*

T to deviate from this desired value, making sure that the temperature of the tube surfaces within the furnace does not, at any time, exceed the value T_m.

Observe the presence of the three objectives related to *safety, product quality, and production rate*, namely: furnace temperature limit T_m the required target temperature T^*, for the furnace "product", and the crude oil throughput F_0 , respectively.

The following dialogue between a ***plant engineer*** (PE) and the ***control engineer*** (CE) explain how to evaluate an acceptable solution to typical industrial control problems (The above furnace as an example).

CE: What are your operation objectives?

PE: We would like to deliver the crude oil to the fractionation unit downstream at a consistent target temperature T^*. The value of this *set-point* is usually determined by the crude oil type, and desired refinery throughput; it therefore changes every 2-3 days. Also, we have an *upper limit constraint* (T_m) on how high the furnace tube temperatures can get.

CE: So, of your two *process outputs*, F_o and T, the former is set externally by the fractionator, while the latter is the one you are concerned about controlling?

PE: Yes.

CE: Your control objective is therefore to *regulate* the process output T as well as deal with the *servo* problem of *set-point* changes every 2-3 days?

PE: Yes.

CE: Of your *input variables* which ones do you really have control over?

PE: Only the air flowrate, and the fuel flowrate; and even then, we usually preset the air flowrate and change only the fuel flowrate when necessary. Our main *control variable* is the air-to-fuel ratio.

CE: The other input variables, the crude oil feed rate F, and inlet temperature T_i, are therefore *disturbances?*

PE: Yes.

CE: Any other process variables of importance that I should know of?

PE: Yes, the fuel supply pressure P_F and the fuel's heat content λ_F; they vary significantly, and we don't have any control over these variations. They are also *disturbances.*

CE: What sort of *instrumentation* do you have for *data acquisition* and *control action implementation?*

PE: We have thermocouples for measuring the temperatures T and T_i ; flow meters for measuring F, Q_F; and a control valve on the fuel line. We have an *optical pyrometer* installed for monitoring the furnace tube temperature. An *alarm* is tripped if the temperature gets within a few degrees of the upper limit constraint.

CE: Do you have a process model available for this furnace?

PE: No; but there's an operator who understands the process behavior quite well. We have tried running the process on manual (control) using this operator, but the results weren't acceptable. The record shown below, taken off the outlet temperature strip-chart recorder, is fairly representative. This is the response to a step increase in the inlet feedrate **F** (See Figure 2).

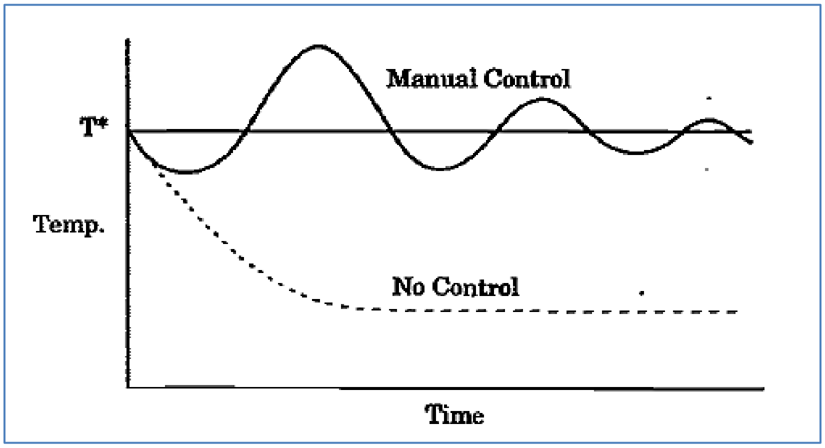


Fig. 2. System performance under manual control

CE: Do you have an idea of what might be responsible?

PE: Yes. We think it has to do with basic human limitations; his anticipation of the effect of the feed disturbance is ingenious, but imperfect, and he just couldn't react fast enough, or accurately enough, to the influence of the additional disturbance effects of variations in fuel supply pressure and heat content.

CE: Let's start with a simple feedback system then. Let's install a temperature controller that uses measurements of the furnace outlet temperature T to adjust the fuel flowrate Q_F accordingly [Figure 3(a)]. We will use a PID controller with these controller parameter values to start with (proportional band = 70%, reset rate = 2 repeats/min, derivative time = 0). Feel free to retune the controller if necessary.

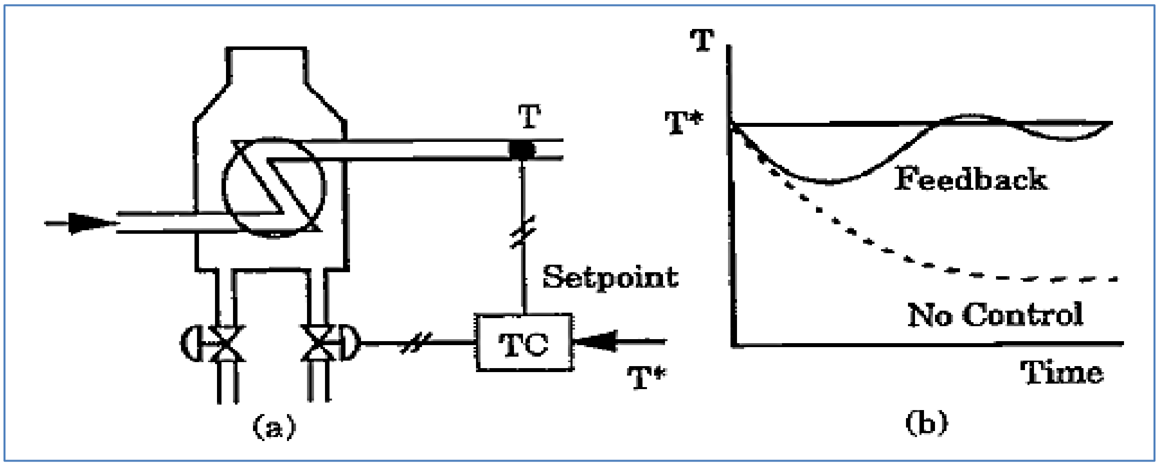


Fig. 2. The feedback control system

PE: The performance of the feedback system [see Figure 2(b)], even though better than with manual control, is still not acceptable; too much low-temperature feed is sent to the fractionator during the first few hours following each throughput increase.

CE: (After a little thought) What is needed is a means by which we can change fuel flow the instant we detect a change in the feed flowrate. Try this *feed-forward* control strategy by itself first (Figure 3); augment this with *feedback* only **if** you find it necessary (Figure 4(a)).

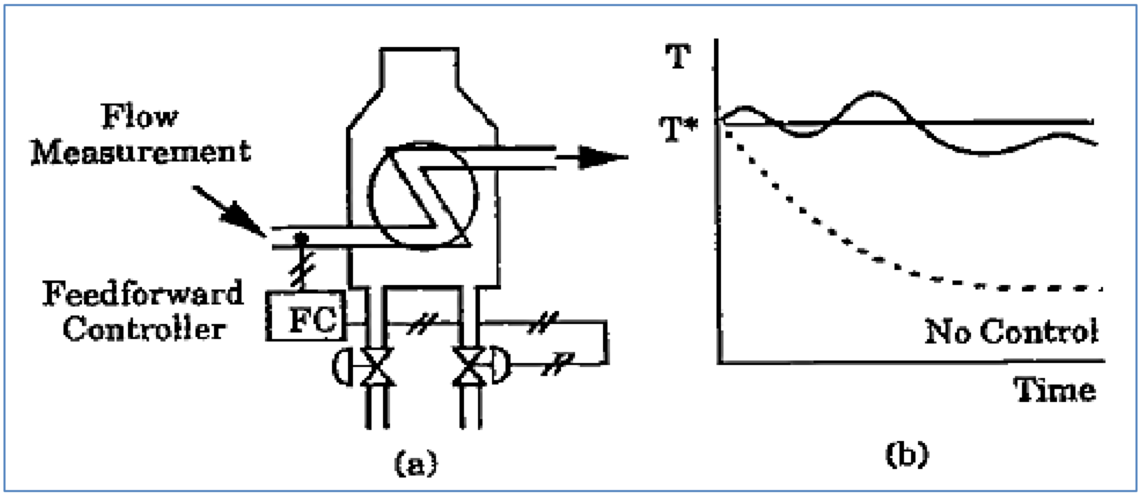


Fig. 3. The feed-forward control system

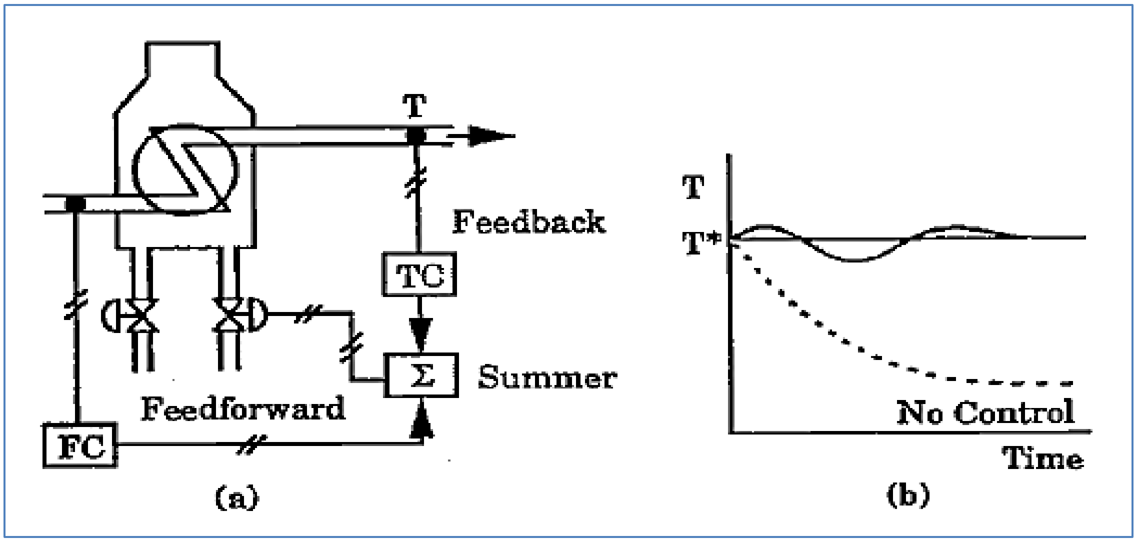


Fig. 4. The feed-forward/feedback control system

PE: With the *feedforward* strategy by itself, there was the definite advantage of quickly compensating for the effect of the disturbance, at least initially. The main problem was the nonavailability of the furnace outlet temperature measurement **to** the controller, with the result that we had *offsets.* Since we can't afford the persistent offset, we had to activate the feedback system. As expected the addition of feedback rectified this problem (Figure 4(b)).

PE: We have one major problem left the furnace outlet temperature still fluctuates, sometimes rather unacceptably, whenever we observe variations in the fuel delivery pressure. In addition, we are pretty sure that the variations **in** the fuel's heat content contributes to these fluctuations, but we have no easy way of *quantitatively* monitoring these heat content variations. At this point, however, they don't seem to be as significant as supply pressure variations.

CE: Let's focus on the problem caused by the variations in fuel supply pressure. It is easy to see why this should be a problem. The controller can only adjust the valve on the fuel line; and even though we expect that specific valve positions should correspond to specific fuel flowrates, this will be so only if the delivery pressure is constant. Any fluctuations in delivery pressure means that the controller will not get the *fuel flowrate* it asks for. We must install an additional *loop* to ensure that the temperature controller gets the actual flowrate change it demands; a mere change in valve position will not ensure this. We will install a *flow controller* in between the temperature controller and the control valve on the fuel line. The task of this *inner loop controller* will be to ensure that the fuel flowrate demanded by the temperature controller is actually delivered to the furnace regardless of supply pressure variations. The addition of this *cascade control system* should work well. (See Figure 5 for the final control system and its performance.)

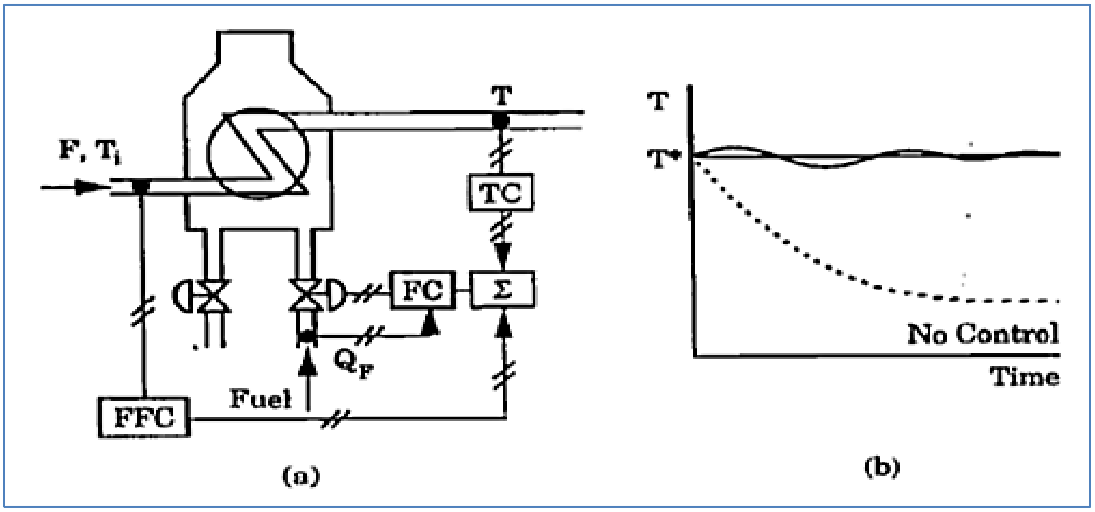


Fig. 5. The final control system (feed-forward/feedback-plus-cascade)

The basic concepts and terminology of process control:

The state of affairs within, or in the immediate environment of, a typical processing unit is usually indicated by such quantities as *temperature, flowrates in and out of containing vessels, pressure, composition, etc.* These are referred to as the *variables* of the process, or *process variables.* Recall that in our discussion of the furnace control problem we referred to such variables as these. It is customary to classify these variables according to whether they simply *provide information about process conditions,* or whether they are Capable of *influencing process conditions.* On the first level, therefore, there are two categories of process variables: *input* and *output* variables.

Input variables are those that independently stimulate the system and can thereby induce change in the internal conditions of the process.

Output variables are those by which one obtains information about the internal state of the process.

It is appropriate, at this point, to introduce what is called a *state* variable and distinguish it from an *output* variable. *State variables* are generally recognize as:

That minimum set of variables essential for completely describing the internal state (or condition) of a process.

The state variables are therefore the *true* indicators of the internal state of the process system. The actual *manifestation* of these *internal states* by measurement is what yields an *output.* Thus the *output* variable is, in actual fact, some *measurement* either of a single state variable or a combination of state variables.

On a second level, it is possible to further classify *input* variables as follows:

1. Those input variables that are at our disposal to manipulate freely as we choose are called *manipulated* (or *control) variables.*

2. Those over which we have no control (i.e., those whose values we are in no position to decide at will) are called *disturbance variables.*

Finally, we must note that some process variables *(output* as well as *input* variables) are directly available for measurement while some are not. Those process variables whose values are made available by direct on-line measurement are classified as *measured variables;* the others are called *unmeasured variables* (see Figure 6). Although output variables are defined as measurements, it is possible that some outputs are not measured on-line (no instrument is installed on the process) but require infrequent samples to be taken to the laboratory for analysis. Thus for control system design these are usually considered unmeasured outputs in the sense that the measurements are not available frequently enough for control purposes.

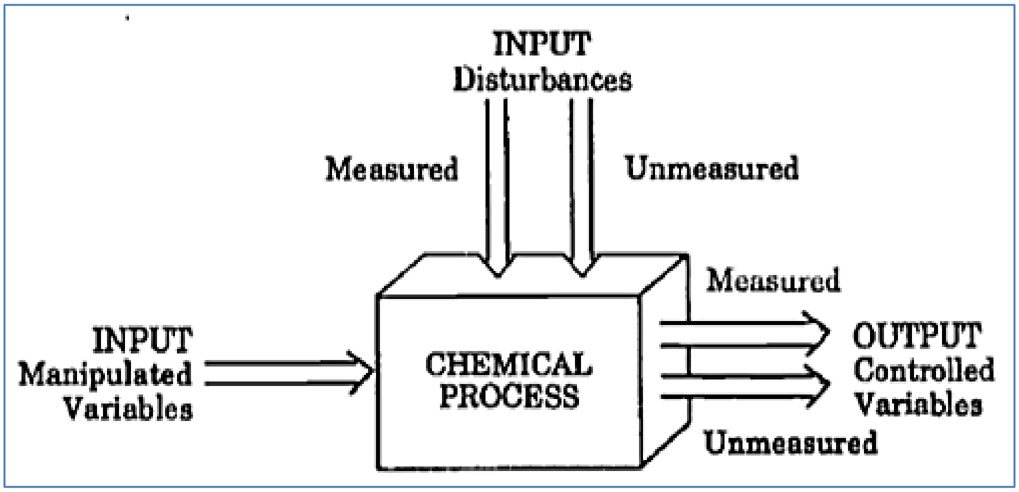


Fig. 6. The variables of a process

For each process, the process control problem has been characterized by identifying three important types of process variables.

- **Controlled variables (CVs)**: The process variables that are controlled. The desired value of a controlled variable is referred to as its set point.
- **Manipulated variables (MVs)**: The process variables that can be adjusted in order to keep the controlled variables at or near their set points. Typically, the manipulated variables are flow rates.
- **Disturbance variables (DVs) or Load variables (LVs)**: Process variables that affect the controlled variables but cannot be manipulated. Disturbances generally are related to changes in the operating environment of the process: for example, its feed conditions or ambient temperature. Some disturbance variables can be measured on-line, but many cannot such as the crude oil composition for Process (c), a thermal cracking furnace.

The specification of CVs, MVs, and DVs is a critical step in developing a control system. The selections should be based on process knowledge, experience, and control objectives.

***H.W. 2.1**: What are the CV, MV, and DV for a storage tank.*

Processes Control:

Illustrative examples:

Example 1: The variables of an industrial furnace

As shown in Fig. 1, it is clear that T, the outlet temperature, is *our output variable.* Next, we note that the value of this variable is affected by a host of other variables that must be carefully considered in order to classify them properly. Of all the variables that can affect the value of **T**, only Q_A, the air flowrate, and Q_F, the fuel flowrate, can be manipulated at will; they are therefore the *manipulated variables.* The other variables, F (the inlet feedrate), T_i (the inlet temperature), P_F (the fuel supply pressure), and λ_F (the fuel's heat content), all vary in a manner that we cannot control; hence they are all *disturbance variables. This* process, therefore, has *one output variable, two manipulated input variables, and four disturbance variables.*

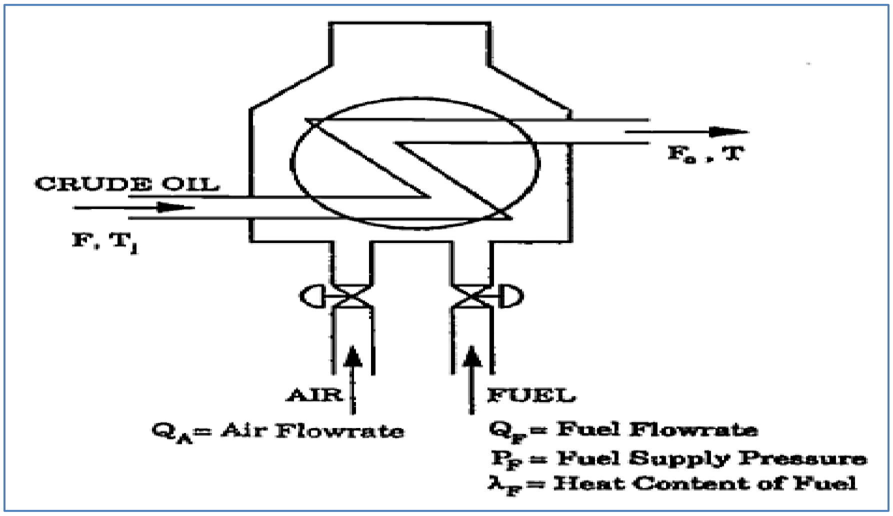


Fig. 1. Crude oil preheater furnace

Example 2: A blending process

A simple blending process is used to introduce some important issues in control system design. Blending operations are commonly used in many industries to ensure that final products meet customer specifications. A continuous, stirred-tank blending system is shown in Fig. 2. The control objective is to blend the two inlet streams to produce an outlet stream that has the desired composition. Stream 1 is a mixture of two chemical species, A and B. We assume that its mass flow rate w_1 is constant, but the mass fraction of **A**, x_1, varies with time. Stream 2 consists of pure **A** and thus $x_2 = 1$. The mass flow rate of Stream 2, w_2, can be manipulated using a control valve. The mass fraction of **A** in the outlet stream is denoted by x and the desired value (set point) by x_{sp}. Thus for this control problem, the controlled variable is x, the manipulated variable is w_2, and the disturbance variable is x_1.

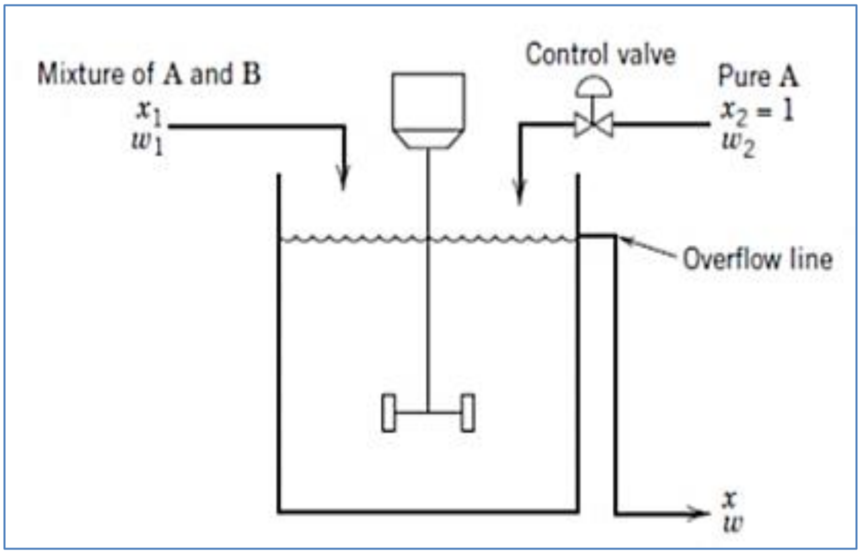


Fig. 2. Stirred-tank blending system

Design Question. *If the nominal value of* x_1 *is* X_1, *what nominal flow rate* W_2 *is required to produce the desired outlet concentration,* x_{sp}*?*

To answer this question, we consider the steady-state material balances:

Overall balance:

$$0 = W_1 + W_2 - W \qquad (1)$$

Component A balance:

$$0 = W_1X_1 + W_2X_2 - WX \qquad (2)$$

The capital symbol denotes its nominal steady-state value, for example, the value used in the process design. According to the process description, $X_2 = 1$ and $X = x_{sp}$. Solving Eq. 1 for W, substituting these values into Eq. 2, and rearranging gives

$$W_2 = W_1[(x_{sp} - X_1)/(1 - x_{sp})] \qquad (3)$$

Equation 3 is the design equation for the blending system. If our assumptions are correct and if $x_1 = X_1$, then this value of w_2 will produce the desired result, $x = x_{sp}$. But what happens if conditions change?

Control Question. *Suppose that inlet concentration x_1 varies with time. How can we ensure that the outlet composition x remains at or near its desired value, x_{sp}?*

As a specific example, assume that x_1 increases to a constant value that is larger than its nominal value, X_1. It is clear that the outlet composition will also increase due to the increase in inlet composition. Consequently, at this new steady state, $x > x_{sp}$. Next we consider several strategies for reducing the effects of x_1 disturbances on x.

Method 1. *Measure x and adjust w_2*. It is reasonable to measure controlled variable x and then adjust w_2 accordingly. For example, if x is too high, w_2 should be reduced; if x is too low, w_2 should be increased. This control strategy could be implemented by a person (*manual control*). However, it would normally be more convenient and economical to automate this simple task (*automatic control*). Method 1 can be implemented as a simple control algorithm (or control law),

$$w_2(t) = W_2 + K_c\,[x_{sp} - x(t)] \qquad (4)$$

where K_c is a constant called the *controller gain*. The symbols, $w_2(t)$ and $x(t)$, indicate that w_2 and x change with time. Equation 4 is an example of *proportional control*, because the change in the flow rate, $w_2(t) - W_2$, is proportional to the deviation from the set point, $x_{sp} - x(t)$. Consequently, a large deviation from set point produces a large corrective action, while a small deviation results in a small corrective action. Note that we require K_c to be positive because w_2 must increase when x decreases, and vice versa. However, in other control applications, negative values of K_c are appropriate, as will discuss later.

A schematic diagram of Method 1 is shown in Fig. 3. The outlet concentration is measured and transmitted to the controller as an electrical signal. (Electrical signals are shown as dashed lines in Fig. 4.) The controller executes the control law and sends an appropriate electrical signal to the control valve. The control valve opens or closes accordingly. Later, we consider process instrumentation and control hardware in more detail.

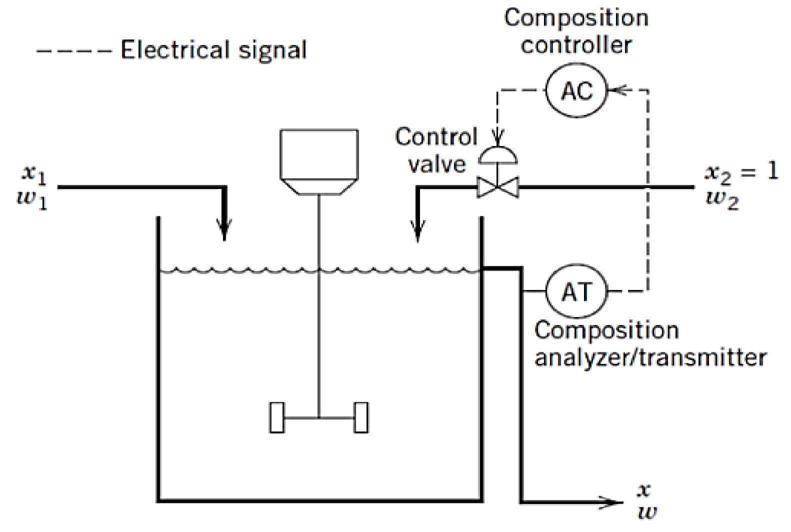


Figure 1.4 Blending system and Control Method 1.

Fig. 3. Blending system and control Method 1

Method 2. *Measure* x_1*, adjust* w_2. As an alternative to Method 1, we could measure disturbance variable x_1 and adjust w_2 accordingly. Thus, if $x_1 > X_1$, we would decrease w_2 so that $w_2 < W_2$. If $x_1 < X_1$, we would increase w_2. A control law based on Method 2 can be obtained from Eq. 3 by replacing X_1 with $x_1(t)$ and W_2 with $w_2(t)$:

$$w_2(t) = W_1[(x_{sp} - x_1(t))/(1 - x_{sp})] \qquad (5)$$

The schematic diagram for Method 2 is shown in Fig. 4. Because Eq. 1 is valid only for steady-state conditions, it is not clear just how effective Method 2 will be during the transient conditions that occur after an x_1 disturbance.

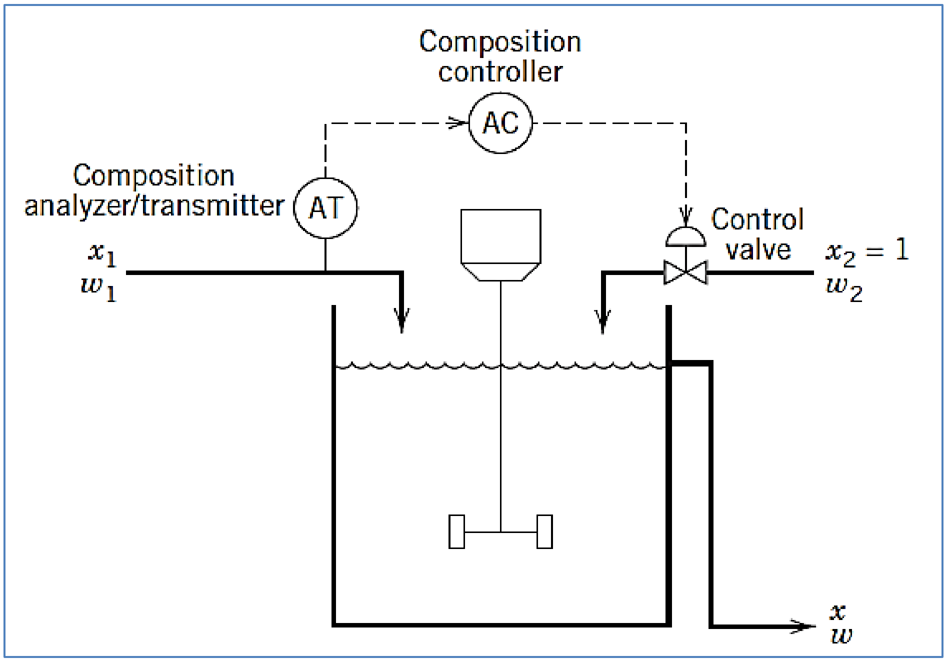


Fig. 4. Blending system and control Method 2

H.W. 3.1: Suggest other methods to solve this problem

As earlier noted, the dynamic *(i.e.,* ever changing) nature of chemical processes makes it imperative that we have some means of effectively monitoring, and inducing change in, the process variables of interest. In a typical chemical process, the *process control system* is the entity that is charged with the responsibility for monitoring outputs, making decisions about how best to manipulate inputs so as to obtain desired output behavior, and effectively implementing such decisions on the process.

It is therefore convenient to break down the responsibility of the control system into the following three major tasks:

1. Monitoring process output variables by *measurement*
2. Making rational *decisions* regarding what corrective action is needed on the basis of the information about the current and desired state of the process
3. Effectively *implementing* these decisions on the process

When these tasks are carried out manually by a human operator we have a ***manual control system****;* on the other hand, a control system in which these tasks are carried out in an *automatic* fashion by a machine is known as an ***automatic control system;*** in particular, when the machine involved is a computer, we have a ***computer control system***.

With the possible exception of the manual control system, all other control systems require certain hardware elements for carrying out each of the above itemized tasks. Later we will introduce the hardware elements.

Processes Control:

Control System Hardware Elements:

The hardware elements required for the investigation of the control system's tasks of measurements, decision making, and corrective action application typically fall into the following categories: sensors, controllers, transmitters and final control elements.

Sensors: The first task, that of gaining information about the status of the process output variables, is carried out by sensors (also called measuring devices or primary elements). In most process control applications, the sensors are usually needed for pressure, temperature, liquid level, flow, and composition measurements. Typical examples are: thermocouples (for temperature measurements), differential pressure cells (for liquid level measurements), gas/liquid chromatographs (for composition measurements), etc.

Controllers: The decision maker, and hence the "heart" of the control system, is the controller; it is the hardware element with "built-in" capacity for performing the only task requiring some form of "intelligence." The controller hardware may be pneumatic in nature (in which case it operates on air signals), or it may be electronic (in which case, it operates on electrical signals). Electronic controllers are more common in more modem industrial process control applications. The pneumatic and electronic controllers are limited to completely simple operations "Which we shall have cause to discuss more fully later. When more complex control operations are required, the digital computer is usually used as a controller.

Transmitters: How process information gained by the sensor gets to the controller, and the controller decision gets back to the process, is the responsibility of devices known as transmitters. Measurement and control signals may be transmitted as air pressure signals, or as electrical signals. Pneumatic transmitters are required for the former, and electrical ones for the latter.

Final Control Elements: Final control elements have the task of actually applying, on the process, the control command issued by the controller. Most final control elements are control valves (usually pneumatic, i.e., they are air-driven), and they occur in various shapes, sizes, and have several modes of specific operation. Some other examples of final control elements include: *variable speed fans, pumps, and compressors; conveyors; and relay switches.*

Other Hardware Elements: In transmitting information back and forth between the process and the controller, the need to *convert* one type of signal to another type is often unavoidable. For example, it will be necessary to convert the electrical signal from an electronic controller to a pneumatic signal needed to operate a control valve. The devices used for such signal transformations are called ***transducers*** and as will be further discussed later, various types are available for various signal transformations. Also, for computer control applications, it is necessary to have devices known as *analog-to-digital* (A/D) and *digital-to-analog (D/*A) *converters.* This is because while the rest of the control system operates on *analog* signals (electric voltage or pneumatic pressure), the computer operates digitally, giving out, and receiving, only binary numbers. *A/D* converters make the process information available in recognizable form to the computer, while the *D/*A converters make the computer commands accessible to the process.

H.W. 4.1: *Which of the above instruments are the most important than others? Explain your answer briefly.*

Classification of process control strategies:

Depending primarily upon the structure of the decision-making process in relation to the information-gathering and decision-implementation ends, a process control system can be configured in several different ways. Let us introduce some of the most common configurations.

Feedback Control: The control system illustrated in Figure 1 operates by feeding process output information back to the controller. Decisions based on such "fed back" information is then applied on the process. This is known as a *feedback* control structure, and it is one of the simplest, and by far the ·most common, control structures employed in chemical process control. It was introduced for the furnace example. Observe that it makes use of current information about the output of the process to determine what action to take in regulating process behavior. We must note, however, that with such a structure, the effect of any disturbance entering the process must first be registered by the process as an upset in its output before corrective control action can be taken; i.e., controller decisions are taken "after the fact." For feedback control, the disturbance variable is not measured.

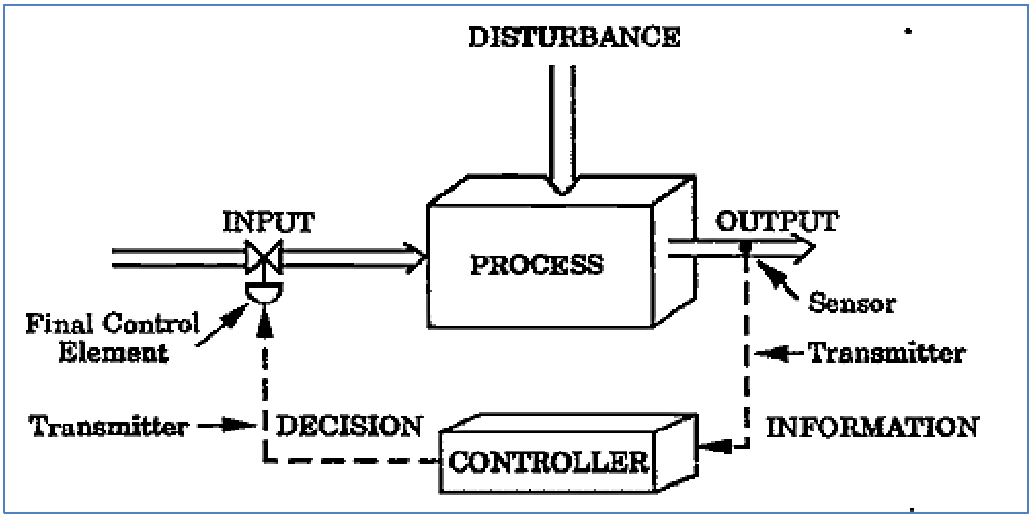


Fig. 1. The feedback control configuration

Feedforward Control**:** In Figure 2 we have a situation in which it is information about an incoming disturbance that gets directly communicated to the controller instead of actual system output information. With this configuration, the controller decision is taken *before* the process is affected by the incoming disturbance. This is the *feedforward control* structure (compare with Figure 1) since the controller decision is based on information that is being “fed forward”. For feedback control, the disturbance variable is measured.

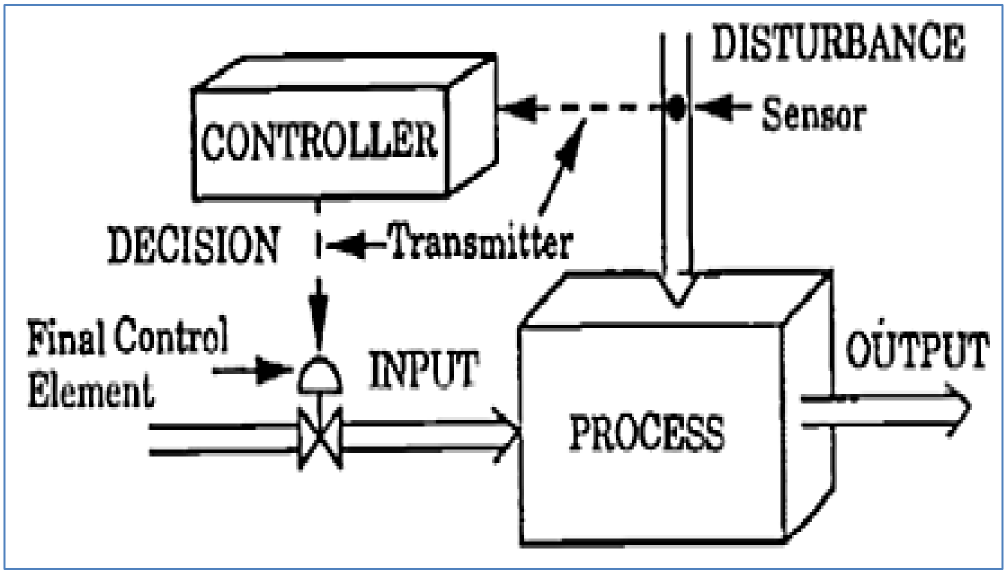


Fig. 2. The feedforward control configuration

As we shall see later, feedforward control has proved indispensable in dealing with certain process control problems. The main feature of the feedforward configuration is the choice of measuring the *disturbance* variable rather than the output variable that we desire to regulate. The potential advantage of this strategy has already been noted. Further reflection on this strategy will, however, also reveal a potential drawback: the controller has *no information* about the conditions existing at the process output, the actual process variable we are concerned about regulating. Thus the controller detects the entrance of disturbances and before the process is upset attempts to compensate for their effects somehow (typically based on an imperfect process model); however, the controller is unable to determine the accuracy of this compensation, since this strategy does not call for a measurement of the process output.

It is important to make a distinction between ***negative feedback*** and ***positive feedback***. In the engineering literature, negative feedback refers to the desirable situation in which the corrective action taken by the controller forces the controlled variable toward the set point. On the other hand, when positive feedback occurs, the controller makes things worse by forcing the controlled variable farther away from the set point. For example, in the blending control problem, positive feedback takes place if $\mathbf{K_c < 0}$, because $\mathbf{w_2}$ will increase when x increases. Clearly, it is of paramount importance to ensure that a feedback control system incorporates negative feedback rather than positive feedback.

Open-Loop Control**:** When, as shown in Figure 3, the controller decision is not based upon any measurement information collected from any part of the process, but upon some sort of internally generated strategy, we have an open-loop control structure. This is because the controller makes decisions without the advantage of information that "closes the loop" between the output and input variables of the process, as is the case with the feedback control configuration (see Figure 1.) This otherwise vital loop is "open." However, this does not necessarily constitute a handicap. Perhaps the most common example of an open-loop control system can be found in the simple timing device used for some traffic lights. Regardless of the volume of traffic, the timer is set such that the period of time for which the light remains green, yellow, or red is predetermined.

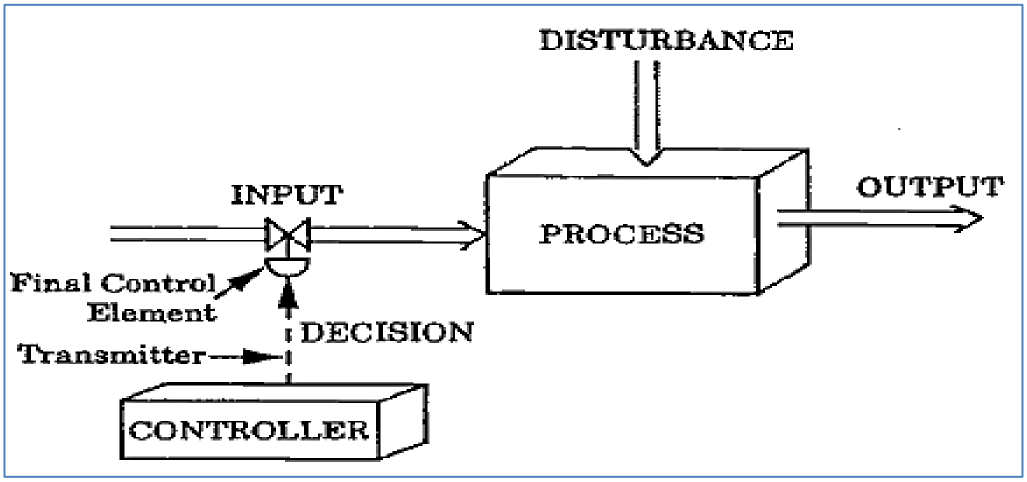


Fig. 3. The open loop control configuration

Important process variables that have been selected to receive the attention of the control system typically have target values at which they are required to be maintained. These target values are called ***set-points***. Maintaining these process variables at their prescribed set-points is, of course, the main objective of the process control system, be it manual or automatic. However, output variables deviate from their set-points:

1. Either as a result of the effect of disturbances, or
2. Because the set-point itself has changed.

We have *regulatory control* when the control system's task is solely that of counteracting the effect of disturbances in order to maintain the output at its set-point (as was the case in the furnace example). When the objective is to cause the output to track the changing set-point, we have *servo control* (see Figure 4.)

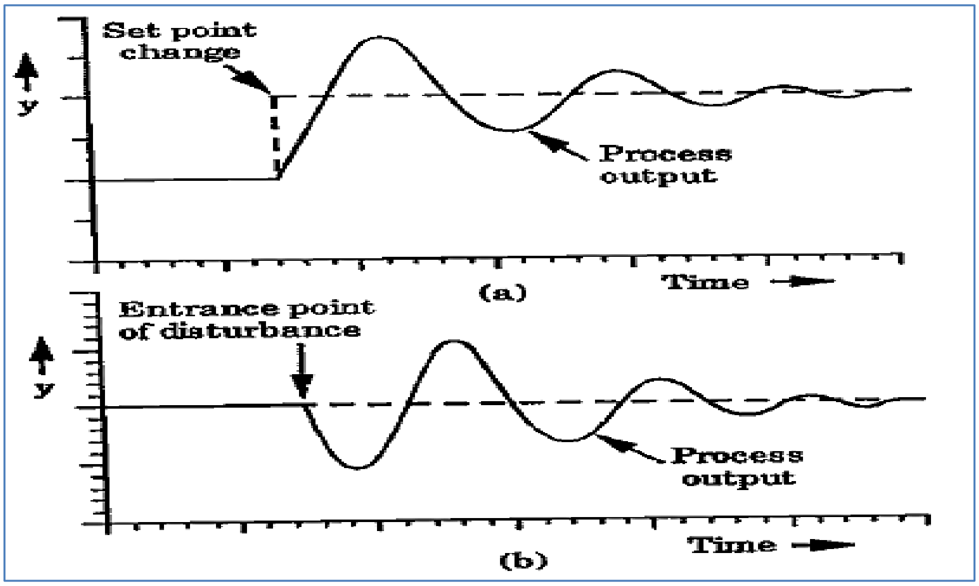


Fig. 4. Possible process responses (a) servo; (b) regulatory control

An example of the feedback control strategy is the blending system (**Method 1**), while (**Method 2**) applied the feedforward control strategy. However, in industrial applications, it is generally uneconomical to attempt to measure all potential disturbance variables. A more practical approach is to use a combined feedforward‑feedback control system, in which feedback control provides corrective action for unmeasured disturbances, while feedforward control reacts to measured disturbances before the controlled variable is upset. Consequently, in industrial applications, feedforward control is normally used in combination with feedback control.

Processes Control:

Process Control Diagrams:

Next, we consider the equipment that is used to implement control strategies. For the stirred-tank mixing system under feedback control (Method 1), the exit concentration ***x*** is controlled and the flow rate **w_2** of pure species A is adjusted using proportional control. To consider how this feedback control strategy could be implemented, a block diagram for the stirred-tank control system is shown in Fig. 1.

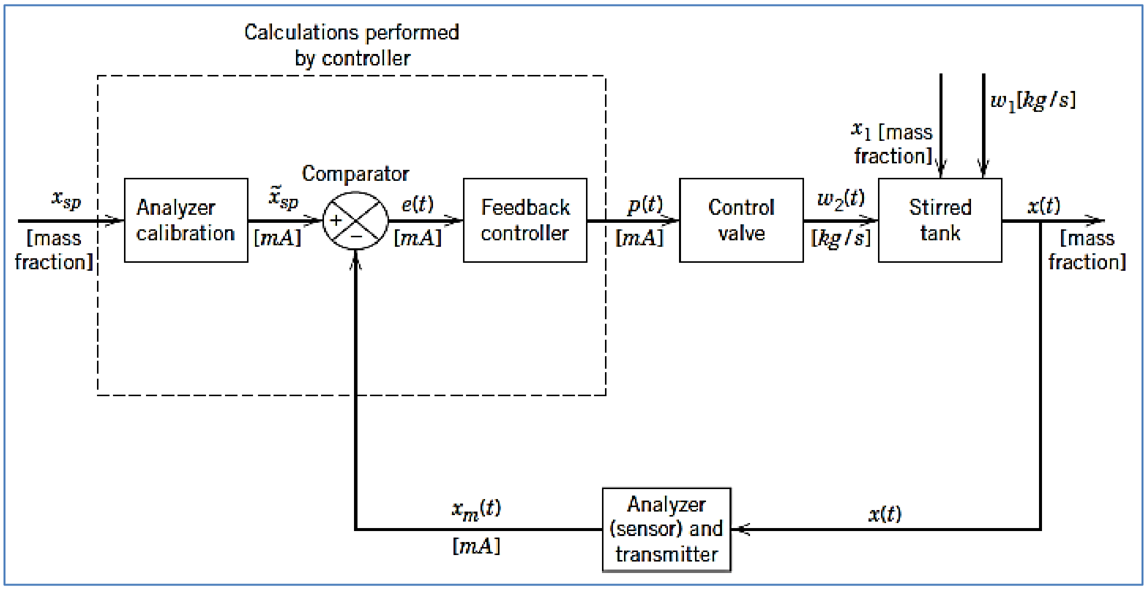


Fig. 1. Block diagram for the outlet composition feedback control system

The operation of the feedback control system can be summarized as follows:

1. **Analyzer and transmitter***:* The tank exit concentration is measured by an analyzer and then the measurement is converted to a corresponding electrical current signal by a transmitter.
2. **Feedback controller**: The controller performs three distinct calculations. First, it converts the actual set point x_{sp} into an equivalent internal signal $\tilde{x}_{sp}$. Second, it calculates an error signal ***e*(t)** by subtracting the measured value $x_m(t)$ from the set point $\tilde{x}_{sp}$, that is, $\mathbf{e(t)} = \tilde{x}_{sp} - x_m(t)$. Third, controller output p(t) is calculated from the proportional control law similar to Eq. 4 (Lecture 3).

3. **Control valve**: The controller output $p(t)$ in this case is a DC current signal that is sent to the control valve to adjust the valve stem position, which in turn affects flow rate $w_2(t)$. (The controller output signal is traditionally denoted by p because early controllers were pneumatic devices with pneumatic (pressure) signals as inputs and outputs.)

The block labeled “control valve” has $p(t)$ as its input signal and $w_2(t)$ as its output signal, which illustrates that the signals on a block diagram can represent either a physical variable such as $w_2(t)$ or an instrument signal such as $p(t)$. Each component in Fig. 1 exhibits behavior that can be described by a differential or algebraic equation. One of the tasks facing a control engineer is to develop suitable mathematical descriptions for each block; the development and analysis of such dynamic models are considered later. In particular, networks of digital computers can be used to implement thousands of feedback and feedforward control loops.

Example 5.1. A Distillation Column:

The blending control system in the previous section is quite simple, because there is only one controlled variable and one manipulated variable. For most practical applications, there are multiple controlled variables and multiple manipulated variables. As a representative example, the distillation column shown in Fig. 2, with five controlled variables and five manipulated variables.

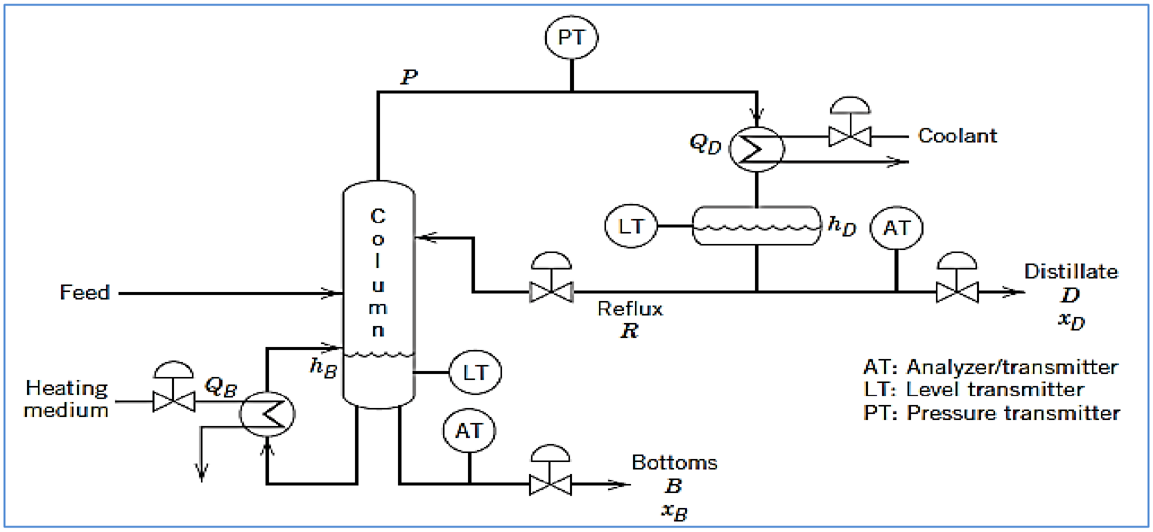


Fig. 2. Controlled and manipulated variables for a typical distillation column.

The controlled variables are product compositions, $\boldsymbol{x_D}$ and $\boldsymbol{x_B}$, column pressure, $\boldsymbol{P}$, and the liquid levels in the reflux drum and column base, $\boldsymbol{h_D}$ and $\boldsymbol{h_B}$. The five manipulated variables are product flow rates, **D** and **B**, reflux flow, **R**, and the heat duties for the condenser and reboiler, $\boldsymbol{Q_D}$ and $\boldsymbol{Q_B}$. The heat duties are adjusted via the control valves on the coolant and heating medium lines. The feed stream is assumed to come from an upstream unit. Thus, the feed flow rate cannot be manipulated, but it can be measured and used for feedforward control.

A conventional multi-loop control strategy for this distillation column would consist of five feedback control loops. Each control loop uses a single manipulated variable to control a single controlled variable. In control applications, for which conventional multi-loop control systems are not satisfactory, an alternative approach, multivariable control, can be advantageous. In multivariable control, each manipulated variable is adjusted based on the measurements of at least two controlled variables rather than only a single controlled variable, as in multi-loop control. The adjustments are based on a dynamic model of the process that indicates how the manipulated variables affect the controlled variables. Consequently, the performance of multivariable control, or any model-based control technique, will depend heavily on the accuracy of the process model.

We have introduced the basic concepts of process dynamics and process control. The process dynamics determine how a process responds during transient conditions, such as plant start-ups and shutdowns, grade changes, and unusual disturbances. Process control enables the process to be maintained at the desired operating conditions, safely and economically, while satisfying environmental and product quality requirements. Without effective process control, it would be impossible to operate large-scale industrial plants. Three physical examples, a Furnace, continuous blending system and a distillation column, have been used to introduce basic control concepts, notably, feedback and feedforward control.

H.W. 5.1. *Solve the following questions.*

1. Which of the following statements are true? For the false statements, explain why you think they are false:

(a) Feedforward and feedback control require a measured variable.

(b) For feedforward control, the measured variable is the variable to be controlled.

(c) Feedback control theoretically can provide perfect control (i.e., no deviations from set point) if the process model used to design the control system is perfect.

(d) Feedback control takes corrective action for all types of process disturbances, both known and unknown.

(e) Feedback control is superior to feedforward control.

2. Consider a home heating system consisting of a natural gas-fired furnace and a thermostat. In this case, the process consists of the interior space to be heated. The thermostat contains both the temperature sensor and the controller. The furnace is either on (heating) or off. Draw a schematic diagram for this control system. On your diagram, identify the controlled variables, manipulated variables, and disturbance variables. Be sure to include several possible sources of disturbances that can affect room temperature.

3. Does a typical microwave oven utilize feedback control to set the cooking temperature or to determine if the food is “cooked”? If not, what technique is used? Can you think of any disadvantages to this approach, for example, in cooking foods?

4. Describe how a bicycle rider utilizes concepts from both feedforward control and feedback control while riding a bicycle.

Processes Control:

Theoretical models of chemical processes:

In the following lectures, we consider the derivation of unsteady-state models of chemical processes from physical and chemical principles. Unsteady-state models are also referred to as ***dynamic models***. We first consider the basis for dynamic models and then present a general strategy for deriving them from first principles such as conservation laws. Then dynamic models are developed for several representative processes. Finally, we describe how dynamic models that consist of sets of ordinary differential equations and algebraic relations can be solved.

Dynamic models play a central role in the subject of process dynamics and control. The models can be used to:

- *Improve understanding of the process*: Dynamic models and computer simulation allow transient process behavior to be investigated without having to disturb the process.
- *Train plant operating personnel*: Process simulators play a critical role in training plant operators to run complex units and to deal with dangerous situations or emergency scenarios. By interfacing a process simulator to standard process control equipment, a realistic training environment is created. This role is analogous to flight training simulators used in the aerospace industry.
- *Develop a control strategy for a new process*: A dynamic model of the process allows alternative control strategies to be evaluated. For example, a dynamic model can help identify the process variables that should be controlled and those that should be manipulated. Preliminary controller tuning may be derived using a model, prior to plant start-up using empirical models. For model-based control strategies, the process model is an explicit element of the control law.
- *Optimize process operating conditions*: It can be advantageous to recalculate the optimum operating conditions periodically in order to maximize profit or minimize

cost. A steady-state process model and economic information can be used to determine the most profitable operating conditions.

Models can be classified based on how they are obtained:

(a) Theoretical models are developed using the principles of chemistry, physics, and biology.

(b) Empirical models are obtained by fitting experimental data.

(c) Semi-empirical models are a combination of the models in categories (a) and (b); the numerical values of one or more of the parameters in a theoretical model are calculated from experimental data.

Theoretical models offer two very important advantages: they provide physical insight into process behavior, and they are applicable over wide ranges of conditions. However, there are disadvantages associated with theoretical models. They tend to be expensive and time-consuming to develop. In addition, theoretical models of complex processes typically include some model parameters that are not readily available, such as reaction rate coefficients, physical properties, or heat transfer coefficients.

Although empirical models are easier to develop and to use in controller design than theoretical models, but they have a serious disadvantage: empirical models typically do not extrapolate well. More specifically, empirical models should be used with caution for operating conditions that were not included in the experimental data used to fit the model. The range of the data is typically quite small compared to the full range of process operating conditions.

Semi-empirical models have three inherent advantages: (***i***) they incorporate theoretical knowledge, (***ii***) they can be extrapolated over a wider range of operating conditions than purely empirical models, and (***iii***) they require less development effort than theoretical models. Consequently, semi-empirical models are widely used in industry.

The following example concerned with the development of theoretical models from first principles such as conservation laws.

Example 6.1. Develop an unsteady-state model for a stirred-tank blending system that will allow us to analyze the more general situation where process variables vary with time and accumulation terms must be included.

Ans*.**: As an illustrative example, we consider the isothermal stirred-tank blending system in Fig. 1. The inlet stream 2 is not necessarily pure A (that is, $x_2 \neq 1$). Now the volume of liquid in the tank ***V can vary with time because the overflow line has been omitted, and the exit flow rate is not necessarily equal to the sum of the inlet flow rates.

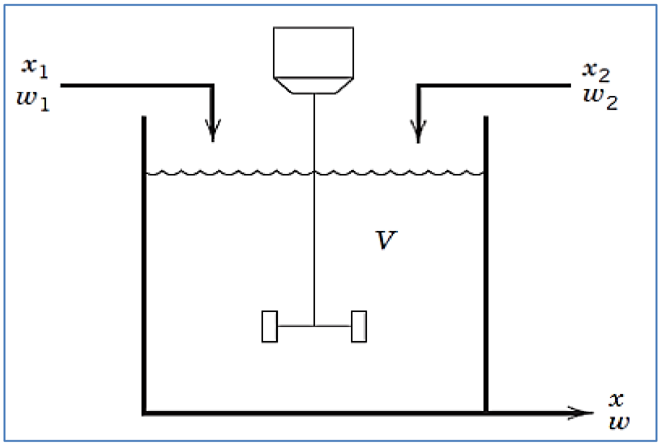


Fig. 1. The blending process

An unsteady-state mass balance for the blending system in Fig. 1 has the form:

{rate of accumulation of mass in the tank}={rate of mass in}−{rate of mass out} (1)

The mass of liquid in the tank can be expressed as the product of the liquid volume ***V*** and the density ***ρ***. Consequently, the rate of mass accumulation is simply ***d(Vρ)/dt***, and Eq. 1 can be written as:

$$\frac{d(V\rho)}{dt} = w_1 + w_2 - w \tag{2}$$

For the perfect mixing assumption, the rate of accumulation of component ***A*** is ***d(Vρx)/dt***, where ***x*** is the mass fraction of ***A***. The unsteady-state component balance is:

$$\frac{d(V\rho x)}{dt} = w_1 x_1 + w_2 x_2 - wx \qquad (3)$$

Equations 2 and 3 provide an unsteady-state model for the blending system. The corresponding steady-state model was derived in previous lectures. It also can be obtained by setting the accumulation terms in Eqs. 2 and 3 equal to zero,

$$0 = W_1 + W_2 - W \qquad (4)$$

$$0 = W_1 X_1 + W_2 X_2 - W X \qquad (5)$$

In general, a steady-state model is a special case of an unsteady-state model that can be derived by setting accumulation terms equal to zero. So, A dynamic model can be used to characterize the transient behavior of a process for a wide variety of conditions.

H.W. 6.1. *Write your assumptions for the a stirred-tank blending system shown in example 6.1 according to the listed conditions.*

It is important to remember that a process model is nothing more than a mathematical abstraction of a real process. The model equations are at best an approximation to the real process. A systematic procedure for developing dynamic models from first principles is summarized in Table 1 as follows:

Table 1 *A Systematic Approach for Developing Dynamic Models*

1. State the modeling objectives and the end use of the model. Then determine the required levels of model detail and model accuracy.
2. Draw a schematic diagram of the process and label all process variables.
3. List all of the assumptions involved in developing the model. The model should be no more complicated than necessary to meet the modeling objectives.
4. Determine whether spatial variations of process variables are important. If so, a partial differential equation model will be required.
5. Write appropriate conservation equations (mass, component, energy, and so forth).
6. Introduce equilibrium relations and other algebraic equations (from thermodynamics, transport phenomena, chemical kinetics, equipment geometry, etc.).

7. Perform a degrees of freedom analysis to ensure that the model equations can be solved.
8. Simplify the model. It is often possible to arrange the equations so that the output variables appear on the left side and the input variables appear on the right side. This model form is convenient for computer simulation and subsequent analysis.
9. Classify inputs as disturbance variables or as manipulated variables.

For process control problems, dynamic models are derived using unsteady-state conservation laws. Theoretical models of chemical processes are based on conservation laws such as the conservation of mass and energy. Consequently, we now consider important conservation laws and use them to develop dynamic models for representative processes.

Conservation of mass

{rate of mass accumulation}={rate of mass in}−{rate of mass out} (6)

Conservation of component i

*{rate of component **i** accumulation}={rate of component **i** in}−{rate of component **i** out} + {rate of component **i** produced}* (7)

The last term on the right-hand side of Eq. 7 represents the rate of generation (or consumption) of component ***i*** as a result of chemical reactions. Conservation equations can also be written in terms of molar quantities, atomic species, and molecular species.

Conservation of energy

The general law of energy conservation is also called the First Law of Thermodynamics. It can be expressed as:

{rate of energy accumulation}={rate of energy in by convection}−{rate of energy out by convection}+{net rate of heat addition to the system from the surroundings}+{net rate of work performed on the system by the surroundings} (8)

H.W. 6.2. *Is there a conservation equation of energy for a component **i** ?Explain your answer briefly.*

The total energy of a thermodynamic system, U_{tot}, is the sum of its internal energy, kinetic energy, and potential energy:

$$U_{tot} = U_{int} + U_{KE} + U_{PE} \qquad (9)$$

For the processes and examples considered in this course, it is appropriate to make two assumptions:

1. Changes in potential energy and kinetic energy can be neglected, because they are small in comparison with changes in internal energy.
2. The net rate of work can be neglected, because it is small compared to the rates of heat transfer and convection.

For these reasonable assumptions, the energy balance in Eq. 8 can be written as:

$$\frac{dU_{\text{int}}}{dt} = -\Delta(w\hat{H}) + Q \qquad (10)$$

where U_{int} is the internal energy of the system, $\hat{H}$ is the enthalpy per unit mass, w is the mass flow rate, and Q is the rate of heat transfer to the system. The Δ operator denotes the difference between outlet conditions and inlet conditions of the flowing streams. Consequently, the $[-\Delta(w\hat{H})]$ term represents the enthalpy of the inlet stream(s) minus the enthalpy of the outlet stream(s). The analogous equation for molar quantities is

$$\frac{dU_{\text{int}}}{dt} = -\Delta(\tilde{w}\tilde{H}) + Q \qquad (11)$$

Where $\tilde{H}$ is the enthalpy per mole and $\tilde{w}$ is the molar flow rate.

Next, we show that the dynamic model of the blending process in Eqs. 2 and 3 can be simplified. For this analysis, we introduce the additional assumption that the density of the liquid, ρ, is a constant. For constant ρ, Eqs. 2 and 3 become:

$$\rho\frac{dV}{dt} = w_1 + w_2 - w \qquad (12)$$

$$\rho\frac{d(Vx)}{dt} = w_1x_1 + w_2x_2 - w\,x \qquad (13)$$

After solving Eqs. 12 and 13, a more convenient model form is obtained:

$$\frac{dV}{dt} = \frac{1}{\rho}(w_1 + w_2 - w) \qquad (14)$$

$$\frac{dx}{dt} = \frac{w_1}{\rho V}(x_1 - x) + \frac{w_2}{\rho V}(x_2 - x) \qquad (15)$$

H.W. 6.3. *Derive Eq. 15.*

The dynamic model in Eqs. 14 and 15 is quite general and is based on only two assumptions: perfect mixing and constant density. For special situations, the liquid volume ***V*** is constant (i.e., ***dV/dt = 0***), and the exit flow rate equals the sum of the inlet flow rates, $\boldsymbol{w = w_1 + w_2}$. These conditions might occur when:

1. An overflow line is used in the tank as shown in the previous lecture.
2. The tank is closed and filled to capacity.
3. A liquid-level controller keeps V essentially constant by adjusting a flow rate.

In all three cases, Eq. 14 reduces to the same form as Eq. 4, not because each flow rate is constant, but because $\boldsymbol{w = w_1 + w_2}$ at all times. In order to obtain a solution to the **ODE** model, we must specify the inlet compositions ($\boldsymbol{x_1}$ and $\boldsymbol{x_2}$) and the flow rates ($\boldsymbol{w_1}$, $\boldsymbol{w_2}$, and ***w***) as functions of time. After specifying initial conditions for the dependent variables, ***V(0)*** and ***x(0)***, we can determine the transient responses, ***V(t)*** and ***x(t)***. The derivation of an analytical expression for ***x(t)*** when ***V*** is constant is illustrated in Example 6.2.

Example 6.2. A stirred-tank blending process with a constant liquid holdup of $\mathbf{2\ m^3}$ is used to blend two streams whose densities are both approximately **900** $\mathbf{kg/m^3}$. The density does not change during mixing.

(a) Assume that the process has been operating for a long period of time with flow rates of $\boldsymbol{w_1}$ **= 500 kg/min** and $\boldsymbol{w_2}$ **= 200 kg/min**, and feed compositions (mass fractions) of $\boldsymbol{x_1}$ **= 0.4** and $\boldsymbol{x_2}$ **= 0.75**. What is the steady-state value of ***x***?

(b) Suppose that w_1 changes suddenly from **500** to **400 kg/min** and remains at the new value. Determine an expression for ***x(t)*** and plot it.

(c) Repeat part (b) for the case where w_2 (instead of w_1) changes suddenly from **200** to **100 kg/min** and remains there.

(d) Repeat part (c) for the case where x_1 suddenly changes from **0.4** to **0.6** (in addition to the change in w_2).

Ans.:

(a) Denote the initial steady-state conditions by X, W, and so on. For the initial steady state, Eqs. 4 and 5 are applicable. Solve Eq. 5 for X:

$$X = \frac{W_1X_1 - W_2X_2}{W} = \frac{500 \times 0.4 + 200 \times 0.75}{700} = 0.5$$

(b) The component balance in Eq. 3 can be rearranged (for constant V and ρ) as

$$\tau\frac{dx}{dt} + x = \frac{w_1x_1 + w_2x_2}{w} = C^* \; where \; x(0) = 0.5$$

At steady-state, the right side equal **0.5** and for three cases (b, c, and d) the $\tau = (V\rho/W) = (2 * 900/600) = 3$ **min**

The last equation could be solved to be:

$$x(t) = 0.5\, e^{-t/3} + C^* (1- e^{-t/3})$$

For case (b):

$$C^* = \frac{400 \times 0.4 + 200 \times 0.75}{600} = 0.517$$

$$x(t) = 0.5\, e^{-t/3} + 0.517\, (1- e^{-t/3})$$

(c) For the step change in w_2,

$$C^* = \frac{500 \times 0.4 + 100 \times 0.75}{600} = 0.458$$

$$x(t) = 0.5\, e^{-t/3} + 0.458\, (1- e^{-t/3})$$

(d) Similarly, for the simultaneous changes in x_1 and w_2, $C^* = 0.625$. Thus, the solution is

$$x(t) = 0.5\, e^{-t/3} + 0.625\,(1 - e^{-t/3})$$

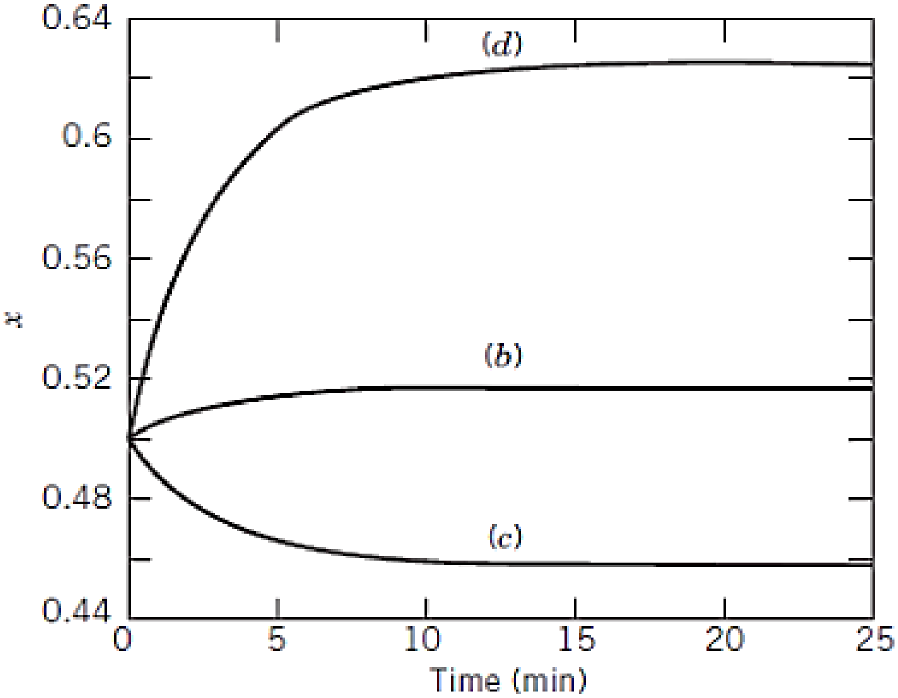


Fig. 2 Exit composition responses of a stirred-tank blending process to step changes in: (b) Flow rate w_1 (c) Flow rate w_2 (d) Flow rate w_2 and inlet composition x_1

Processes Control:

Degrees of freedom analysis

To simulate a process, we must first ensure that its model equations (differential and algebraic) constitute a solvable set of relations. In order for the model to have a unique solution, the number of unknown variables must equal the number of independent model equations. An equivalent statement is that all of the available degrees of freedom must be utilized. The number of degrees of freedom, N_F, can be calculated from the expression

$$N_F = N_V - N_E \qquad (1)$$

where N_V is the total number of process variables (distinct from constant process parameters) and N_E is the number of independent equations. A degrees of freedom analysis allows modeling problems to be classified according to the following categories:

1. $N_F = 0$: The process model is exactly specified. If $N_F = 0$, then the number of equations is equal to the number of process variables and the set of
equations has a solution.
2. $N_F > 0$: The process is underspecified. If $N_F > 0$, then $N_V > N_E$, so there are more process variables than equations.
3. $N_F < 0$: The process model is over-specified. For $N_F < 0$, there are fewer process variables than equations, and consequently the set of equations has no solution.

Note that $N_F = 0$ is the only satisfactory case. If $N_F > 0$, then a sufficient number of input variables have not been assigned numerical values. Then additional independent model equations must be developed in order for the model to have an exact solution. The steps in the degrees of freedom analysis are summarized as follows:

1. List all quantities in the model that are known constants (or parameters that can be specified) on the basis of equipment dimensions, known physical properties, and so on.
2. Determine the number of equations N_E and the number of process variables, N_V. Note that time t is not considered to be a process variable, because it is neither a process input nor a process output.

3. Calculate the number of degrees of freedom, $N_F = N_V - N_E$.
4. Identify the output variables that will be obtained by solving the process model.
5. Identify the input variables that must be specified as either disturbance variables or manipulated variables, in order to utilize the N_F degrees of freedom.

In Step 4, the output variables include the dependent variables in the ordinary differential equations. For Step 5, the N_F degrees of freedom are assigned by specifying a total of input variables to be either disturbance variables or manipulated variables.

Example 7.1. Analyze the degrees of freedom for the blending model of Eq. 3 (Lecture 6) for the special condition where volume V is constant.

***Ans*.**: For this example, there are

2 parameters: V, ρ

4 variables ($N_V = 4$): x, x_1, w_1, w_2

1 equation ($N_E = 1$): Eq. 3 (Lecture 6)

The degrees of freedom are calculated as $N_F = 4 - 1 = 3$.

Thus, we must identify three input variables that can be specified as known functions of time in order for the equation to have a unique solution. The dependent variable x is an obvious choice for the output variable in this simple example. Consequently, we have

1 output: x

3 inputs: x_1, w_1, w_2

The three degrees of freedom can be utilized by specifying the inputs as

2 disturbance variables: x_1, w_1

1 manipulated variable: w_2

Because all of the degrees of freedom have been utilized, the single equation is exactly specified and can be solved.

Hint: A linear model cannot contain any nonlinear combinations of variables (e.g., a product of two variables) or any variable raised to a power other than one.

Example 7.1. Analyze the degrees of freedom of the blending system model in Eqs. 14 and 15 (Lecture 6). Is this set of equations linear, or nonlinear, according to the usual definition?

Ans.: In this case, volume is now considered to be a variable rather than a constant parameter. Consequently, for the degrees of freedom analysis, we have

1 parameter: **ρ**
7 variables ($N_V = 7$): $V, x, x_1, x_2, w, w_1, w_2$
2 equations ($N_E = 2$): Eqs. 14 and 15 (Lecture 6)

Thus, $N_F = 7 - 2 = 5$. The dependent variables on the left side of the differential equations, V and x, are the model outputs. The remaining ***five*** variables must be chosen as inputs. Note that a physical output, effluent flow rate w, is classified as a mathematical input, because it can be specified arbitrarily. Any process variable that can be specified arbitrarily should be identified as an input. Thus, we have

2 outputs: V, x
5 inputs: x_1, x_2, w, w_1, w_2

Because the two outputs are the only variables to be determined in solving the system of two equations, no degrees of freedom are left. The system of equations is exactly specified and hence solvable. To utilize the degrees of freedom, the five inputs are classified as either disturbance variables or manipulated variables. A reasonable classification is

3 disturbance variables: w_1, x_1, x_2
2 manipulated variables: w, w_2
For example, w could be used to control V and w_2 to control x. Note that Eq. 14 (Lecture 6) is a linear ODE, while Eq. 15 (Lecture 6) is a nonlinear ODE as a result of the products and quotients.

For the simple process discussed so far, the stirred-tank blending system, energy effects were not considered due to the assumed isothermal operation. Next, we illustrate how dynamic models can be developed for processes where energy balances are important.

Model 1: Stirred-Tank Heating Process: Constant Holdup

Consider the stirred-tank heating system shown in Fig. 1. The liquid inlet stream consists of a single component with a mass flow rate w_i and an inlet temperature T_i. The tank contents are agitated and heated using an electrical heater that provides a heating rate, Q.

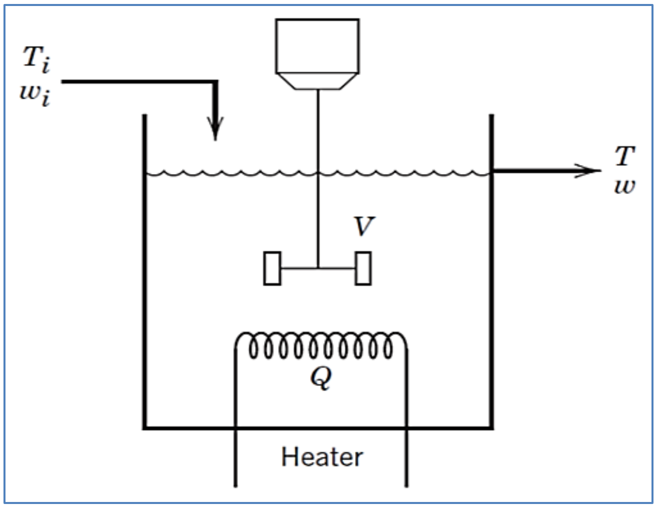


Fig. 1. Stirred-tank heating process with constant holdup, ***V*.**

A dynamic model will be developed based on the following assumptions:

1. Perfect mixing; thus, the exit temperature T is same as the temperature of the tank contents.
2. The inlet and outlet flow rates are equal; thus, $w_i = w$ and the liquid holdup V is constant.
3. The density ρ and heat capacity C of the liquid are assumed to be constant. Thus, their temperature dependence is neglected.
4. Heat losses are negligible.

H.W. 7.1. *Explain how could the assumption No.4 be applied?*

In general, dynamic models are based on conservation laws. For this example, it is clear that we should consider an energy balance, because thermal effects predominate. A mass balance is not required in view of Assumptions 2 and 3.

Next, we show how the general energy balance in Eq. 10 (Lecture 6) can be simplified for this particular example. For a pure liquid at low or moderate pressures, the internal energy is approximately equal to the enthalpy, $U_{\mathrm{int}} \approx H$, and H depends only on temperature. Consequently, in the subsequent development, we assume that $U_{int} = H$ and $\hat{U}_{int} = \hat{H}$ where the caret $(\hat{\ })$ means per unit mass. As known in thermodynamic, a differential change in temperature, dT, produces a corresponding change in the internal energy per unit mass, $d\hat{U}_{int}$,

$$d\hat{U}_{int} = d\hat{H} = C\,dT \qquad (2)$$

where C is the constant pressure heat capacity (assumed to be constant). The total internal energy of the liquid in the tank can be expressed as the product of $\hat{U}_{int}$ and the mass in the tank, ρV:

$$U_{int} = \rho\, V\, \hat{U}_{int} \qquad (3)$$

An expression for the rate of internal energy accumulation can be derived from Eqs. 2 and 3:

$$\frac{dU_{int}}{dt} = \rho V C \frac{dT}{dt} \qquad (4)$$

Note that this term appears in the general energy balance of Eq. 10 (Lecture 6). Next, we derive an expression for the enthalpy term that appears on the right-hand side of Eq. 10 (Lecture 6) . Suppose that the liquid in the tank is at a temperature T and has an enthalpy, $\hat{H}$. Integrating Eq. 2 from a reference temperature T_{ref} to T gives

$$\hat{H} - \hat{H}_{ref} = C(T - T_{ref}) \qquad (5)$$

where $\hat{H}_{ref}$ is the value of $\hat{H}$ at T_{ref}. Without loss of generality, we assume that $\hat{H}_{ref} = 0$. Thus, Eq. 5 can be written as

$$\hat{H} = C(T - T_{ref}) \qquad (6)$$

Similarly, for the inlet stream,

$$\hat{H}_i = C(T_i - T_{ref}) \qquad (7)$$

Substituting Eq. 6 and Eq. 7 into the convection term of Eq. 10 (Lecture 6) gives

$$-\Delta(w\hat{H}) = w[C(T_i - T_{ref})] - w[C(T - T_{ref})] \qquad (8)$$

Finally, substitution of Eq. 4 and Eq. 8 into Eq. 10 (Lecture 6) gives the desired dynamic model of the stirred-tank heating system:

$$\rho V C \frac{dT}{dt} = wC(T_i - T) + Q \qquad (9)$$

Note that the T_{ref} terms have canceled, because C was assumed to be constant, and thus independent of temperature. A degrees of freedom analysis for this model gives

3 parameters: V, ρ, C

4 variables: T, T_i, w, Q

1 equation: Eq. 9

Thus, the degrees of freedom are $N_F = 4 - 1 = 3$. The process variables are classified as

1 output variable: T

3 input variables: T_i, w, Q

For control purposes, it is reasonable to classify the three inputs as

2 disturbance variables: T_i, w

1 manipulated variable: Q

Model 2: Stirred-Tank Heating Process: Variable Holdup

Now we consider the more general situation in which the tank holdup can vary with time. This analysis also is based on Assumptions 1, 3, and 4 of the previous case. Now an overall mass balance is needed, because the holdup is not constant. The overall mass balance is

$$\frac{d(V\rho)}{dt} = w_i - w \qquad (10)$$

The energy balance for the current stirred-tank heating system can be derived from Eq. 10 (Lecture 6) in analogy with the derivation of Eq. 9. We again assume that $U_{int} = H$ for the liquid in the tank. Thus, for constant ρ

$$\frac{dU_{int}}{dt} = \frac{dH}{dt} = \frac{d(\rho V\hat{H})}{dt} = \rho\frac{d(V\hat{H})}{dt} \qquad (11)$$

From the definition of $-\Delta(w\hat{H})$ and Eqs. 6 and 7, it follows that

$$-\Delta(w\hat{H}) = w_i\hat{H}_i - w\hat{H} = w_i C(T_i - T_{ref}) - w\,C(T - T_{ref}) \qquad (12)$$

where $\boldsymbol{w_i}$ and $\boldsymbol{w}$ are the mass flow rates of the inlet and outlet streams, respectively. Substituting Eq. 11 and Eq. 12 into the general equation of energy balance which gives

$$\rho\frac{d(V\hat{H})}{dt} = w_i\, C\left(T_i\, -\, T_{ref}\right) - w\, C\left(T\, -\, T_{ref}\right) + Q \qquad (\mathbf{13})$$

Next we simplify the dynamic model. Because $\boldsymbol{\rho}$ is constant, Eq. 10 can be written as

$$\rho\frac{dV}{dt} = w_i - w \qquad (14)$$

The chain rule can be applied to expand the left side of Eq. 13 for constant $\boldsymbol{C}$ and $\boldsymbol{\rho}$:

$$\rho\frac{d(V\hat{H})}{dt} = \rho V\frac{d(\hat{H})}{dt} + \rho\hat{H}\frac{dV}{dt} \qquad (\mathbf{15})$$

After additional steps, Eq. 13 will be as follows:

$$C\left(T\, -\, T_{ref}\right)(w_i - w) + \rho C V\frac{dT}{dt} = w_i C\left(T_i\, -\, T_{ref}\right) - wC\left(T\, -\, T_{ref}\right) + Q \qquad (\mathbf{16})$$

So, the variation of holdup (***V***) and temperature (***T***) relations could be obtained depending on Eqs. 14 and 16 as follows:

$$\frac{dV}{dt} = \frac{1}{\rho}(w_i - w) \qquad (17)$$

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i\, -\, T) + \frac{Q}{\rho C V} \qquad (18)$$

H.W. 7.2. *List the additional steps that had been used to obtained the final form of Eq. 16 which designed for stirred-tank heating process in the case of variable holdup?*

Equations 17 and 18 provide a model that can be solved for the two outputs (***V*** and ***T***) if the two parameters (***ρ*** and ***C***) are known and the four inputs ($\boldsymbol{w_i}$, ***w***, $\boldsymbol{T_i}$, and ***Q***) are known functions of time.

H.W. 7.3. *Find the degree of freedom for the case of the stirred-tank heating process in the case of variable holdup?*

Processes Control:

Degrees of freedom analysis

Model 3: **Liquid Storage Systems**

A typical liquid storage process is shown in Fig. 1 where $\boldsymbol{q_i}$ and $\boldsymbol{q}$ are volumetric flow rates.

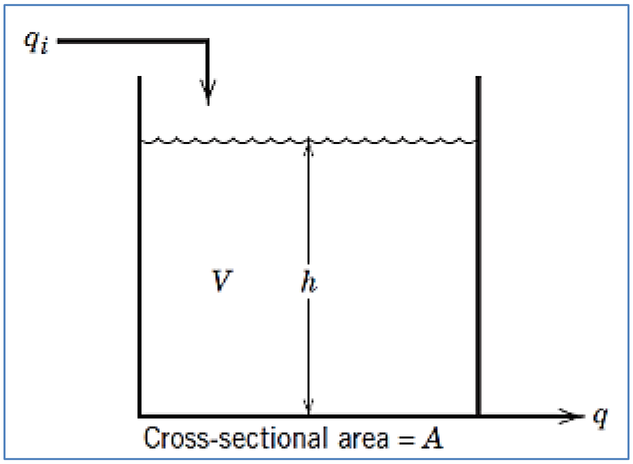


Fig. 1. A liquid-level storage process**.**

A mass balance yields

$$\frac{d(\rho V)}{dt} = \rho q_i - \rho q \qquad (1)$$

Assume that liquid density $\boldsymbol{\rho}$ is constant and the tank is cylindrical with cross-sectional area, $\boldsymbol{A}$. Then the volume of liquid in the tank can be expressed as $\boldsymbol{V = Ah}$, where $\boldsymbol{h}$ is the liquid level (or head). Thus, Eq. 1 becomes

$$A\frac{dh}{dt} = q_i - q \qquad (2)$$

Note that Eq. 2 appears to be a volume balance. However, in general, volume is not conserved for fluids. This result occurs in this example due to the constant density assumption. We refer to volume as a “state” of the system, which is a dependent variable that correlates with a fundamentally conserved quantity (in this case, mass). Similarly, energy balances often yield differential equations for temperature, which is a state variable that correlates with energy. There are three important variations of the liquid storage process:

1. The inlet or outlet flow rates might be constant; for example, exit flow rate q might be kept constant by a constant-speed, fixed-volume (metering) pump. An important consequence of this configuration is that the exit flow rate is then completely independent of liquid level over a wide range of conditions. Consequently, $\boldsymbol{q = \bar{q}}$ where $\boldsymbol{\bar{q}}$ is the steady-state value. For this situation, the tank operates essentially as a flow *integrator*.

2. The tank exit line may function simply as a resistance to flow from the tank (distributed along the entire line), or it may contain a valve that provides significant resistance to flow at a single point. In the simplest case, the flow may be assumed to be linearly related to the driving force, the liquid level, in analogy to Ohm's law for electrical circuits ($\boldsymbol{E = IR}$)

$$\boldsymbol{h = q\, R_v} \qquad (3)$$

where $\boldsymbol{R_v}$ is the resistance of the line or valve. Rearranging Eq. 3 gives the following *flow-head equation*:

$$\boldsymbol{q = \frac{1}{R_v} h} \qquad (4)$$

Substituting Eq. 4 into Eq. 2 gives a first-order differential equation:

$$\boldsymbol{A\frac{dh}{dt} = q_i - \frac{1}{R_v} h} \qquad (5)$$

This model of the liquid storage system exhibits dynamic behavior similar to that of the stirred-tank heating system.

3. A more realistic expression for flow rate $\boldsymbol{q}$ can be obtained when a fixed valve has been placed in the exit line and turbulent flow can be assumed. The driving force for flow through the valve is the pressure drop $\boldsymbol{\Delta P}$:

$$\Delta \boldsymbol{P} = \boldsymbol{P} - \boldsymbol{P_a} \qquad (6)$$

where $\boldsymbol{P}$ is the pressure at the bottom of the tank and $\boldsymbol{P_a}$ is the pressure at the end of the exit line. We assume that $\boldsymbol{P_a}$ is the ambient pressure. If the valve is considered to be an orifice, a mechanical energy balance, or *Bernoulli equation*, can be used to derive the relation

$$\boldsymbol{q} = \boldsymbol{C_v^*}\sqrt{\frac{\boldsymbol{P} - \boldsymbol{P_a}}{\boldsymbol{\rho}}} \qquad (7)$$

where $\boldsymbol{C_v^*}$ is a constant. The value of $\boldsymbol{C_v^*}$ depends on the particular valve and the valve setting (how much it is open). The pressure $\boldsymbol{P}$ at the bottom of the tank is related to liquid level $\boldsymbol{h}$ by a force balance

$$\boldsymbol{P} = \boldsymbol{P_a} + \frac{\boldsymbol{\rho g}}{\boldsymbol{g_c}}\boldsymbol{h} \qquad (8)$$

where the acceleration of gravity g is constant. Substituting Eqs. 7 and 8 into Eq. 2 yields the dynamic model

$$\boldsymbol{A}\frac{\boldsymbol{dh}}{\boldsymbol{dt}} = \boldsymbol{q_i} - \boldsymbol{C_v}\sqrt{\boldsymbol{h}} \qquad (9)$$

Where $\boldsymbol{C_v} \triangleq \boldsymbol{C_v^*}\sqrt{\boldsymbol{g}/\boldsymbol{g_c}}$. This model is nonlinear due to the square root term.

The liquid storage processes discussed above could be operated by controlling the liquid level in the tank or by allowing the level to fluctuate without attempting to control it. For the latter case (operation as a surge tank), it may be of interest to predict whether the tank would overflow or run dry for particular variations in the inlet and outlet flow rates. Thus, the dynamics of the process may be important even when automatic control is not utilized.

Model 4: The Continuous Stirred-Tank Reactor (CSTR)

Continuous stirred-tank reactors (CSTR) have widespread application in industry and embody many features of other types of reactors. CSTR models tend to be simpler than models for other types of continuous reactors such as tubular reactors and packed-bed reactors. Consequently, a CSTR model provides a convenient way of illustrating modeling principles for chemical reactors. Consider a simple liquid-phase, irreversible chemical reaction where chemical species A reacts to form species B. The reaction can be written as $A \rightarrow B$. We assume that the rate of reaction is first-order with respect to component A,

$$r = k\,C_A \qquad (10)$$

where r is the rate of reaction of A per unit volume, k is the reaction rate constant (with units of reciprocal time), and C_A is the molar concentration of species A. For single-phase reactions, the rate constant is typically a strong function of reaction temperature given by the Arrhenius relation,

$$k = k_0 e^{-E/RT} \qquad (11)$$

where k_0 is the frequency factor, E is the activation energy, and R is the gas constant. The expressions in Eqs. 10 and 11 are based on theoretical considerations, but model parameters k_0 and E are usually determined by fitting experimental data. Thus, these two equations can be considered to be semi-empirical relations, according to the definition in the previous lecture.

The schematic diagram of the CSTR is shown in Fig. 2. The inlet stream consists of pure component A with molar concentration, C_{Ai}. A cooling coil is used to maintain the reaction mixture at the desired operating temperature by removing heat that is released in the exothermic reaction.

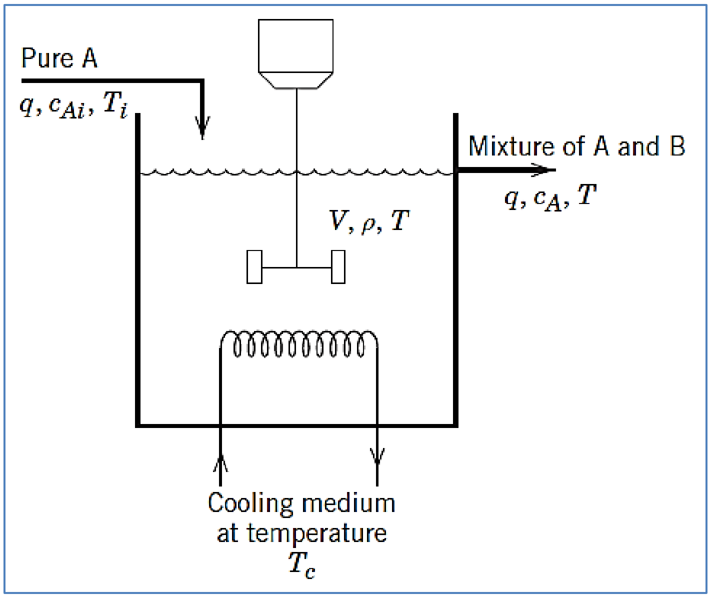


Fig. 2. A nonisothermal continuous stirred-tank reactor**.**

Our initial CSTR model development is based on three assumptions:

1. The CSTR is perfectly mixed.
2. The mass densities of the feed and product streams are equal and constant. They are denoted by ρ.
3. The liquid volume V in the reactor is kept constant by an overflow line.

For these assumptions, the unsteady-state mass balance for the CSTR is

$$\frac{d(\rho V)}{dt} = \rho q_i - \rho q \qquad (12)$$

Because V and ρ are constant, Eq. 12 reduces to

$$q_i = q \qquad (13)$$

Thus, even though the inlet and outlet flow rates may change due to upstream or downstream conditions, Eq. 13 must be satisfied at all times. In Fig. 2, both flow rates are denoted by the symbol q. For the stated assumptions, the unsteady-state component balances for species A (in molar concentration units) is

$$V\frac{dC_A}{dt} = q(C_{Ai} - C_A) - VkC_A \qquad (14)$$

H.W. 8.1. *Derive Eq. 14?*

Next, we consider an unsteady-state energy balance for the CSTR. But first we make five additional assumptions:

4. The thermal capacitances of the coolant and the cooling coil wall are negligible compared to the thermal capacitance of the liquid in the tank.
5. All of the coolant is at a uniform temperature, T_c. (That is, the increase in coolant temperature as the coolant passes through the coil is neglected.)
6. The rate of heat transfer from the reactor contents to the coolant is given by

$$Q = UA(T_c - T) \qquad (15)$$

where U is the overall heat transfer coefficient and A is the heat transfer area. Both of these model parameters are assumed to be constant.
7. The enthalpy change associated with the mixing of the feed and the liquid in the tank is negligible compared with the enthalpy change for the chemical reaction. In other words, the heat of mixing is negligible compared to the heat of reaction.
8. Shaft work and heat losses to the ambient can be neglected.

The following form of the CSTR energy balance is convenient for analysis and can be derived from Eqs. 10 and 11 and Assumptions 1-8,

$$V\rho C\frac{dT}{dt} = wC(T_i - T) + (-\Delta H_R)VkC_A + UA(T_C - T) \qquad (16)$$

where ΔH_R is the heat of reaction per mole of A that is reacted.

In summary, the dynamic model of the CSTR consists of Eqs. 10 to 12, 14, 15, and 16. This model is nonlinear as a result of the many product terms and the exponential temperature dependence of k in Eq. 11. Consequently, it must be solved by numerical integration techniques. The CSTR model will become considerably more complex if

1. More complicated rate expressions are considered. For example, a mass action kinetics model for a second-order, irreversible reaction, $2A \rightarrow B$, is given by

$$r = k_2 C^2_A \qquad (17)$$

2. Additional species or chemical reactions are involved. If the reaction mechanism involved production of an intermediate species, $2A \rightarrow B^* \rightarrow B$, then unsteady-state component balances for both A and B^* would be necessary (to calculate C_A and C^*_B), or balances for both A and B could be written (to calculate C_A and C_B). Information concerning the reaction mechanisms would also be required.

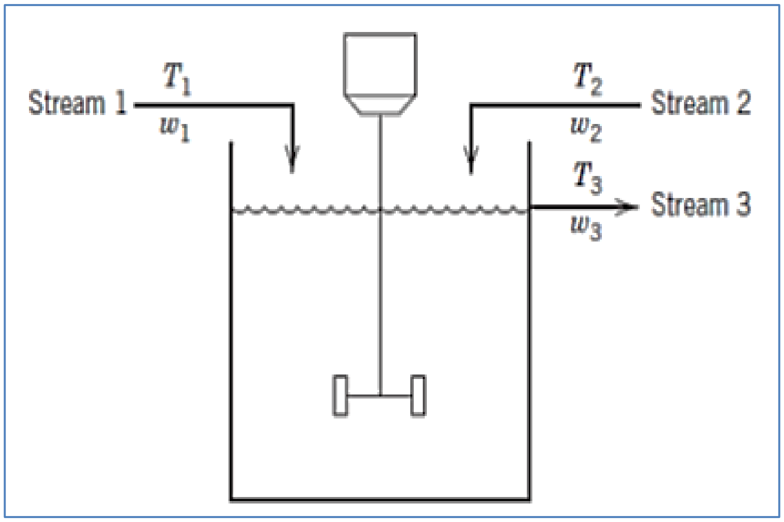
Reactions involving multiple species are described by high-order, highly coupled, nonlinear reaction models, because several component balances must be written.

H.W. 8.2. *Find the degree of freedom for the case of the CSTR?*

Processes Control:

Examples:

Example 10.1. A perfectly stirred, constant-volume tank has two input streams, both consisting of the same liquid. The temperature and flow rate (w_i) of each of the streams can vary with time, but liquid properties are constant (When they could be varied?).



(a) Derive a dynamic model that will describe transient operation. Make a degrees of freedom analysis assuming that both Streams 1 and 2 come from upstream units (i.e., their flow rates and temperatures are known functions of time).

(b) Simplify your model, if possible, to one or more differential equations by eliminating any algebraic equations. Also, simplify any derivatives of products of variables.

Ans.:

a) Overall mass balance:

$$\frac{d(\rho V)}{dt} = w_1 + w_2 - w_3 \tag{1}$$

Energy balance:

$$C\frac{d\left[\rho V(T_3 - T_{ref})\right]}{dt} = w_1 C(T_1 - T_{ref}) + w_2 C(T_2 - T_{ref}) - w_3 C\left(T_3 - T_{ref}\right) \tag{2}$$

Because ρ = constant and $V = \overline{V}$ = constant, Eq. 1 becomes:

$$w_3 = w_1 + w_2 \tag{3}$$

b) From Eq. 2, substituting Eq. 3

$$\rho C\overline{V}\frac{d(T_3 - T_{ref})}{dt} = \rho C\overline{V}\frac{dT_3}{dt} = w_1 C(T_1 - T_{ref}) + w_2 C(T_2 - T_{ref}) - (w_1 + w_2)C(T_3 - T_{ref}) \tag{4}$$

Constants C and T_{ref} can be cancelled:

$$\rho\overline{V}\frac{dT_3}{dt} = w_1 T_1 + w_2 T_2 - (w_1 + w_2)T_3 \tag{5}$$

The simplified model now consists only of Eq. 5.

<u>Degrees of freedom for the simplified model:</u>

Parameters : ρ, $\overline{V}$

Variables : w_1, w_2, T_1, T_2, T_3
$N_E = 1$
$N_V = 5$

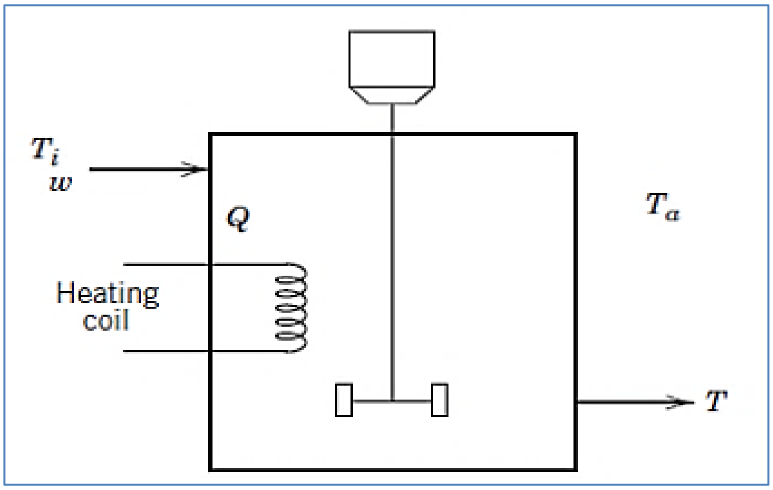
Thus, $N_F = 5 - 1 = 4$

Because w_1, w_2, T_1 and T_2 are determined by upstream units, we assume they are known functions of time:

$w_1 = w_1(t)$
$w_2 = w_2(t)$
$T_1 = T_1(t)$
$T_2 = T_2(t)$

Thus, N_F is reduced to 0.

Example 10.2. A completely enclosed stirred-tank heating process is used to heat an incoming stream whose flow rate varies. The heating rate from this coil and the volume are both constant. Assume that liquid properties (ρ, C_p), and U are constants, A_s is the surface area for heat losses to ambient and $T_i > T_a$ (inlet temperature is higher than ambient temperature). Develop a mathematical model (differential and algebraic equations) that describes the exit temperature if heat losses to the ambient occur and if the ambient temperature (T_a) and the incoming stream's temperature (T_i) both can vary.



Ans.:

Energy balance:

$$C_p \frac{d\left[\rho V(T - T_{ref})\right]}{dt} = wC_p(T_i - T_{ref}) - wC_p(T - T_{ref}) - UA_s(T - T_a) + Q$$

Simplifying

$$\rho V C_p \frac{dT}{dt} = wC_p T_i - wC_p T - UA_s(T - T_a) + Q$$

$$\rho V C_p \frac{dT}{dt} = wC_p(T_i - T) - UA_s(T - T_a) + Q$$

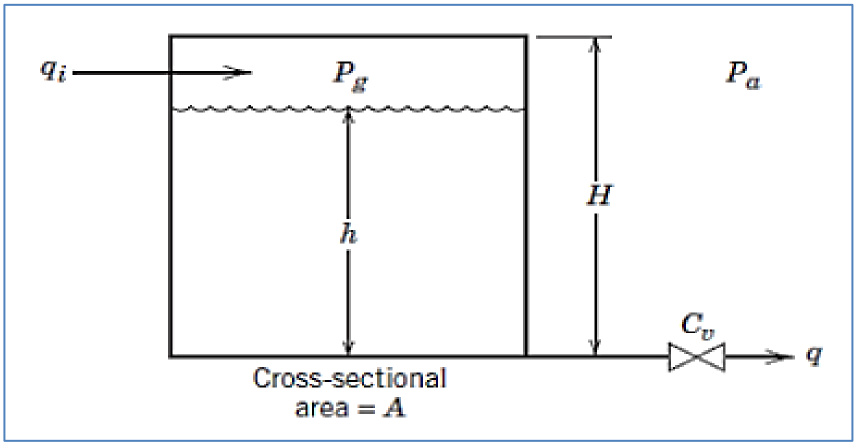
H.W. 10.1. *Discuss qualitatively what you expect to happen as* $\boldsymbol{T_i}$ *and* ***w*** *increase (or decrease) and the relation between heat supplied to the liquid and that losses to environment.*

Example 10.3. Consider a liquid flow system consisting of a sealed tank with non-condensable gas above the liquid as shown in following figure. Derive an unsteady-state model relating the liquid level $\boldsymbol{h}$ to the input flow rate $\boldsymbol{q_i}$. Is operation of this system independent of the ambient pressure $\boldsymbol{P_a}$? What about for a system open to the atmosphere? You may make the following assumptions:

(i) The gas obeys the ideal gas law. A constant amount of mg/M moles of gas is present in the tank.

(ii) The operation is isothermal.

(iii) A square root relation holds for flow through the valve.



Ans.:

Assume constant liquid density, ρ . The mass balance for the tank is

$$\frac{d(\rho A h + m_g)}{dt} = \rho(q_i - q)$$

Because ρ, A, and m_g are constant, this equation becomes

$$A\frac{dh}{dt} = q_i - q \qquad (1)$$

The square-root relationship for flow through the control valve is

$$q = C_v \left(P_g + \frac{\rho g h}{g_c} - P_a \right)^{1/2} \tag{2}$$

From the ideal gas law,

$$P_g = \frac{(m_g / M)RT}{A(H-h)} \tag{3}$$

where T is the absolute temperature of the gas.

Equation 1 gives the unsteady-state model upon substitution of q from Eq. 2 and of P_g from Eq. 3:

$$A\frac{dh}{dt} = q_i - C_v \left[\frac{(m_g / M)RT}{A(H-h)} + \frac{\rho g h}{g_c} - P_a \right]^{1/2} \tag{4}$$

Because the model contains P_a, operation of the system is not independent of P_a. For an open system $P_g = P_a$ and Eq. 2 shows that the system is independent of P_a.

Example 10.3. A closed stirred-tank reactor with two compartments as shown in the following figure. The basic idea is to feed the reactants continuously into the first compartment, where they will be preheated by energy liberated in the exothermic reaction, which is anticipated to occur primarily in the second compartment. The wall separating the two compartments is quite thin, thus allowing heat transfer; the outside of the reactor is well insulated; and a cooling coil is built into the second compartment to remove excess energy liberated in the reaction. Both tanks are completely full at all times. Tests are to be conducted initially with a single-component feed (i.e., no reaction) to evaluate the reactor's thermal characteristics.

(a) Develop a dynamic model for this process under the conditions of no reaction. Assume that q_0, T_i, and T_c all may vary.

(b) Make a degrees of freedom analysis for your model-identifying all parameters, outputs, and inputs that must be known functions of time in order to obtain a solution.

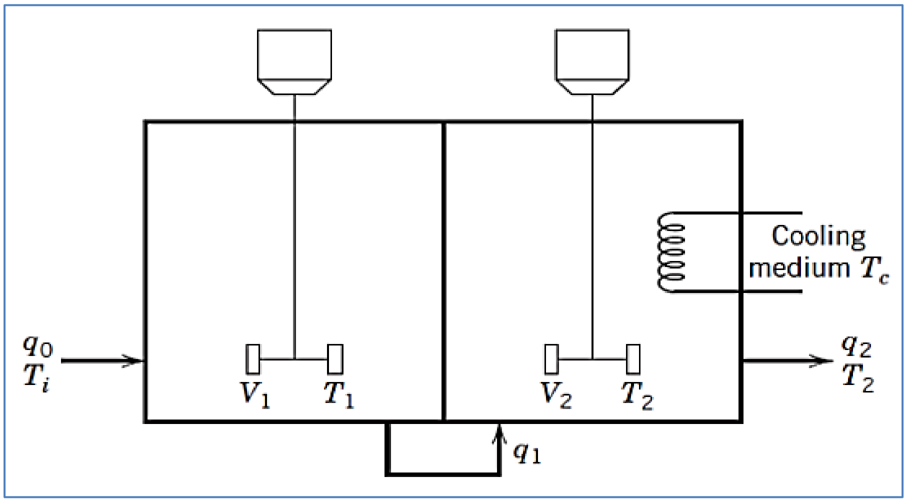
Use the following symbols:

U_t, A_t : Overall heat transfer coefficient and surface area between compartments.

U_c, A_c : Overall heat transfer coefficient and surface area of cooling tube.

V_1: Volume of Compartment 1.

V_2: Volume of Compartment 2.



Ans.:

a) Assumptions:

1. Each compartment is perfectly mixed.
2. ρ and C are constant.
3. No heat losses to ambient.

Compartment 1:

Overall balance (No accumulation of mass):

$$0 = \rho q - \rho q_1 \quad \text{thus} \quad q_1 = q \tag{1}$$

Energy balance (No change in volume):

$$V_1 \rho C \frac{dT_1}{dt} = \rho q C (T_i - T_1) - UA(T_1 - T_2) \tag{2}$$

Compartment 2:

Overall balance:

$$0 = \rho q_1 - \rho q_2 \quad \text{thus} \quad q_2 = q_1 = q \tag{3}$$

Energy balance:

$$V_2 \rho C \frac{dT_2}{dt} = \rho q C (T_1 - T_2) + UA(T_1 - T_2) - U_c A_c (T_2 - T_c) \tag{4}$$

b) Eight parameters: $\rho, V_1, V_2, C, U, A, U_c, A_c$

Five variables: T_i, T_1, T_2, q, T_c

Two equations: (2) and (4)

Thus $N_F = 5 - 2 = 3$

2 outputs = T_1, T_2

3 inputs = T_i, T_c, q (specify as functions of t)

Processes Control:

Laplace transforms of representative functions:

In previous lectures, we developed a number of mathematical models that describe the dynamic operation of selected processes. Solving such models-that is, finding the output variables as functions of time for some change in the input variable(s) requires either analytical or numerical integration of the differential equations. Sometimes considerable effort is involved in obtaining the solutions. One important class of models includes systems described by linear ordinary differential equations (ODEs). Such *linear systems* represent the starting point for many analysis techniques in process control.

Now, we introduce a mathematical tool, the ***Laplace transform***, which can significantly reduce the effort required to solve and analyze linear differential equation models. A major benefit is that this transformation converts ordinary differential equations to algebraic equations, which can simplify the mathematical manipulations required to obtain a solution or perform an analysis. The Laplace transform of a function f(t) is defined as:

$$\boldsymbol{F(s) = L[f(t)] = \int_{0}^{\infty} f(t)e^{-st}dt} \qquad (1)$$

where ***F(s)*** is the symbol for the Laplace transform, ***s*** is a complex independent variable, ***f(t)*** is some function of time to be transformed, and ***L*** is an operator, defined by the integral. The function ***f(t)*** must satisfy mild conditions that include being piecewise continuous for $0 < t < \infty$; this requirement almost always holds for functions that are useful in process modeling and control. When the integration is performed, the transform becomes a function of the Laplace transform variable ***s***.

The inverse Laplace transform (L^{-1}) operates on the function ***F(s)*** and converts it to ***f(t)***. Notice that ***F(s)*** contains no information about ***f(t)*** for $t < 0$. Hence, f(t) = $L^{-1}\{F(s)\}$ is not defined for t < 0.

One of the important properties of the Laplace transform and the inverse Laplace transform is that they are linear operators; a linear operator satisfies the Principle of Superposition:

$$L[ax(t) + by(t)] = aX(s) + bY(s) \qquad (2)$$

Derivatives. The transform of a first derivative of f is important because such derivatives appear in dynamic models:

$$L(df/dt) = \int_0^{\infty} (df/dt)e^{-st}dt = sL\big(f(t)\big) - f(0) = sF(s) - f(0) \qquad (3)$$

In many process modeling applications, functions are defined so that they are *zero* at *initial time*-that is, $f(0) = 0$. In these cases, Eq. 3 simplifies to $L(df/dt) = sF(s)$. The Laplace transform for higher-order derivatives can be found by repeated application of Eq. 3:

$$L\left(\frac{d^nf}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \qquad (4)$$

where $f^{(i)}(0)$ is the ith derivative evaluated at $t=0$.

The Laplace transform can be derived easily for most simple functions, as shown below.

Constant Function. For $f(t) = a$ (a constant),

$$F(s) = L(a) = \int_0^{\infty} a\, e^{-st}dt = -\frac{a}{s}e^{-st}\Big|_0^{\infty} = 0 - \left(-\frac{a}{s}\right) = \frac{a}{s} \qquad (5)$$

Step Function. The unit step function, defined as is an important input that is used frequently in process dynamics and control.

$$P(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \qquad (6)$$

The Laplace transform of the unit step function is the same as that obtained for the constant above when ***a = 1***:

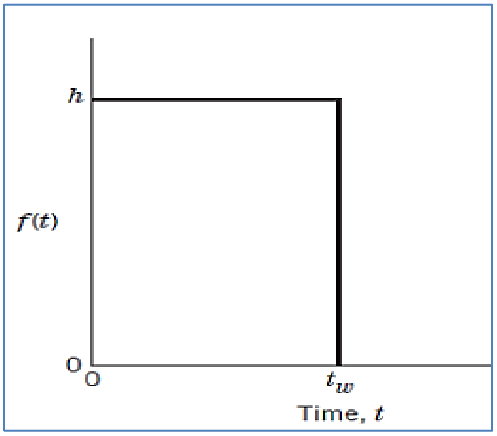
$$L[P(t)] = \frac{1}{s} \qquad (7)$$

The step function incorporates the idea of ***initial time***, ***zero time***, or ***time zero*** for the function, which refers to the time at which ***P(t)*** changes from **0** to **1**.

Exponential Functions. The Laplace transform of an exponential function is important because exponential functions appear in the solution to most linear differential equations. For an exponential, e^{-bt}, with $b > 0$,

$$L(e^{-bt}) = \int_0^\infty e^{-bt}\, e^{-st} dt = \int_0^\infty e^{-(b+s)t} dt = \frac{1}{b+s}\left[-e^{-(b+s)t}\right]\Big|_0^\infty = \frac{1}{s+b} \qquad (8)$$

The Rectangular Pulse Function. An illustration of the rectangular pulse is shown in Fig. 1. The pulse has height ***h*** and width t_w. This type of signal might be used to depict the opening and closing of a valve regulating flow into a tank. The flow rate would be held at ***h*** for a duration of t_w units of time.



The area under the curve in Fig. 1 could be interpreted as the amount of material delivered to the tank (= $h.t_w$). Mathematically, the function ***f(t)*** is defined as

$$f(t) = \begin{cases} 0 & t < 0 \\ h & 0 \le t < t_w \\ 0 & t \ge t_w \end{cases} \qquad (9)$$

$$\boldsymbol{F(s)} = \int_{0}^{\infty} \boldsymbol{f(t)\, e^{-st} dt} = \int_{0}^{t_w} \boldsymbol{h\, e^{-st} dt} = -\frac{\boldsymbol{h}}{\boldsymbol{s}} \boldsymbol{e^{-st}} \bigg|_{0}^{t_w} = \frac{\boldsymbol{h}}{\boldsymbol{s}} (\boldsymbol{1} - \boldsymbol{e^{-t_w s}}) \qquad (10)$$

For a unit rectangular pulse, $\boldsymbol{h = 1/t_w}$ and the area under the pulse is unity.

Impulse Function. A limiting case of the unit rectangular pulse is the *impulse* or *Dirac delta* function, which has the symbol $\boldsymbol{\delta(t)}$. This function is obtained when $\boldsymbol{t_w \rightarrow 0}$ while keeping the area under the pulse equal to ***unity***. A pulse of infinite height and infinitesimal width results. Mathematically, this can be accomplished by substituting $\boldsymbol{h = 1/t_w}$ into Eq. 10; the Laplace transform of $\boldsymbol{\delta(t)}$ is

$$\boldsymbol{L}[\boldsymbol{\delta(t)}] = \mathbf{lim}_{t_w \rightarrow \infty} \frac{\boldsymbol{1}}{\boldsymbol{t_w s}} (\boldsymbol{1} - \boldsymbol{e^{-t_w s}}) = \boldsymbol{1} \qquad (11)$$

If the impulse magnitude (i.e., area $\boldsymbol{t_w h}$) is a constant $\boldsymbol{a}$ rather than unity, then

$$\boldsymbol{L}[\boldsymbol{\delta(t)}] = \boldsymbol{a}$$

A physical example of an impulse function is the rapid injection of dye or tracer into a fluid stream, where ***f(t)*** corresponds to the concentration or the flow rate of the tracer. This type of signal is sometimes used in process testing, for example, to obtain the residence time distribution of a piece of equipment.

Table 1 lists important functions of time, including derivatives, and their Laplace transform equivalents. Because the Laplace transform converts any function f(t) to F(s) and the inverse Laplace transform converts F(s) back to f(t), the table provides an organized way to carry out these transformations.

Table 1 Laplace Transforms for Various Time-Domain Functions (for t ≥ 0 only)

$f(t)$	$F(s)$
1. $\delta(t)$ (unit impulse)	1
2. $S(t)$ (unit step)	$\frac{1}{s}$
3. t (ramp)	$\frac{1}{s^2}$
4. t^{n-1}	$\frac{(n-1)!}{s^n}$
5. e^{-bt}	$\frac{1}{s+b}$
6. $\frac{1}{\tau}e^{-t/\tau}$	$\frac{1}{\tau s+1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n>0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$	$\frac{1}{(\tau s+1)^n}$
9. $\frac{1}{b_1-b_2}(e^{-b_2 t}-e^{-b_1 t})$	$\frac{1}{(s+b_1)(s+b_2)}$
10. $\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11. $\frac{b_3-b_1}{b_2-b_1}e^{-b_1 t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2 t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$	$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13. $1-e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

15. $\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
16. $\sin(\omega t+\phi)$	$\dfrac{\omega \cos \phi + s \sin \phi}{s^2+\omega^2}$
17. $e^{-bt} \sin \omega t$ } b, ω real	$\dfrac{\omega}{(s+b)^2+\omega^2}$
18. $e^{-bt} \sin \omega t$	$\dfrac{s+b}{(s+b)^2+\omega^2}$
19. $\dfrac{1}{\tau\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1-\zeta^2}\, t/\tau)$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20. $1+\dfrac{1}{\tau_2-\tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \ne \tau_2)$	$\dfrac{1}{s(\tau_1 s+1)(\tau_2 s+1)}$
21. $1-\dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1-e^{-\zeta t/\tau}\left[\cos(\sqrt{1-\zeta^2}\, t/\tau) + \dfrac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)\right]$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1+\dfrac{\tau_3-\tau_1}{\tau_1-\tau_2} e^{-t/\tau_1} + \dfrac{\tau_3-\tau_2}{\tau_2-\tau_1} e^{-t/\tau_2}$ $(\tau_1 \ne \tau_2)$	$\dfrac{\tau_3 s+1}{s(\tau_1 s+1)(\tau_2 s+1)}$
24. $\dfrac{df}{dt}$	$sF(s)-f(0)$
25. $\dfrac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t-\theta)S(t-\theta)$	$e^{-\theta s} F(s)$
27. $\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{1}{s} F(s)$

Solution of differential equations by Laplace transform techniques

The procedure used to solve a differential equation is quite simple. First, Laplace transform both sides of the differential equation, substituting values for the initial conditions in the derivative transforms. Rearrange the resulting algebraic equation, and solve for the transform of the dependent (output) variable. Finally, find the inverse of the transformed output variable. The solution method is illustrated by means of several examples.

Example 11.1. Solve the differential equation $5\frac{dy}{dt} + 4y = 2 \qquad y(0) = 1$, using Laplace transform.

Ans.: First, take the Laplace transform of both sides of this equation

$$L(5\frac{dy}{dt} + 4y) = L(2)$$

Using the Principle of Superposition, each term can be transformed individually:

$$L(5\frac{dy}{dt}) + L(4y) = L(2)$$

$$L\left(5\frac{dy}{dt}\right) = 5L\left(\frac{dy}{dt}\right) = 5(sY(s) - 1) = 5sY(s) - 5$$

$$L(4y) = 4L(y) = 4Y(s)$$

$$L(2) = \frac{2}{s}$$

Substitute the individual terms:

$$5sY(s) - 5 + 4Y(s) = \frac{2}{s}$$

Rearrange this equation and factor out Y(s):

$$Y(s)(5s + 4) = 5 + \frac{2}{s}$$

Or

$$Y(s) = \frac{(5s + 2)}{s(5s + 4)}$$

Take the inverse Laplace transform of both sides of this equation:

$$L^{-1}Y(s) = L^{-1}[\frac{(5s + 2)}{s(5s + 4)}]$$

The inverse Laplace transform of the right side of this equation can be found by using Table 1. First, divide the numerator and denominator by **5** to put all factors in the ***s + b*** form corresponding to the table entries:

$$y(t) = L^{-1}[\frac{(s+0.4)}{s(s+0.8)}]$$

Because entry 11 in the table, *(s+*b_3*)/[(s+*b_1*)(s+*b_2*)]*, matches this equation with b_1=0.8, b_2 = 0, and b_3 = 0.4, the solution can be written immediately:

$$y(t) = 0.5 + 0.5e^{-0.8\,t}$$

Example 11.2. Solve the differential equation $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1$,using Laplace transform with initial conditions y(0) = $y^{/}$0) = $y^{//}$(0) =0 where the primes denote derivatives.

Ans.: Take Laplace transforms, term by term, using Table 1:

$$L(\frac{d^3y}{dt^3}) = s^3Y(s)$$

$$L\left(6\frac{d^2y}{dt^2}\right) = 6\,s^2Y(s)$$

$$L\left(11\frac{dy}{dt}\right) = 11\,s\,Y(s)$$

$$L(6y) = 6\,Y(s)$$

$$L(1) = \frac{1}{s}$$

Rearranging and factoring Y(s), we obtain

$$Y(s)(s^3 + 6\,s^2 + 11\,s + 6) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^3 + 6\,s^2 + 11\,s + 6)}$$

This problem always arises for higher-order differential equations, because the order of the denominator polynomial (*characteristic polynomial*) of the transform is equal to the order of the original differential equation, and no table entries are higher than third order in the denominator. Therefore, a ***partial fraction expansion*** (***PFE***) is used based on the *characteristic polynomial*. The factors of the characteristic polynomial correspond to the roots of the characteristic polynomial set equal to zero. The input factors may be quite simple. Once the factors are obtained, the Laplace transform is then expanded into partial fractions. As an example, consider

$$Y(s) = \frac{s+5}{s^2+5s+4}$$

The denominator can be factored into a product of first-order terms, (s + 1)(s + 4). This transform can be expanded into the sum of two partial fractions:

$$\frac{s+5}{(s+1)(s+4)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4}$$

where α_1 and α_2 are unspecified coefficients that must satisfy this equation. Now, multiply both sides of the equation by one of the denominator terms (***s* + *b*$_i$**) and then set ***s*=-*b*$_i$**, which causes all terms except one to be multiplied by zero. Multiplying this equation by ***s* + *1*** and then letting ***s* = *-1*** gives

$$\alpha_1 = \left.\frac{s+5}{s+4}\right|_{\boldsymbol{s=-1}} = \frac{\mathbf{4}}{\mathbf{3}}$$

Similarly, after multiplying by (s + 4) and letting s = - 4, the expansion gives

$$\alpha_2 = \left.\frac{s+5}{s+1}\right|_{\boldsymbol{s=-4}} = -\frac{\mathbf{1}}{\mathbf{3}}$$

After substituting the values of where α_1 and α_2 in the equation,

$$\frac{s+5}{(s+1)(s+4)} = \frac{4/3}{s+1} + \frac{-1/3}{s+4}$$

Note that several entries in Table 1 have the ***τs* + *1*** format. Now, we will complete Example 11.2 using this technique as follows:

$$Y(s) = \frac{1}{s(s^3 + 6\,s^2 + 11\,s + 6)}$$

The characteristic polynomial set equals $[s(s^3 + 6\,s^2 + 11\,s + 6)]$

$$s(s^3 + 6\,s^2 + 11\,s + 6) = s(s+1)(s+2)(s+3)$$

$$Y(s) = \frac{1}{s(s+1)(s+2)(s+3)}$$

$$Y(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+2} + \frac{\alpha_4}{s+3}$$

After solution, α_1 = 1/6, α_2 = −1/2, α_3 = 1/2, α_4 = −1/6.

$$y(t) = L^{-1}\left(\frac{1/6}{s} + \frac{-1/2}{s+1} + \frac{1/2}{s+2} + \frac{-1/6}{s+3}\right)$$

$$y(t) = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

H.W. 11.1*. Find the Laplace transforms of the following functions,*

(a) *f*(*t*) = 5 + *e*−3*t* + *te*−4*t*

(b) *f*(*t*) = sin(4*t*) + *t* − 3 + *e*−(*t*−3) + 5/*t*

H.W. 11.2*. Using* ***PFE*** *where required, find x(t) for,*

$$Y(s) = \frac{s(s+1)}{(s+2)(s+3)(s+4)}$$

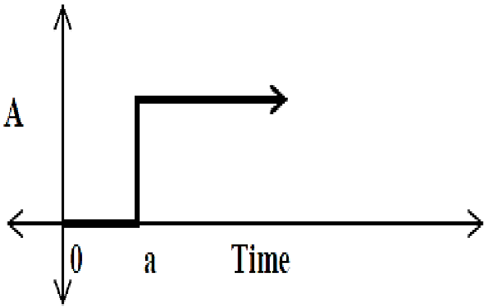
Processes Control:

Laplace transforms of representative functions:

Step Function with time delay

$$\boldsymbol{f(t)} = \begin{cases} \boldsymbol{0} & \boldsymbol{t < a} \\ \boldsymbol{A} & \boldsymbol{t \geq a} \end{cases} \qquad (1)$$

$$\boldsymbol{F(s) = L(A) = \int_a^{\infty} A\, e^{-st} dt = -\frac{A}{s} e^{-st}\Big|_a^{\infty} = \frac{A}{s}\, e^{-as}} \qquad (2)$$

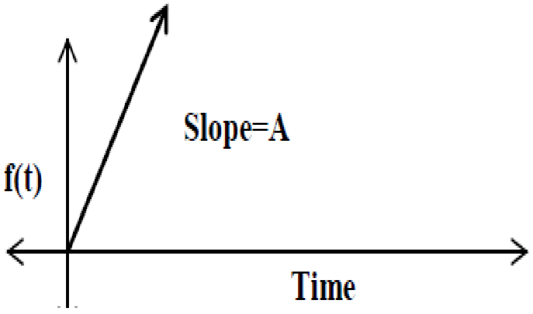


Ramp function

$$\boldsymbol{f(t)} = \begin{cases} \boldsymbol{0} & \boldsymbol{t < 0} \\ \boldsymbol{At} & \boldsymbol{t \geq 0} \end{cases} \qquad (3)$$

$$L\{At\} = \int_0^{\infty} Ate^{-st} = -\frac{At}{s}e^{-st}\Big|_0^{\infty} + \int \frac{A}{s} e^{-st} dt = -\frac{At}{s} e^{-st}\Big|_0^{\infty} + \frac{A}{s}\frac{-1}{s} e^{-st}\Big|_0^{\infty}$$

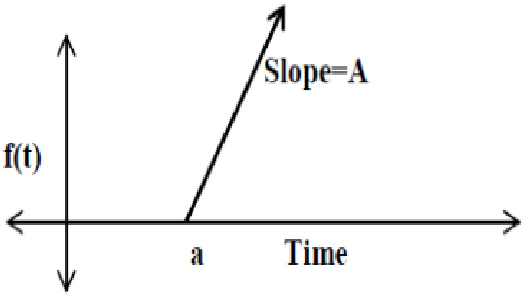
$$= \frac{-A}{s}(\infty e^{-\infty} - 0e^{-0}) - \frac{A}{s^2}(e^{-\infty} - e^{-0}) = \frac{A}{s^2} \qquad (4)$$



Ramp function with time delay

$$f(t) = \begin{cases} 0 & t < a \\ At & t \geq a \end{cases} \tag{5}$$

$$F(s) = L(At) = \int_a^\infty At\, e^{-st} dt = \frac{A}{s^2}\, e^{-as} \tag{6}$$

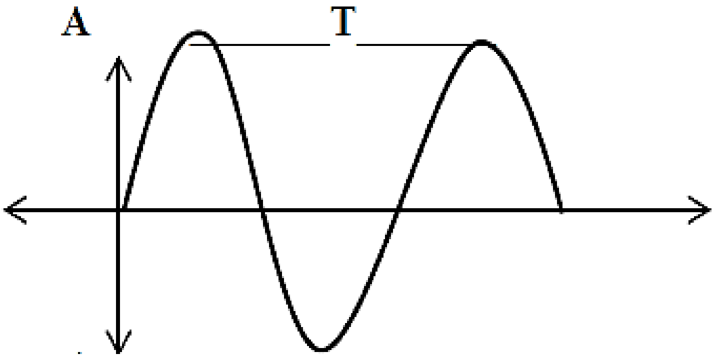


Sine function

$$f(t) = \begin{cases} 0 & t < 0 \\ A \sin \omega t & t \geq 0 \end{cases} \tag{7}$$

$$F(s) = L(A \sin \omega t) = \int_0^\infty (A \sin \omega t)\, e^{-st} dt = \frac{A\omega}{s^2 + \omega^2} \tag{8}$$

Where: $\omega = 2\pi f$ and $T = {}^{1}/_{f}$

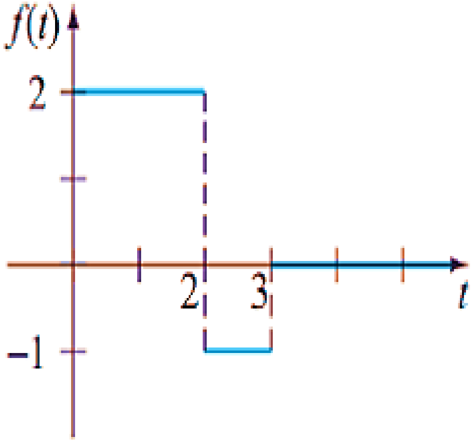


Cosine function

$$f(t) = \begin{cases} 0 & t < 0 \\ A\cos\omega t & t \geq 0 \end{cases} \qquad (9)$$

$$F(s) = L(A\cos\omega t) = \int_{0}^{\infty} (A\cos\omega t)\, e^{-st} dt = \frac{As}{s^2+\omega^2} \qquad (10)$$

Example 12.1. Find the Laplace transform for



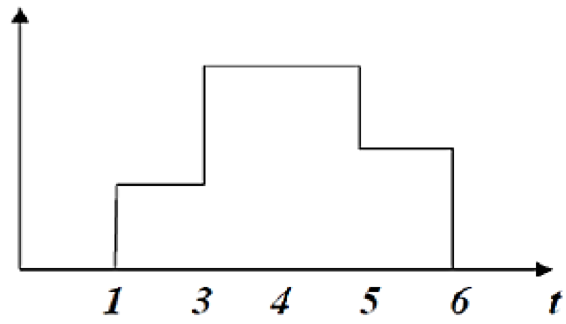
Ans.: As observed that this function looks like unit step at ***t=0***, but unit function knows only about **0** and **1**, so, we have ***f(t)= 2*** at *zero time* which means that we have to use ***2u(t)***. Then in time ***t=2*** its value falls down from ***f(t)= 2*** to ***f(t)= -1*** (i.e. **3** down at ***t=2***) which means we have to add [***-3u(t-2)***]. Finally the value at ***t=3*** rises **1** higher, which brings member [***u(t−3)***].

$$f(t)=2u(t) - 3u(t-2) + u(t-3)$$

So far we collected unit step functions to express function from the graph.

$$F(s) = L(f(t)) = L(2u(t) - 3u(t-2) + u(t-3))$$
$$= L(2u(t)) + L(3u(t-2)) + L(u(t-3)) = \frac{2}{s} - \frac{3}{s}e^{-2s} + \frac{1}{s}e^{-3s}$$

Example 12.2. Find the Laplace transform for



Ans.: As observed in this function, there is a delay; therefore, ***f(t)=0*** at ***t=0***. Then at ***t=1***, its values rises to be ***f(t)= 1u(t-1).*** At ***t=3***, ***f(t)*** rises ***2*** points to be [***2u(t-3)***]. Then it fall downs one point at ***t=5*** to be [***1u(t-5)***]. Finally, it falls down ***2*** points to be [***2u(t-6)***].

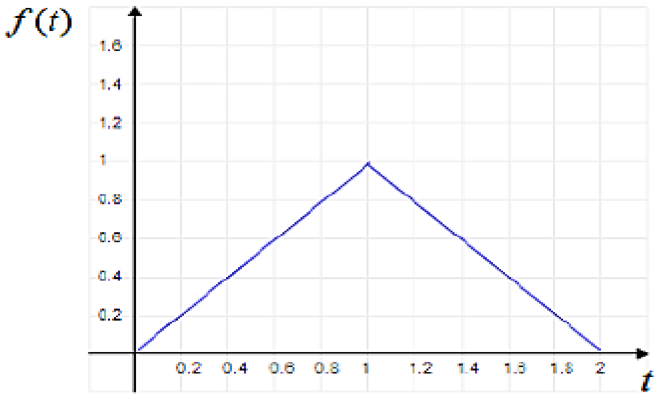
$$f(t)=\begin{cases} 0 & t<1 \\ 1 & 1\le t\le 3 \\ 3 & 3\le t\le 5 \\ 2 & 5\le t\le 6 \\ 0 & t\ge 6 \end{cases}$$

$$f(t)=0u(t-0)+1u(t-1)+2u(t-3)-1u(t-5)-2u(t-6)$$

$$F(s)=\frac{1}{s}e^{-s}+\frac{2}{s}e^{-3s}-\frac{1}{s}e^{-5s}-\frac{2}{s}e^{-6s}=\frac{1}{s}(e^{-s}+2e^{-3s}-e^{-5s}-2e^{-6s})$$

Example 12.3. : Find F(s) for

$$f(t)=\begin{cases} 0 & t<0 \\ t & 0<t<1 \\ t-2 & 1<t<2 \\ 0 & t>2 \end{cases}$$

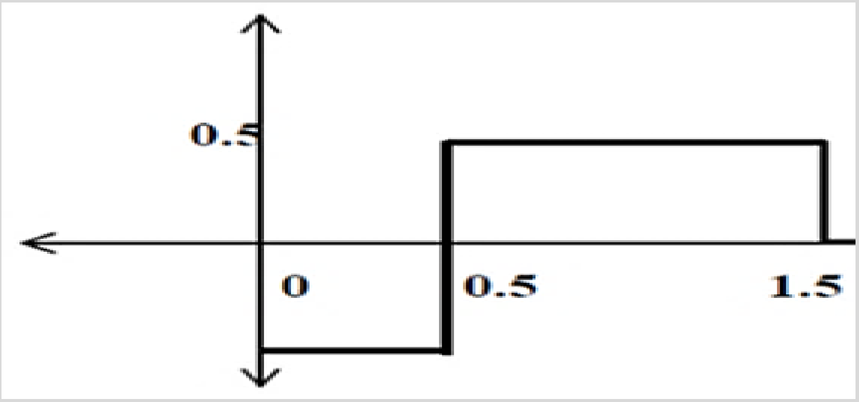


Ans.: Here, at first, the function obeys as a ramp function from ***t=0*** to ***t=1*** to be [***1tU(t)***]. When it raises toward ***1*** or falls from ***1*** it value will decreases twice [***2(t-1)U(t-1)***]. Then it stabled when it reaches ***t=2*** to be ***[(t-2)U(t-2)***]:

$$
\begin{aligned}
&f(t) = tU(t) - t(t-1)U(t-1) - (t-1)U(t-1) + (t-2)U(t-2)\\
&= tU(t) - 2(t-1)U(t-1) + (t-2)U(t-2)\\
&L[f(t)] = L[tU(t) - 2(t-1)U(t-1) + (t-2)U(t-2)]\\
&= L[tU(t)] - L[2(t-1)U(t-1)] + L[(t-2)U(t-2)]\\
&\qquad\qquad = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}
\end{aligned}
$$

Hint: *Any function starting with ramp shape will possess negative at many times as it raises away from time axis.*

H.W. 12.1. *Find the Laplace transform for*



Example 12.4. : Show that

$$\frac{1}{(s+7)(s+3)}=\frac{-\frac{1}{4}}{s+7}+\frac{\frac{1}{4}}{s+3}$$

Ans.:

$$\frac{1}{(s+7)(s+3)}=\frac{A}{s+7}+\frac{B}{s+3}$$

Step1: Remove Fractions

Multiply both sides of the equation by the denominator on the left hand side

$$\frac{1\times(s+7)(s+3)}{(s+7)(s+3)}=\frac{A(s+7)(s+3)}{s+7}+\frac{B(s+7)(s+3)}{s+3}$$

$$\Rightarrow 1=A(s+3)+B(s+7)$$

Step2: Choose s values to find A and B

The equation above is true for ***all*** values of s. We can choose s values to make things simple:

Choose s = -3 so that (s+3) = 0 and we have

$$1=A(0)+B(4)\;\Rightarrow\;1=4B\;\Rightarrow\;B=\frac{1}{4}$$

Choose s = -7 so that (s+7) = 0 and we have

$$1=A(-4)+B(0)\;\Rightarrow\;1=-4A\;\Rightarrow\;A=-\frac{1}{4}$$

Step3: Substitute A and B into the original expression

$$\frac{1}{(s+7)(s+3)}=\frac{-\frac{1}{4}}{s+7}+\frac{\frac{1}{4}}{s+3}$$

Example 12.5. : Show that

$$\frac{3s-1}{(s+1)(s+2)}=\frac{-4}{s+1}+\frac{7}{s+2}$$

Ans.:

write $\frac{3s-1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$

Multiply both sides by $(s+1)(s+2)$:

$$\frac{3s-1}{(s+1)(s+2)}(s+1)(s+2)=\frac{A}{s+1}(s+1)(s+2)+\frac{B}{s+2}(s+1)(s+2)$$

$$\Rightarrow 3s-1=A(s+2)+B(s+1)$$

If choose s = -2

$$\Rightarrow 3(-2)-1=A(-2+2)+B(-2+1)\Rightarrow -7=A.0+B(-1)\Rightarrow -7=-B$$

$$\Rightarrow B=7$$

If choose s = -1 $\Rightarrow 3(-1)-1=A(-1+2)+B(-1+1)\Rightarrow -4=A(1)+B(0)\Rightarrow -4=A$

$$\Rightarrow A=-4$$

Therefore $\frac{3s-1}{(s+1)(s+2)}=\frac{-4}{s+1}+\frac{7}{s+2}$

Example 12.6.

Find the fixed constants A, B, C so that the partial fraction decomposition can be completed: $\frac{4s+3}{s(s-1)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+3}$

Ans.:

$$\frac{4s+3}{s(s-1)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+3}$$

Multiplying both sides of this equation by the left hand side denominator we deduce that,

$$s(s-1)(s+3)\frac{4s+3}{s(s-1)(s+3)}=s(s-1)(s+3)\frac{A}{s}+s(s-1)(s+3)\frac{B}{s-1}+s(s-1)(s+3)\frac{C}{s+3}$$

$$\Rightarrow 4s+3=A(s-1)(s+3)+Bs(s+3)+Cs(s-1)$$

Now in turn, put , $s=0$, $s=1$, $s=-3$. You will find at each stage that **2** of the A, B, C terms will vanish:

If choose $s=0$

$$4(0)+3=A(0-1)(0+3)+B(0)+C(0)\Rightarrow 3=-3A\Rightarrow A=1$$

If choose $s=1$

$$4(1)+3=A(0)+B(1)(1+3)+C(0)\Rightarrow 7=4B\Rightarrow B=\frac{7}{4}$$

If choose $s=-3$

$$4(-3)+3=A(0)+B(0)+C(-3)(-3-1)\Rightarrow -9=12C\Rightarrow B=\frac{-9}{12}$$

Therefore :

$$\frac{4s+3}{s(s-1)(s+3)}=\frac{[-1]}{s}+\frac{\left[\frac{7}{4}\right]}{s-1}+\frac{\left[\frac{-3}{4}\right]}{s+3}=-\frac{1}{s}+\frac{7}{4(s-1)}-\frac{3}{4(s+3)}$$

Example 12.7.

Write in terms of partial fractions $\frac{8s-1}{(s-1)^2}$

***Ans.*:**

In this case $\frac{8s-1}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}$

Step1 we multiply both sides by $(s-1)^2$ to remove fractions

$$\frac{8s-1}{(s-1)^2}(s-1)^2=\frac{A}{s-1}(s-1)^2+\frac{B}{(s-1)^2}(s-1)^2 \Rightarrow 8s-1=A(s-1)+B$$

Step2: we can choose s = 1 as before so that (s-1) = 0 and we get

$8-1=B \;\Rightarrow\; B=7$

We cannot choose another s value to directly find A however. There is more than one approach to finding A but the easiest method is called "***equating coefficients***". In this case, we note that there must be the same "amount of s" on both sides of the equation. On the left hand side we have 8s and on the right we have As, so that A must be 8.

Step3; write the answer down

$$\frac{8s-1}{(s-1)^2}=\frac{8}{s-1}+\frac{7}{(s-1)^2}$$

H.W. 12.2. *Show that,*

$$\frac{2s+1}{(s-1)(s+2)^2}=\frac{1}{3(s-1)}-\frac{1}{3(s+2)}+\frac{1}{(s+2)^2}$$

Final Value Theorem

The asymptotic value of y(t) for large values of time **y(∞)** can be found from Eq. 3-53, providing that $\mathbf{lim}_{s\to 0}[sY(s)]$exists for all **Re(s) ≥ 0**:

$$\lim_{t\to\infty} y(t)=\lim_{s\to 0}\,[sY(s)] \tag{11}$$

In the practical application of this theorem, the limit of ***f(t)*** that is found by use of the theorem is correct only if ***f(t)*** is bounded as ***t*** approaches infinity.

Initial Value Theorem

Analogous to the Final Value Theorem, the Initial Value Theorem can be stated as

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} [sY(s)] \tag{12}$$

Example 12.8. Apply the Initial and Final Value Theorems to the following transfer function

$$x(s) = \frac{1}{s(s^3 + 3s^2 + 3s + 1)}$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sx(s) = \lim_{s \to 0} \frac{s \times 1}{s(s^3 + 3s^2 + 3s + 1)}$$

$$= \lim_{s \to 0} \frac{1}{(s^3 + 3s^2 + 3s + 1)} = 1$$

H.W. 12.3. Apply the Initial and Final Value Theorems to the following transfer function

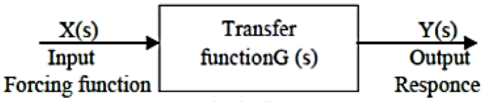
$$Y(s) = \frac{5s + 2}{s(5s + 4)}$$

Processes Control:

Transfer Function Models

A transfer function model characterizes the dynamic relationship of two process variables, a dependent variable (or output variable) and an independent variable (or input variable). For example, in a continuous chemical reactor, the output variable could be the exit concentration and the input variable, a feed flow rate.

$$\mathrm{T.F} = \frac{\text{Laplace transform of the output (responce)}}{\text{Laplace transform of the input (forcing function distubance)}}$$



Block diagram

$$\mathrm{T.F} = \mathrm{G(s)} = \frac{\mathrm{y(s)}}{\mathrm{x(s)}}$$

Where this definition is applied to linear systems.

First-Order systems

Systems whose dynamic behavior is modeled by first-order differential equations of the type:

$$\boldsymbol{a_1 \frac{dy}{dx} + a_0 y = b\, x(t)} \qquad (\mathbf{1})$$

(where y(t) represents the output and u(t) represents the input) are referred to as first-order systems. It is customary to rearrange Eq. 1 to read:

$$\boldsymbol{\tau \frac{dy}{dx} + y = K\, x(t)} \qquad (\mathbf{2})$$

where, for $\boldsymbol{a_0 \neq 0}$, the new parameters $\boldsymbol{\tau}$ and $\boldsymbol{K}$ are given by:

$$\boldsymbol{\tau = \frac{a_1}{a_0} \quad and \;\; K = \frac{b}{a_0}} \qquad (\mathbf{3})$$

The characteristic parameters of a first-order system may now be identified as $\boldsymbol{\tau}$ and $\boldsymbol{K}$, respectively called the steady-state gain and the time constant.

Taking Laplace transforms of Eq. 2 and rearranging gives:

$$y(s) = \left(\frac{K}{\tau s + 1}\right) x(s) \qquad (4)$$

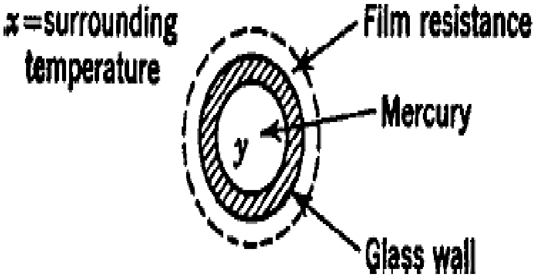
It is therefore clear that the general transfer function t for a first-order system is given by:

$$G(s) = \frac{y(s)}{x(s)} = \left(\frac{K}{\tau s + 1}\right) \qquad (5)$$

Physical Examples of First-Order Systems

Mercury Thermometer

Consider the thermometer to be located in a flowing stream of fluid for which the temperature x varies with time. Our problem is to calculate the response or the time variation of the thermometer reading y for a particular change in surrounding temperature x. The result of the analysis of the thermometer be general and therefore applicable to other first-order systems.



1. All the resistance to heat transfer offered by the glass and mercury is neglected.
2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.
3. The glass wall containing the mercury does not expand or contract during the transient response.

It is assumed that the thermometer is initially at steady state. This means that, before time zero, there is ***no change*** in temperature with time. At time zero the thermometer will be subjected to some change in the surrounding temperature ***x(t)***. By applying the ***unsteady-state*** energy balance

rate of accumulation= Input rate - output rate

we get the result

$$h\,A\,(x - y) - 0 = m\,C\frac{dy}{dt} \qquad (6)$$

where ***A*** = surface area of bulb for heat transfer; ***C*** = heat capacity of mercury; ***m*** = mass of mercury in bulb; ***t*** = time; ***h*** = film coefficient of heat transfer which will assume to be for a particular installation of the thermometer.

The unsteady-sate form of Eq. 6 is referred as a first-order differential equation. At time less than zero (**t<0**), it will be considered as steady-state:

$$\boldsymbol{h\,A\,(x_s - y_s) = 0 \quad t < 0} \qquad (7)$$

Equation 7 gives that x = y at steady state. Now, to find the ***deviation variables*** Subtracting Eq. (7) from Eq. (6) gives

$$\boldsymbol{h\,A\,[(x - x_s)(y - y_s)] = m\,C\frac{d(y - y_s)}{dt}} \qquad (8)$$

$$\boldsymbol{h\,A\,(X - Y) = m\,C\frac{dY}{dt}} \qquad (9)$$

$$\boldsymbol{X - Y = \frac{m\,C}{h\,A}\frac{dY}{dt} = \tau\,\frac{dY}{dt}} \qquad (10)$$

Where the parameter $\boldsymbol{\tau}$ is called the time constant of the system and has the units of time. Taking Laplace transforms of Eq. 10 and rearranging gives:

$$\boldsymbol{X(s) - Y(s) = \tau\,sY(s)}$$

$$\boldsymbol{\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}} \qquad (11)$$

It is the ratio of the Laplace transform of the deviation in thermometer reading to the Laplace transform of the deviation in the surrounding temperature.

Liquid Level Systems

The process under consideration is the simple cylindrical liquid receiver shown in Figure 1. Liquid flows in at the rate $\boldsymbol{F_i}$ and is permitted to flow out at a possibly different flow rate ***F***. The tank's cross-sectional area is considered uniform, with a constant value $\boldsymbol{A_c}$.

$$A_c\boldsymbol{\frac{dh}{dt} = F_i - F = F_i - c\,h} \qquad (12)$$

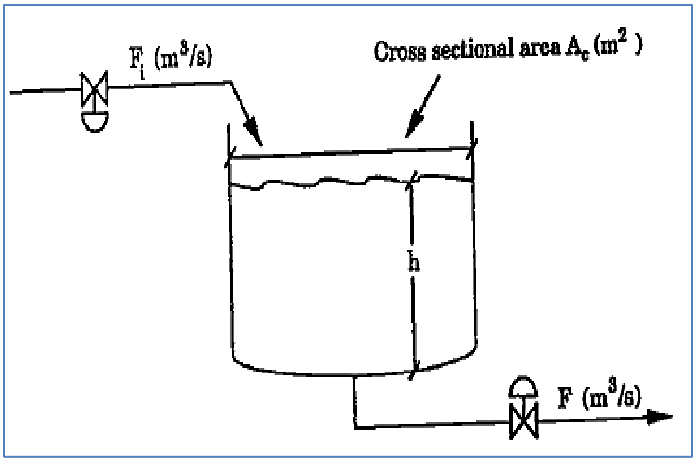


Fig. 1. The liquid receiver

At steady state, the time derivative vanishes, and if the inflow and outflow have respective steady-state values of F_{is} and F_s , then Eq. (12) becomes:

$$\boldsymbol{0 = F_{is} - c\,h_s} \qquad (13)$$

Subtracting Eq. (13) from Eq. (12) now gives:

$$A_c \frac{d(h-h_s)}{dt} = (F_i - F_{is}) - (c\,h - c\,h_s) \qquad (14)$$

$$\frac{dy}{dt} = -\frac{c}{A_c} y + \frac{u}{A_c}$$

$$sy(s) = -\frac{c}{A_c}\,y(s) + \frac{1}{A_c} u(s) \qquad (15)$$

$$G(s) = \frac{y(s)}{u(s)} = \frac{1/c}{\left({A_c}/{c}\right) s + 1} \qquad (16)$$

Observe now that for this system, the time constant is A_c/c, while the steady-state gain is $1/c$.

Isothermal Continuous Stirred Tank Reactor

The process shown in Figure 2 below, a first-order, irreversible reaction A → B , is taking place in an isothermal CSTR. Making the usual assumption of constant volume, and perfect mixing, it is easy to show that an appropriate mathematical model for this process, obtained from a component mass balance on **A**, is:

$$\frac{dy}{dt} + \left(k + \frac{1}{\theta}\right) y = \frac{1}{\theta} u \qquad (17)$$

Here, **y** is the concentration of A in the reactor, C_A, in terms of deviations from its initial steady-state value C_{As}; u is the inlet concentration of **A**, C_{Af} (also in deviation variables); k is the reaction rate constant; and θ is the reactor residence time (**V/F**).

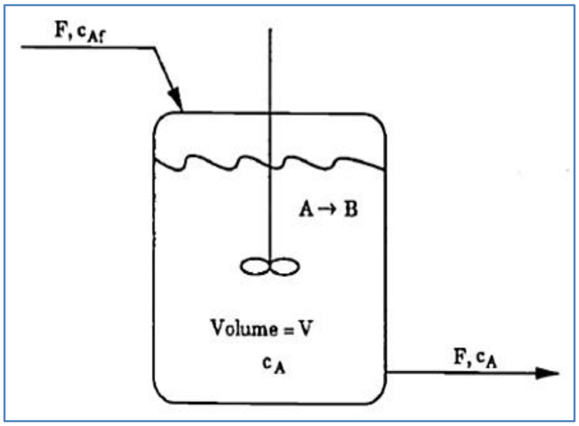


Fig. 2. The isothermal CSTR

Laplace transformation and subsequent rearrangement of the first-order differential equation in Eq. (18) gives the transfer function for this process as:

$$G(s) = \frac{y(s)}{u(s)} = \frac{\left(\frac{1}{\theta k + 1}\right)}{\left(\frac{\theta}{\theta k + 1}\right)s + 1} \qquad (18)$$

***H.W. 13.1**. Derive Eq. (18).*

So that in this case, the time constant and steady-state gain are respectively given by:

$$\tau = \frac{\theta}{\theta k + 1}$$

$$K = \frac{1}{\theta k + 1}$$

The Stirred Heating Tank

The process shown in Figure 3 is the stirred heating tank. Liquid at temperature T_i flowing into the tank at a volumetric *flowrate* F, is heated by steam flowing through the steam coil arrangement at a rate Q (mass/time). The heated fluid, now at temperature T, is withdrawn at the same volumetric rate as in the inlet stream. The tank volume is V; the liquid density and specific heat capacity are, respectively, ρ, and C_P. The latent heat of vaporization of steam is λ.

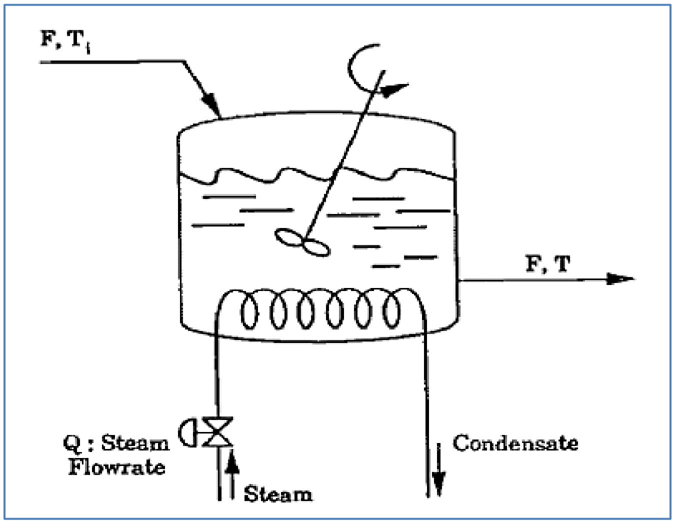


Fig. 3. The stirred heating tanke

Assumptions

1. We will assume that the content of the tank is well mixed so that the liquid temperature in the tank and the temperature of liquid in the outlet stream are the same.
2. The physical properties ρ, C_P, and λ do not vary significantly with temperature.
3. All the heat given up by the steam (through condensation) is received by the liquid content of the tank; i.e., no heat from the steam is accumulated in the coils.
4. Heat losses to the atmosphere are negligible.

Overall Mass Balance:

$$\frac{d}{dt}(\rho V) = \rho F - \rho F = 0 \qquad (19)$$

Thus the material balance equation merely tells us that we have a *constant volume* process.

Overall Energy Balance

Recall Eq. (18 Lecture 7) with some modification to be

$$\boldsymbol{\frac{dT}{dt} = -\frac{1}{\theta}\,T + \frac{\lambda}{\rho V C_p}Q + \frac{1}{\theta}\,T_i} \qquad where\ \boldsymbol{\theta} = \boldsymbol{\frac{V}{F}}\ is\ the\ residence\ time \qquad (20)$$

Under steady-state conditions, nothing changes with time within the system any longer. As such, time derivatives in the process models vanish, and Eq. (20) becomes:

$$\boldsymbol{0 = -\frac{1}{\theta}\,T_s + \frac{\lambda}{\rho V C_p}Q_s + \frac{1}{\theta}\,T_{is}} \qquad (21)$$

If we now define the following variables, in terms of deviations from the steady-state values indicated above:

$$\boldsymbol{y = T\text{-}T_s;} \qquad \boldsymbol{u = Q\text{-}Q_s;} \qquad \boldsymbol{d = T_i - T_{is}}$$

and assuming that the tank temperature is available for measurement directly, then subtracting Eq. (21) from Eq. (20) gives:

$$\boldsymbol{\frac{dy}{dt} = -\frac{1}{\theta}\,y + \beta u + \frac{1}{\theta}\,d} \qquad where\ \boldsymbol{\beta} = \boldsymbol{\frac{\lambda}{\rho V C_p}} \qquad (22)$$

Where $\boldsymbol{y(0)=0}$ is the proper initial condition if $\boldsymbol{T = T_s}$ at $\boldsymbol{t = 0}$. Then taking Laplace transforms gives:

$$\boldsymbol{y(s) = \left(\frac{\beta\theta}{\theta s + 1}\right)u(s) + \left(\frac{1}{\theta s + 1}\right)d(s)} \qquad \boldsymbol{(23)}$$

This, of course, is of the form:

$$\boldsymbol{y(s) = G(s)u(s) + G(s)d(s)} \qquad \boldsymbol{(24)}$$

with the process transfer function, $\boldsymbol{G(s)}$, and the disturbance transfer function, $\boldsymbol{G_d(s)}$, given by:

$$\boldsymbol{G(s) = \left(\frac{\beta\theta}{\theta s + 1}\right)} \qquad \text{The effect of the input u on the system output}$$

$$\boldsymbol{G(s) = \left(\frac{1}{\theta s + 1}\right)} \qquad \text{The effect of the diturbance d on the system output}$$

Processes Control:

Response of First-Order Systems to Various Inputs

In what follows, we shall utilize the general first-order transfer function form introduced in (Lecture 13) to obtain the response of first-order systems to such input functions as were discussed before, i.e., step, rectangular pulse, impulse, ramp, and sinusoidal functions.

$$\boldsymbol{y}(\boldsymbol{s}) = \left(\frac{\boldsymbol{K}}{\boldsymbol{\tau s} + \mathbf{1}}\right)\boldsymbol{u}(\boldsymbol{s}) \qquad where\ \boldsymbol{y}(\boldsymbol{s}) = ? \quad for\ different\ input\ \boldsymbol{x}(\boldsymbol{s}) \qquad (\mathbf{1})$$

Step Response

When a step function of magnitude A is applied as an input to a first-order system, the dynamic response of the system output is obtainable from:

$$\boldsymbol{y}(\boldsymbol{s}) = \left(\frac{\boldsymbol{K}}{\boldsymbol{\tau s} + \mathbf{1}}\right)\frac{\boldsymbol{A}}{\boldsymbol{s}} \qquad (\mathbf{2})$$

By partial fraction expansion, Eq. (2) becomes:

$$\boldsymbol{y}(\boldsymbol{s}) = \boldsymbol{AK}\left(\frac{\mathbf{1}}{\boldsymbol{s}} - \frac{\boldsymbol{\tau}}{\boldsymbol{\tau s} + \mathbf{1}}\right) \qquad (\mathbf{3})$$

which, upon inverting back to the time domain, gives the required step response:

$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{AK}\left(\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{t}/\boldsymbol{\tau}}\right) \qquad (\mathbf{4})$$

A sketch of this response and the input (forcing) function is shown in Figure 1. There are a number of important points to note about the response of *first-order systems* to *step inputs*:

1. The output, ***y(t)***, attains the value ***AK*** at steady state. The value of the input function, ***u(t)***, responsible for this response is ***A***. Observe, therefore, that the ultimate change in the output as ***t*** → ∞ is the magnitude of the input step function ***A*** multiplied by the constant ***K***. (This obviously also implies that the ratio of the steady-state value of the output to that of the input (***AK***/ ***A***) is ***K***.) This is the reason for calling ***K*** the *steady-state gain*.

2. When ***t*** = $\boldsymbol{\tau}$, ***y(t) = 0.632 AK***; i.e., ***y(t)*** attains 63.2% of its ultimate value in $\boldsymbol{\tau}$ time units. It is easy to establish that by the time ***t*** = 3.9$\boldsymbol{\tau}$, ***y(t)*** has reached 98% of its ultimate value, reaching the 99% mark when ***t*** = 4.6$\boldsymbol{\tau}$.

3. The slope of the step response at the origin, obtained by differentiating Eq. (4) with respect to ***t***, and setting ***t*** = 0, is given by:

$$\left(\frac{dy}{dt}\right)\Big|_{t=0} = \frac{AK}{\tau} \qquad (5)$$

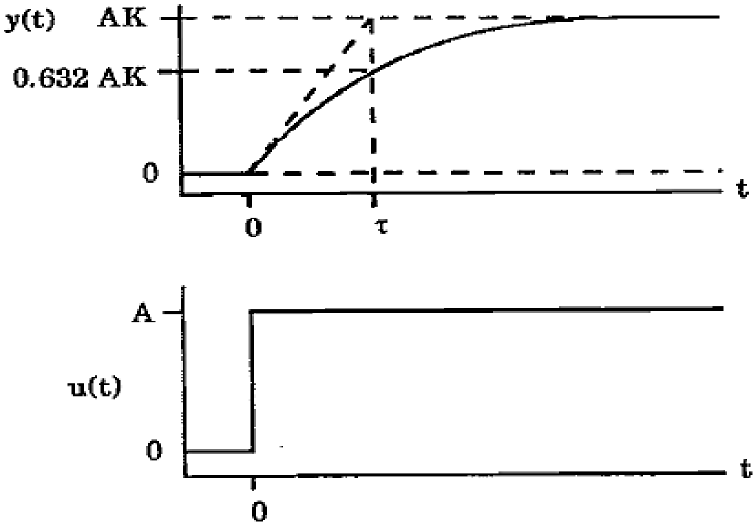
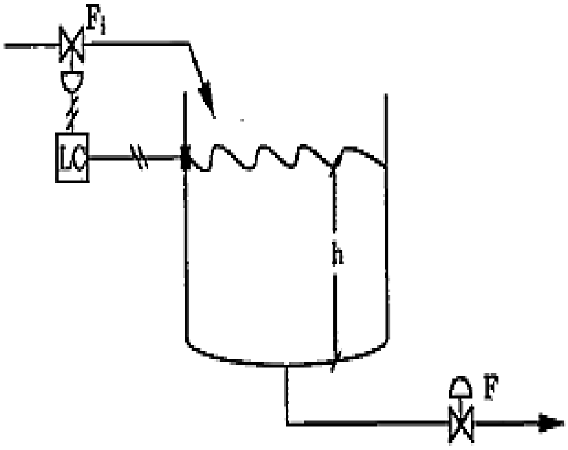


Fig. 1. Response of a first-order system to the step function

H.W. 14.1*. What will be the mathematical relation between* ***t*** *and* $\boldsymbol{\tau}$ *when step response y(t)=1?*

So, if the initial rate of change of ***y(t)*** were maintained, the ultimate value of ***AK*** would be attained in exactly $\boldsymbol{\tau}$ time units, providing an interesting interpretation of the first-order system time constant $\boldsymbol{\tau}$. The magnitude of the step response is indicated by ***K***, while $\boldsymbol{\tau}$ indicates the speed of response.

Observe that the larger the value of ***K***, the larger the ultimate value of the process output in response to a unit step change; the converse is also true. On the other hand, observe that the larger the value of $\boldsymbol{\tau}$, the more sluggish the process response; and conversely, the smaller the value of $\boldsymbol{\tau}$ the faster the process response. In the extreme case with $\boldsymbol{\tau}$ = 0 the process responds in the fastest possible manner, i.e., instantaneously, and **y= AK** for all times.



Example 14.1. A **250 liter** tank used for liquid storage is configured as shown in the Figure. The cross-sectional area of the tank is **0.25 m^2**, and it may be assumed uniform. The outlet valve resistance, which we shall assume to be linear for simplicity, has a value **c = 0.1 m^2/min**. The entire system was initially at steady state with the inlet flowrate F_i at **37 liters/min (0.037 m^3/min)**.

1. What is the initial value of the liquid level in the tank?
2. If the inlet flowrate was suddenly changed to **87 liters/min (0.087 m^3 /min)**, obtain an expression for how the liquid level in the tank will vary with time.
3. When will the liquid level in the tank be at the **0.86 m** mark? And to what final value will the liquid level ultimately settle?

Ans.: As found before that the unsteady-sate balance:

$$A_c \frac{dh}{dt} = F_i - c\,h$$

1. At steady state:

$$F_{is} = c\,h_s$$

If we now introduce the given values for the flowrate and the resistance **c**, we have the required value for the initial steady-state liquid level:

$$h_s = \frac{F_{is}}{c} = \frac{0.037}{0.1} = 0.37\,m$$

2. Recalling the transfer function for this process given in Eq. (16-Lecture 13), and introducing the given numerical values, we find that the steady-state gain and time constant are given by:

$$\boldsymbol{G(s) = \frac{y(s)}{u(s)} = \frac{1/c}{\left({}^{A_c}\!/_{c}\right) s + 1}}$$

$$K = \frac{1}{c} = \frac{1}{0.1} = 10\ min/m^2$$

$$\tau = \frac{A_c}{c} = \frac{0.25}{0.1} = 2.5\ min$$

The magnitude of the step input in the flow rate is **(0.087 - 0.037) = 0.05 m^3/ min**. Thus from Eq. (4) the expression representing the change in liquid level in response to this input change is:

$$y(t) = AK\left(1 - e^{-t/\tau}\right) = 0.05 \times 10\ (1 - e^{-\frac{t}{2.5}})$$

$$y(t) = 0.5\ (1 - e^{-0.4t})$$

Note that this is in terms of **y**, the deviation of the liquid level from its initial steady-state value. The actual liquid level time behavior is represented by:

$$h(t) = 0.37 + 0.5\ (1 - e^{-0.4t})$$

H.W. 14.1. *Solve point 3 in Example 14.1.*

Rectangular Pulse Response

The input function in this case is as described before

$$u(t) = \begin{cases} 0 & t < 0 \\ A & 0 \le t < b \\ 0 & t \ge b \end{cases}$$

Since the Laplace transform of this forcing function is given by:

$$u(s) = \int_0^b A\, e^{-st} dt = -\frac{h}{s} e^{-st}\Big|_0^b = \frac{A}{s}(1 - e^{-bs}) \qquad (6)$$

the rectangular pulse response of the first-order process is obtained from:

$$y(s) = \left(\frac{K}{\tau s + 1}\right) u(s) = \left(\frac{K}{\tau s + 1}\right)\frac{A}{s}(1 - e^{-bs})$$

$$y(s) = \frac{AK}{s(\tau s + 1)}(1 - e^{-bs}) \qquad (7)$$

The inversion of this apparently complicated function is made easier by splitting the terms into two parts as follows:

$$\boldsymbol{y(s)=\frac{AK}{s(\tau s+1)}-\frac{AK}{s(\tau s+1)}e^{-bs}} \qquad \boldsymbol{(8)}$$

Eq. (8) is easily inverted into the time domain to give the following function which takes on different values over different time intervals:

$$\boldsymbol{y(t)=\begin{cases}AK(1-e^{-t/\tau}) & t<b\\ AK\left[\left(1-e^{-t/\tau}\right)-(1-e^{-(t-b)/\tau})\right] & t>b\end{cases}}$$

This response is shown graphically in Figure 2. Note the change in direction experienced at $\boldsymbol{t=b}$ in response to the change in the input function at this same instant. It is easy to establish that the value of the peak response at this point is $\boldsymbol{y_{max}=y(b)=AK(1-e^{-b/\tau})}$.

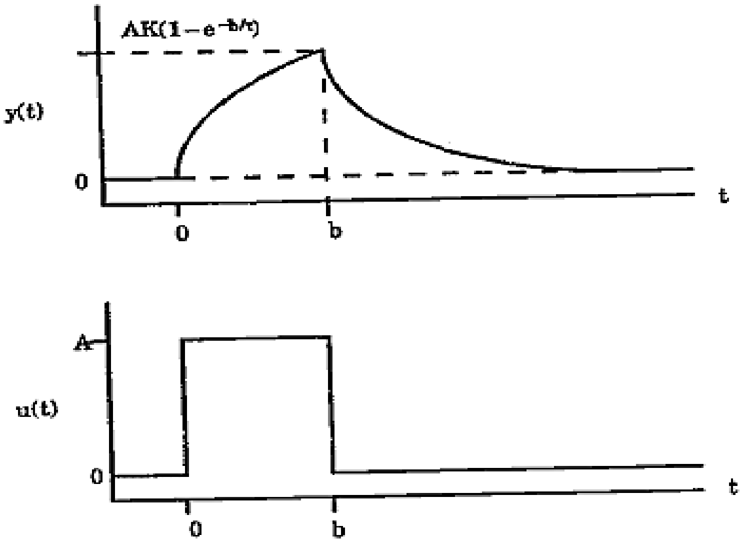


Fig. 2. Rectangular pulse response of a first-order system

Example 14.2. Let us return once again to the system of *Example 14.1*. The flowrate is again to be changed suddenly to **87 liters/min (0.087 m^3 /min)**, but this time we hold it at this value only for **2.5 min**; returning it to its initial value of **37 liters/min (0.037 m^3/min)**. What is the maximum value the liquid level will attain during the course of this experiment?

Ans.:

The expression for the peak value attained by a first-order system in response to a rectangular pulse input of magnitude **A** and duration **b** is given by:

$$y_{max} = y(b) = AK(1 - e^{-b/\tau})$$

(Note that this expression is in terms of deviations from the initial steady-state value for the liquid level.) Since in this example, **A = 0.05, b = 2.5**, and from *Example 14.1* we recall that **K = 10**, τ **= 2.5**, with an initial liquid level of 0.37 m, the peak value for the liquid level is obtained from:

$$h_{max} = 0.37 + 0.5\,(1 - e^{-1}) = 0.686\,m$$

Impulse Response

In this case, recall from Eqs. (3.43) and (3.46) that the input is a Dirac delta function with area A, u(t) = AO(t) and u(s) = A; thus the impulse response is obtained from:

$$u(t) = A\,\delta(t)$$

$$u(s) = L[u(t)] = A \times 1 = A$$

$$y(s) = \left(\frac{K}{\tau s + 1}\right) u(s)$$

$$y(s) = \left(\frac{K}{\tau s + 1}\right) A \qquad (9)$$

which is easily inverted in time to give:

$$y(t) = \frac{KA}{\tau} e^{-t/\tau} \qquad (10)$$

The impulse response is shown in Figure 3.

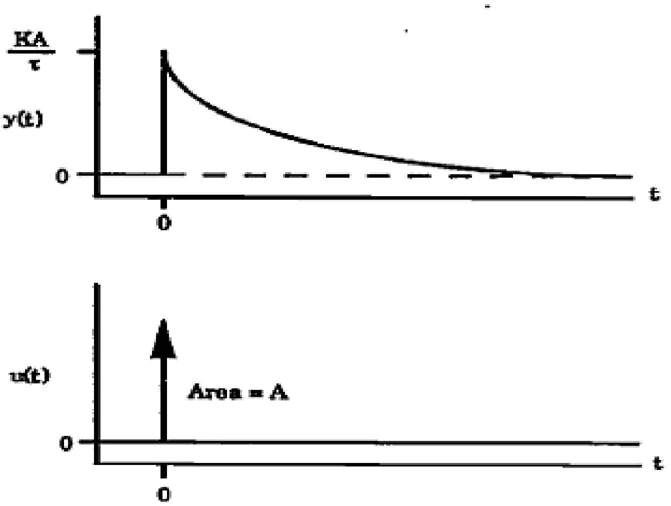


Fig. 3. Impulse response of a first-order system

Note:

1. The impulse response indicates an immediate "*jump*" to a value **AK/τ** at **t = 0** followed by an exponential decay.

2. If y_{step} represents the step response, and $y_{impulse}$ represents the impulse response, then:

$$\frac{d}{dt}\left(y_{step}\right) = y_{impulse} = \frac{AK}{\tau} \qquad (11) \equiv (5)$$

Ramp Response

Recalling that for the ramp input function:

$$u(t) = \begin{cases} 0 & t < 0 \\ At & t \geq 0 \end{cases} \qquad \text{(3 -Lecture 12)}$$

$$u(s) = \frac{A}{s^2}$$

the ramp response is therefore obtained from:

$$y(s) = \left(\frac{K}{\tau s + 1}\right) u(s)$$

$$y(s) = \left(\frac{K}{\tau s + 1}\right) \frac{A}{s^2} \qquad (12)$$

$$y(t) = AK\tau(e^{\frac{-t}{\tau}} + \frac{t}{\tau} - 1) \qquad (13)$$

***H.W. 14.2**. Derive Eq. (13) from Eq. (12).*

Note:

1. As **t** → ∞, ***y(t)*** → ***AK(t - τ).*** Thus the response is asymptotic to a ramp function with slope **AK**, displaced by ***τ*** time units from the origin at ***y=0***.

2. If the step response is represented by y_{step} and the ramp response by y_{ramp} then it is easy to establish that:

$$\frac{d}{dt}\left(y_{ramp}\right) = y_{step} = AK\left(1 - e^{-t/\tau}\right) \qquad (14) \equiv (4)$$

The ramp response is shown in Figure 4.

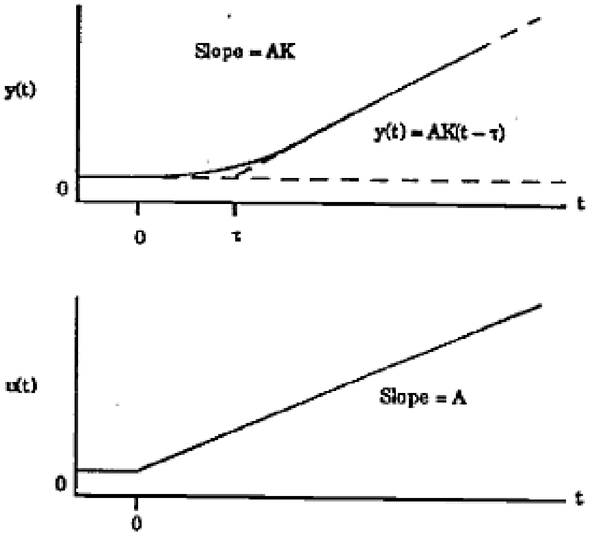


Fig. 4. Ramp response of a first-order system

Processes Control:

Response of First-Order Systems to Various Inputs

Sinusoidal Response

Recalling Eq. (8-Lecture 12) for the Laplace transform of a sine wave input of amplitude **A** and frequency $\boldsymbol{\omega}$, the response of a first-order system to this sinusoidal input function is obtained from:

$$\boldsymbol{y(s)=\left(\frac{K}{\tau s+1}\right)\frac{A\omega}{s^2+\omega^2}} \qquad (1)$$

The inversion of this to the time domain yields

$$\boldsymbol{y(t)=AK\left[\frac{\omega\tau}{(\omega\tau)^2+1}e^{-t/\tau}+\frac{1}{\sqrt{(\omega\tau)^2+1}}sin(\omega t+\phi)\right]} \qquad (2)$$

Where $\boldsymbol{\phi=tan^{-1}(-\omega\tau)}$

***H.W. 15.1**. Derive Eq. (2) from Eq. (1).*

This response is shown in Figure 1. There are a few important points to note in this sinusoidal response:

1. The first term in Eq. (2) is a transient term that decays with a time constant $\boldsymbol{\tau}$, and vanishes as t →∞ ; but the second term persists.

2. After the transients die away (after about four or five time constants in practice) the ultimate response will be a pure sine wave. Thus, we have, from Eq. (2):

$$\boldsymbol{y(t)|_{t\to\infty}=\frac{AK}{\sqrt{(\omega\tau)^2+1}}sin(\omega t+\phi)} \qquad (3)$$

which we shall refer to as the *Ultimate Periodic Response* (**UPR**) because it is to this - a function which is also periodic in its own right - that the sinusoidal response ultimately settles.

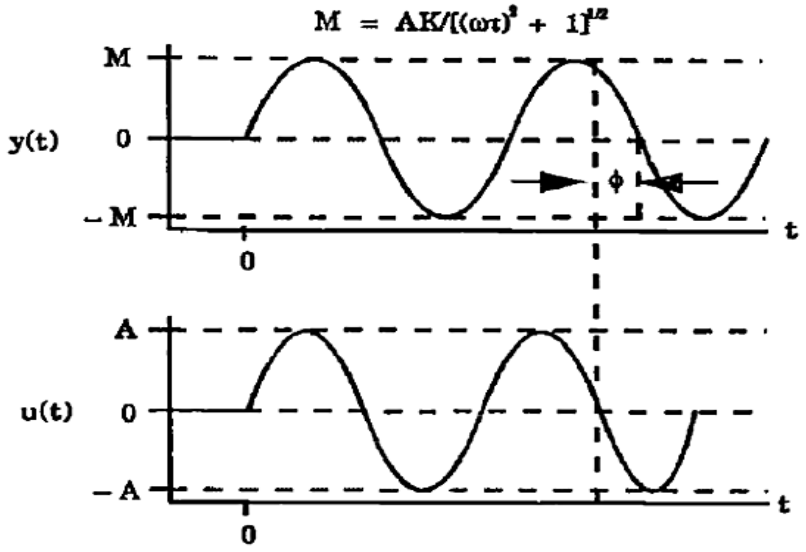


Fig. 1. Sinusoidal response of a first-order system

The following are some important, noteworthy characteristics of the ultimate periodic response (**UPR**) given in Eq. (3):

1. The input function (**u(t) = A sin**$\boldsymbol{\omega t}$) and the **UPR** are both sine waves having the same frequency $\boldsymbol{\omega}$.
2. The **UPR** sine wave lags behind the input signal by an angle $\boldsymbol{\phi}$.
3. The ratio of the **UPR's** amplitude to that of the input sine wave is $[\boldsymbol{K/\sqrt{(\omega\tau)^2+1}}]$, a quantity termed the "**amplitude ratio**" (**AR**) for precisely this reason.

Thus we conclude by noting that the ultimate response of a first-order system to a sine wave, the **UPR**, can be characterized by the following two parameters:

$$\boldsymbol{AR = \frac{K}{\sqrt{(\omega\tau)^2+1}}} \qquad (4)$$

$$\boldsymbol{\phi = tan^{-1}(-\omega\tau)} \qquad (5)$$

respectively known as the amplitude ratio and the phase angle. Note that both quantities are functions of $\boldsymbol{\omega}$, the frequency of the forcing sinusoidal function. Studying the behavior of AR and $\boldsymbol{\phi}$ functions as they vary with $\boldsymbol{\omega}$is the main objective in frequency response analysis. It is customary to refer to the phase angle as a ***phase lag*** if the angle is negative and a ***phase lead*** if the angle is positive. In general, the lag in units of time is given by:-

$$phase\ lag = \frac{|\phi|}{360}\frac{1}{f} \qquad (6)$$

Example 15.1. A mercury thermometer having a time constant of **0.1 min** is placed in a temperature bath at **100°F** and allowed to come to equilibrium with the bath. At time **t=0**, the temperature of the bath begins to vary sinusoidally about its average temperature of **100°F** with an amplitude of **2°F.** If the frequency of oscillation is **10/π cycles/min**, plot the ultimate response of the thermometer reading as a function of time. What is the value of the phase lag?.

Ans.: $\tau = 0.1\ min; f = \frac{10}{\pi}\frac{cycles}{min}$

$$x(t) = \begin{cases} x_s = y_s = 100 & t < 0 \\ 100 + 2 sin\ \omega t & t \geq 0 \end{cases}$$

$$\omega = 2\pi f = 2\pi\left(\frac{10}{\pi}\right) = 20\frac{rad}{min}$$

$$T = \frac{1}{f} = \frac{\pi}{10} = 0.314$$

$$x(t) = x_s + X(t) \quad \rightarrow \quad X(t) = x(t) - x_s$$

$$X(t) = 100 + 2 sin\ \omega t - 100 = 2 sin\ 20t$$

$$X(s) = \frac{2 \times 20}{s^2 + 20^2} = \frac{40}{s^2 + 400}$$

The *Ultimate Periodic Response* (**UPR**) is:

$$y(t)|_{t\to\infty} = \frac{AK}{\sqrt{(\omega\tau)^2 + 1}} sin(\omega t + \phi)$$

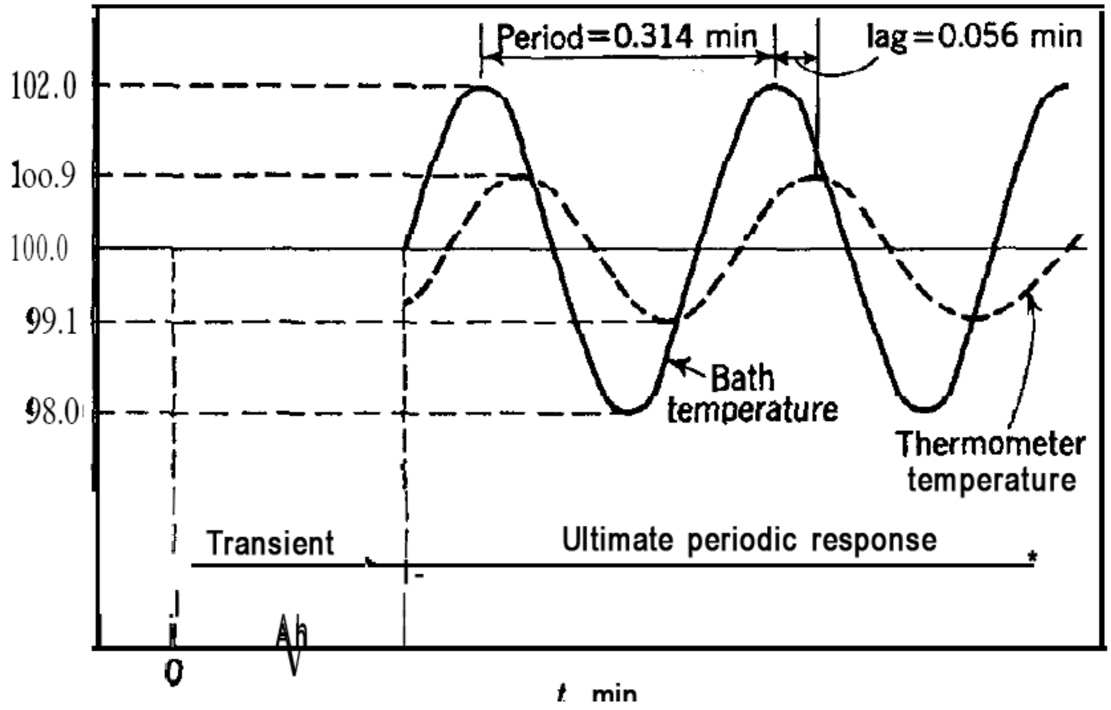
$$\phi = tan^{-1}(-\omega\tau) = tan^{-1}(-20 \times 0.1) = tan^{-1}(-2) = -63.5°$$

$$\boldsymbol{y(t) = \frac{2}{\sqrt{(20 \times 0.1)^2 + 1}} sin\,(20t - 63.5)}$$

$\boldsymbol{y(t) = 0.896\,sin\,(20t - 63.5)}$ *Ultimate Periodic Response*

$$\boldsymbol{phase\ lag = \frac{|\phi|}{360}\frac{1}{f} = \frac{|-63.5|\ cycle}{360}\frac{\pi\ min}{10\ cycle} = 0.056\ min}$$

A frequency of $\frac{10\ cycle}{\pi\ min}$ means that a complete cycle occurs in $(\frac{10}{\pi})^{-1}$min. since cycle is equivalent to **360°** and lag is **63.5°**.



Pure Gain Systems

Consider the first-order system with $\tau = 0$. (This corresponds to a physical system that, theoretically, is infinitely fast in responding to inputs. More realistically, one might imagine a situation in which the first-order system is so fast in responding that it is so small as to be negligible.)

$$\boldsymbol{y(t) = K\,u(t)} \qquad \textbf{(7)}$$

and, either by taking Laplace transforms of Eq. (5.30), we obtain:

$$\boldsymbol{y(s) = K\,u(s)} \qquad (\mathbf{8})$$

so that the transfer function for such a process is identified as:

$$\boldsymbol{G(s) = K} \qquad (\mathbf{9})$$

A process having such characteristics is referred to as a ***pure gain process*** by virtue of the fact that its transfer function involves only one characteristic parameter: ***K***, the process gain term. Such systems are always at steady state, moving instantly from one steady state to another with no transient behavior in between steady states.

There are a few physical processes that truly exhibit pure gain characteristics. One example is the capillary system shown in Figure 2.

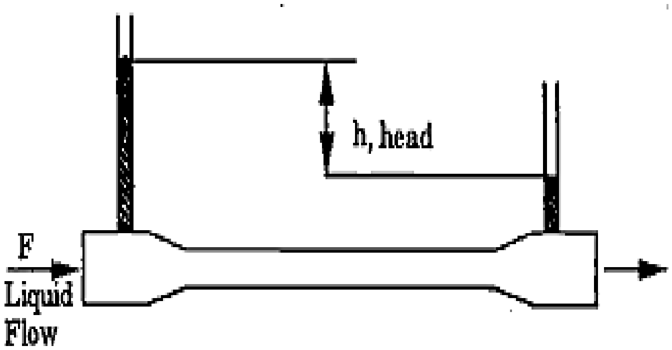


Fig. 2. The capillary system

There is a flow constriction that results in a pressure drop in the incompressible fluid flowing through the capillary. The upstream and downstream pressures are measured by liquid levels in manometers. The value of the head ***h*** is observed to change whenever the liquid flowrate ***F*** changes. Thus in this case, the input variable is ***F***, while the output is ***h***. Owing to the fact that a capillary constitutes a laminar resistance, the head flow relationship is given by the equation:

$$\boldsymbol{h = R\,F} \qquad (\mathbf{10})$$

where ***R*** is the resistance. In terms of deviation from an initial steady-state $\boldsymbol{h_0}$ and $\boldsymbol{F_0}$, Eq. (10) becomes:

$$\boldsymbol{y = R\,u} \qquad (\mathbf{11})$$

With: $\boldsymbol{y = h - h_0}$ and $\boldsymbol{u = F - F_0}$

The important point to note here is that any change in ***F*** is instantaneously transmitted as a change in the head, ***h***. Other physical systems that exhibit a pure gain response include electrical resistors and the mechanical spring. First-order or higher order systems whose dynamics are extremely fast may be conveniently approximated as pure gain processes. For example, a small pneumatic control valve with very rapid response may be approximated as a pure gain process if the other parts of the process have much larger time constants.

Response of Pure Gain System to Various Inputs

From the mathematical representation of the pure gain process in Eq. (7) or (8), it is easy to see that:

The output of a pure gain process is directly proportional to the input, the constant of proportionality being the process gain.

Thus the responses are identical in form to the input (or forcing) functions, differing only in magnitude. The input function is amplified if $K > 1$, attenuated if $K < 1$, and left unchanged if $K = 1$. The following is a catalog of the responses of a pure gain process to various input functions.

Step Response

Input

$$u(t) = \begin{cases} 0; & t < 0 \\ A; & t > 0 \end{cases}$$

Output

$$y(t) = \begin{cases} 0; & t < 0 \\ AK; & t > 0 \end{cases}$$

Impulse Response

Input

$$u(t) = A\,\delta(t)$$

Output

$$y(t) = AK\,\delta(t)$$

Rectangular Pulse Response

Input

$$u(t) = \begin{cases} 0; & t < 0 \\ A; & 0 < t < b \\ 0; & t > b \end{cases}$$

Output

$$y(t) = \begin{cases} 0; & t < 0 \\ AK; & 0 < t < b \\ 0; & t > b \end{cases}$$

Ramp Response

Input

$$u(t) = \begin{cases} 0; & t < 0 \\ At; & t > 0 \end{cases}$$

Output

$$y(t) = \begin{cases} 0; & t < 0 \\ AKt; & t > 0 \end{cases}$$

Sinusoidal Response

Input

$$u(t) = \begin{cases} 0; & t < 0 \\ A\sin\omega t; & t > 0 \end{cases}$$

Output

$$y(t) = \begin{cases} 0; & t < 0 \\ AK\sin\omega t; & t > 0 \end{cases}$$

We note from here that the frequency response of the pure gain process indicates an amplitude ratio $AR = K$, and a phase angle $\phi = 0$. In other words, the **UPR** of the pure gain process is a sine wave whose amplitude is **K** times the amplitude of the input sine wave, and perfectly in phase with the input. This, of course, is in keeping with the characteristic of the pure gain process.

Pure Capacity Systems

Let us return to the general, first-order differential equation used to model the dynamic behavior of first-order systems:

$$\boldsymbol{a_1 \frac{dy}{dt} + a_0 y = b\, u(t)}$$

It is now of interest to investigate what happens when $\boldsymbol{a_0 = 0}$ in this equation. Observe that in this case, the original equation becomes:

$$\boldsymbol{a_1 \frac{dy}{dt} = b\, u(t)} \qquad (\mathbf{12})$$

$$\boldsymbol{\frac{dy}{dt} = K^*\, u(t)} \qquad where \;\; \boldsymbol{K^* = b/a_1} \qquad (\mathbf{13})$$

A process modeled by Eq. (13) is known as a ***pure capacity process***. The solution to Eq. (12) is easily obtained by direct integration; the result is, assuming zero initial conditions:

$$\boldsymbol{y(t) = K^* \int_0^t u(\sigma)\, d\sigma} \qquad (\mathbf{14})$$

Taking Laplace transforms in Eq. (13) gives:

$$\boldsymbol{y(s) = \frac{K^*}{s} u(s)} \qquad (\mathbf{15})$$

from which we obtain the transfer function for the pure capacity system as:

$$\boldsymbol{G(s) = \frac{K^*}{s}} \qquad (\mathbf{16})$$

The pure capacity system is therefore characterized by the presence of the integrator (or capacitance) element ***1/s***, and the parameter ***K****, which may be regarded as an integrator gain. We may thus imagine a pure capacity system to be a first-order system whose *time constant* and *steady-state gain* are both extremely large but for which the ratio of these parameters is a fixed, finite constant.

The most common example of a pure capacity system is a storage (or surge) tank with an outlet pump such as shown in Figure 3. Such a tank is typically used in the process industries for intermediate storage between two processes. The outflow, ***F*** (usually fixed), is set by the pump; its value is therefore independent of the liquid level, ***h***, in the tank. The inflow $\boldsymbol{F_i}$ is usually the input variable, and its value can vary. If $\boldsymbol{A_c}$ is the cross-sectional area of the storage tank (assumed uniform), then a material balance on the tank yields:

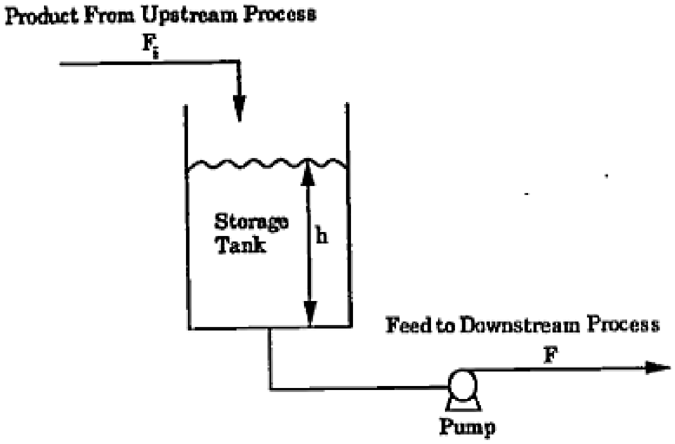


Fig. 3. The storage tank with an outlet pump

$$A_c \frac{dh}{dt} = F_i - F \qquad (17)$$

At steady state, for a fixed value of *F*, Eq. (17) becomes:

$$0 = F_{is} - F \qquad (18)$$

Subtracting Eq. (18) from Eq. (17) and defining the deviation variables $y = h - h_s$, $u = F_i - F_{is}$, we obtain:

$$\frac{dy}{dt} = \frac{1}{A_c} u \qquad (19)$$

from where Laplace transformation gives:

$$y(s) = \frac{1/A_c}{s} u(s) \qquad (20)$$

which is of the same form as Eq. (15), with $\mathbf{K^* = 1/ A_c}$.

Other examples of pure capacity processes include the heating of well-insulated batch systems, the filling of tanks with no outlet, the batch preparation of solutions by addition of chemicals to solvent, etc.

Response of Pure Capacity System to Various Inputs

Once again, by combining the transfer function of the pure capacity system (Eq. 16) with the Laplace transform of the input function in question, we may use the transfer function model in Eq. (15) to derive this process system's response to various input functions.

Step Response

We have, in this case:

$$\boldsymbol{y(s) = \frac{K^{*}}{s}\frac{A}{s} = \frac{AK^{*}}{s^{2}}} \qquad \boldsymbol{(21)}$$

which is easily inverted back to time, giving:

$$\boldsymbol{y(t) = AK^{*}t} \qquad \boldsymbol{(22)}$$

the equation of a ramp function with slope AK*.

The following is a catalog of the pure capacity process response to the other input functions.

Rectangular Pulse Response

$$y(t) = \begin{cases} 0; & t < 0 \\ AK^{*}t; & 0 < t < b \\ AK^{*}b; & t > b \end{cases}$$

A response that starts out as a ramp function, and subsequently settles down to a new steady-state value $\boldsymbol{AK^{*}b}$.

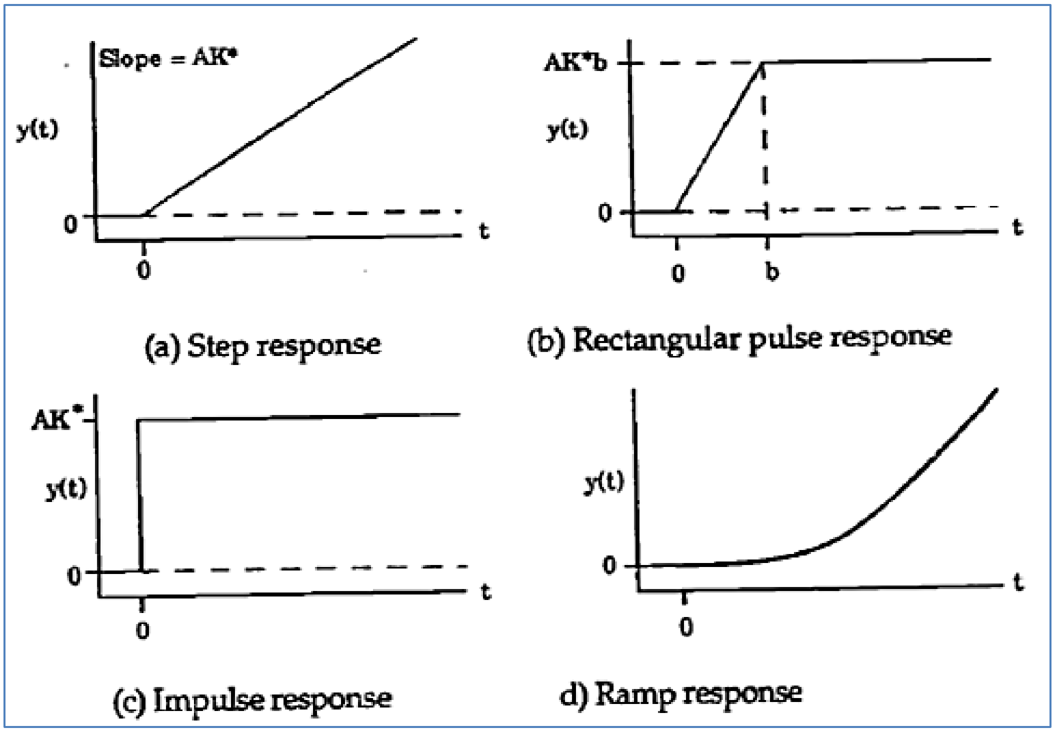
Impulse Response

$$\boldsymbol{y(t) = AK^{*}}$$

a step function.

Ramp Response

$$\boldsymbol{y(t) = \frac{AK^{*}t^{2}}{2}}$$



(a) Step response (b) Rectangular pulse response

(c) Impulse response d) Ramp response

Fig. 3. Pure capacity system responses

Sinusoidal Response

$$y(t) = \frac{AK^*}{\omega}(1 - \cos\omega t)$$

or

$$y(t) = \frac{AK^*}{\omega}[1 + \sin(\omega t - 90°)]$$

implying that the *frequency response* of a pure capacity process has an amplitude ratio and a phase angle given by:

$$AR = \frac{K^*}{\omega} \qquad (23)$$

$$\phi = -90° \qquad (24)$$

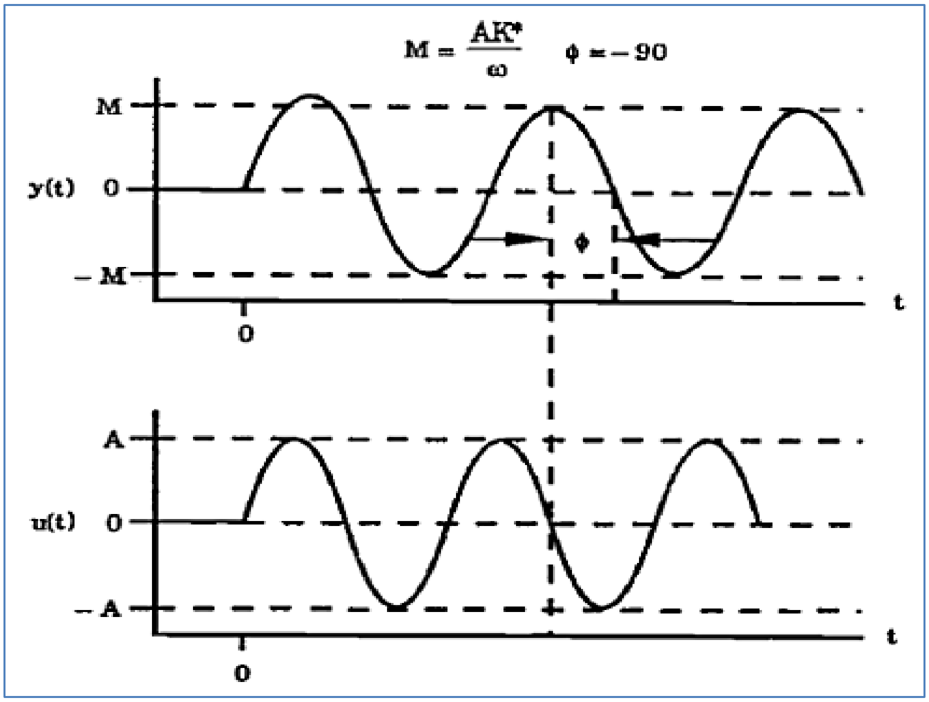


Fig. 4. Sinusoidal response of a pure capacity system

Processes Control:

How to calculate the time constant (τ) for first order system

1) Mathematical method

Using the definitions

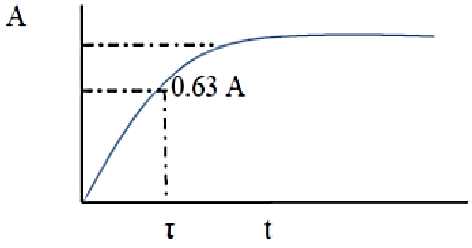
$$\tau = \frac{mC}{hA} \qquad \textit{Thermometer}$$

$$\tau = {}^{A_c}/_{C} = A_c R \qquad \textit{Liquid level tank}$$

$$\tau = \frac{V/F}{{}^{Vk}/_{F} + 1} \qquad \textit{CSTR}$$

$$\tau = {}^{V}/_{F} \qquad \textit{Stirred heating tank}$$

2) Exponential method (Step change in the input variable)



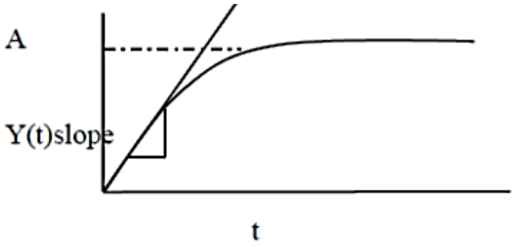
$$Y(t) = A(1 - e^{-t/\tau})$$

$$as\ t \to \infty \qquad Y(\infty) = A(1 - e^{-\infty}) = A$$

$$as\ t \to \tau \qquad Y(\tau) = A(1 - e^{-1}) = A(1 - 0.3678) \approx 0.63A$$

Time constant (τ) is the time required for the response to reach 63% of the its ultimate value.

3. Third method



$$Y(t) = A(1 - e^{-t/\tau})$$

$$\frac{dy}{dt} = -Ae^{-t/\tau}(-1/\tau) = \frac{A}{\tau}e^{-t/\tau}$$

$$\lim_{t\to 0}\frac{dy}{dt} = \frac{A}{\tau}e^{0} = \frac{A}{\tau}$$

Slope of the tangent at t=0 is $\frac{A}{\tau}$

Therfore $\tau = \frac{A}{slope}$

$$Y(t) = A(1 - e^{-t/\tau})$$

$$Y(t) = A - Ae^{-t/\tau}$$

$$Ae^{-t/\tau} = A - Y(t)$$

$$e^{-t/\tau} = \frac{A - Y(t)}{A}$$

$$-t/\tau = \ln\frac{A - Y(t)}{A}$$

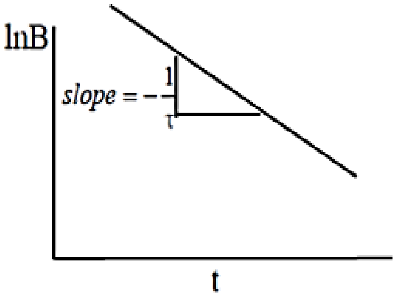
Let $B = \frac{A - Y(t)}{A}$

$$\ln B = \frac{-1}{\tau}t$$

$let\ y = \ln B \quad and\ x = t$

$$slope = -\frac{1}{\tau}$$

Y(t)	$B = \frac{A - Y(t)}{A}$	$\ln B$	t



$$slope = -\frac{1}{\tau}$$

$$\tau = -\frac{1}{slope}$$

Two First-Order Systems in Series

Consider the situation in which two liquid holding tanks are connected in series, each one being of the type identified as a first-order system. Such systems can be configured in one of two ways. Let us consider the first of such configurations shown in Figure 1 below.

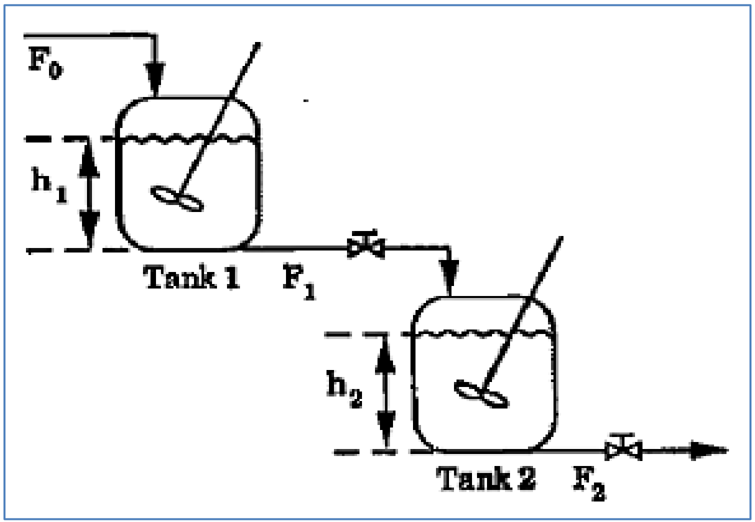


Fig. 1. Two noninteracting tanks in series

This indicates a situation in which the flow out of tank 1 discharges freely into the atmosphere before entering tank 2. The flowrate of this stream is therefore approximately determined by the liquid level in tank 1 and is totally independent of the conditions in tank 2. Thus while the conditions in tank 1 influence the conditions in tank 2, observe carefully that the conditions in tank 2 in no way influence tank 1; i.e., tank 2 does not interact with tank 1. This is the ***noninteracting configuration***.

Contrast the arrangement in Figure 1 with the one shown in Figure 2. In this case, the flow out of tank 1 now depends on the difference between the levels in the two tanks, with the result that variations in the conditions in tank 2 will influence the conditions in tank 1. Such an arrangement gives rise to a situation in which tank 2 does in fact interact with tank 1; it is therefore called the ***interacting configuration***.

As we might expect, the dynamic behavior of the interacting ensemble will differ from the dynamic behavior of the alternative noninteracting arrangement.

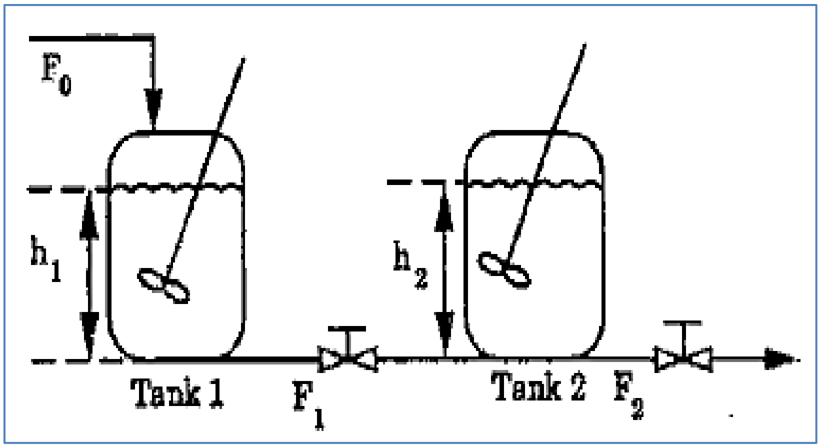


Fig. 2. Two interacting tanks in series

Two Noninteracting Systems in Series

Let us, for simplicity, assume a linear relation between liquid level and flowrate through the outlet valves in the system shown in Figure 1, so that outflows will be directly proportional to the liquid levels. If A_1 and A_2 represent the (uniform) cross-sectional areas of tanks 1 and 2 respectively, then the following mathematical models are obtained by carrying out material balances for each tank:

For Tank 1:

$$A_1 \frac{dh_1}{dt} = F_0 - c_1 h_1 \qquad (1)$$

For Tank 2 the inflow is given by $F_1 = c_1 h_1$; thus we have:

$$A_2 \frac{dh_2}{dt} = c_1 h_1 - c_2 h_2 \qquad (2)$$

These two equations may be rearranged and presented in terms of deviation variables in the usual manner; the results are:

$$\tau_1 \frac{dy_1}{dt} = -y_1 + K_1 u \qquad (3)$$

$$\tau_2 \frac{dy_2}{dt} = -y_2 + K_2 y_1 \qquad (4)$$

where the deviation variables are given by $y_1 = h_1 - h_{1s}$; $y_2 = h_2 - h_{2s}$, and $u = F_0 - F_{0s}$; and the system parameters are given by:

$$\tau_1 = \frac{A_1}{c_1} \quad ; \quad \tau_2 = \frac{A_2}{c_2} \quad ; \quad K_1 = \frac{1}{c_1} \quad ; \quad K_2 = \frac{c_1}{c_2}$$

It is important to note the following facts about this system:

1. Each component tank is modeled by a first-order differential equation; therefore the entire process is an ensemble of two first-order systems in series.
2. The individual transfer function representation for each of the component tanks may be obtained by taking Laplace transforms in Eqs. (3) and (4).

$$y_1(s) = \frac{K_1}{\tau_1 s + 1} u(s) = G_1(s) \qquad (5)$$

$$y_2(s) = \frac{K_2}{\tau_2 s + 1} y_1(s) = G_2(s) \qquad (6)$$

The results are clearly identifying each tank as a first-order system in its own right. Now, for this system, we are interested in studying the effect of changes in the input function F_0 , on the level in the second tank; i.e., we wish to investigate the influence of u on y_2.

The transfer function representation for the composite system relating $y_2(s)$ directly to $u(s)$ is shown in Figure 3 and can be obtained by substituting Eq. (5) for y_1 in Eq. (6); the result is:

$$y_2(s) = \frac{K_2}{\tau_2 s + 1} \frac{K_1}{\tau_1 s + 1} u(s) \qquad (7)$$

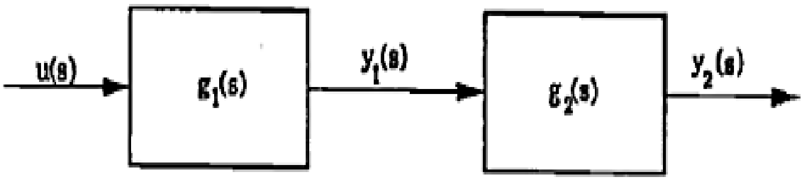


Fig. 3. Block diagramatic representation of two first-order systems connected in series.

The overall transfer function for this process is therefore now given by:

$$G(s) = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (8)$$

with the characteristic parameters:

- **K = K$_1$K$_2$**, the "***combined***" steady-state gain,
- Two time constants, τ_1 and τ_2, the individual time constants of the contributing first-order elements.

The transfer function for this process therefore has two poles (at $s = -1/\tau_1$ and $s = -1/\tau_2$) and no zeros.

A very important point to note here is that the overall transfer function for this process is in fact just a product of the two contributing transfer functions indicated in Eqs. (5) and (6). A block diagramatic representation of this system shown in Figure 2 emphasizes this point.

Unit Step Response of the Noninteracting Systems in Series

We may now use the transfer function of Eq. (7) to obtain expressions for how this system responds to various inputs. When the input function is the *unit step function* **[*u(s)=1/s*]**, for example, the system response may be obtained from:

$$y_2(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{1}{s} \qquad (9)$$

Upon partial fraction expansion Eq. (9) is easily inverted (provided $\tau_1 \neq \tau_2$) to give:

$$y_2(t) = K[1 - \left(\frac{\tau_1}{\tau_1 - \tau_2}\right) e^{-t/\tau_1} - \left(\frac{\tau_2}{\tau_2 - \tau_1}\right) e^{-t/\tau_2} \qquad (10)$$

In the special case when $\tau_1 = \tau_2 = \tau$, Eq. (9) becomes:

$$y_2(s) = \frac{K}{(\tau s + 1)^2} \frac{1}{s} \qquad (11)$$

and Laplace inversion yields:

$$y_2(t) = K(1 - e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau}) \qquad (12)$$

H.W. 16.1. *Find Laplace invers for Eq. (9).*

A sketch of the two-tank system response is shown in Figure 4 and compared with the first-order response behavior of a single tank. [$y_i(t)/K_i$ vs. (t: time)]

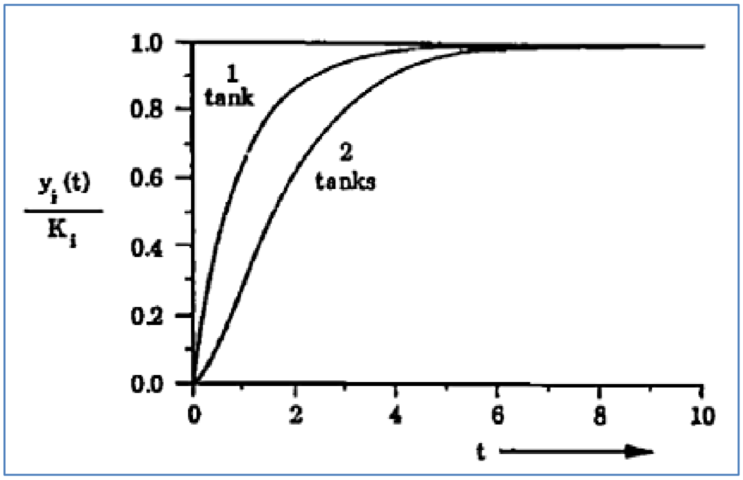


Fig. 4. Unit step responses of the liquid level in the two-tank process

The important points to note about these responses are:

1. While the single first-order system response is instantaneous, the response of the two-tank system shows a sigmoidal behavior characterized by an initial sluggishness (at $t = 0$) followed by a speeding up prior to the final approach to steady state. Note the presence of an inflection point in the response curve.
2. The slope of the two-tank response at the origin is zero. which is to be compared with the nonzero initial slope of the single, first-order response.

The Interacting System

Let us now consider the interacting system arrangement of Figure 2. The material balances for this arrangement are as shown below:

For Tank 1:

$$A_1 \frac{dh_1}{dt} = F_0 - F_1 \qquad (13)$$

For Tank 2:

$$A_2\frac{dh_2}{dt} = F_1 - F_2 \qquad (14)$$

Assuming linear resistances, the flowrate out of tank 2, as before, is still given by $F_2 = c_2h_2$; but now F_1 is given by:

$$F_1 = c_1(h_1 - h_2)$$

The mathematical models Eqs. (13) and (14) therefore become:

$$\frac{A_1}{c_1}\frac{dh_1}{dt} = -h_1 + h_2 + \frac{F_0}{c_1} \qquad (15)$$

$$\frac{A_2}{c_2}\frac{dh_2}{dt} = \frac{c_1}{c_2}h_1 - (1 + \frac{c_1}{c_2})h_2 \qquad (16)$$

As was done in the noninteracting case (in producing Eqs. (3) and (4)), if we now introduce the same constants, and express these equations in terms of the usual deviation variables, taking Laplace transforms and rearranging the resulting expressions will give rise to the following transfer function model:

$$y_2(s) = \frac{K}{\tau_1\tau_2s^2 + (\tau_1 + \tau_2 + K_2\tau_1)s + 1}u(s) \qquad (17)$$

where, once again, $K = K_1K_2$.

***H.W. 16.2**. Find Eq. (17) obtained.*

Observe now that the difference between the transfer function for the noninteracting case (given in Eq. (7)) and the one indicated in Eq. (17) for the interacting case lies in the presence of the term $K_2\tau_1$ in the coefficient of sin the denominator polynomial in Eq. (17). This is appropriately indicative of the interaction of tank 2 (via its steady-state gain K_2) with tank 1 (via its time constant τ_1).

Example 16.1. Consider the situation in which the system of Figure 1 consists of two identical tanks with identical time constants $\tau_1 = \tau_2 = 1$ **minute**. The steady-state gains are also identical, and equal to 1; i.e., $K_1 = K_2 = 1$.

1. Obtain an expression for the response of $\mathbf{y_2}$ (the level in the second tank as a deviation from its initial steady-state value) to a unit step change in the inlet flowrate to tank 1.
2. If these same tanks are now arranged as in Figure 2, obtain the unit step response for this interacting configuration and compare with the response obtained in (1) for the noninteracting case.

Ans.:

1. In this case since $\boldsymbol{\tau_1 = \tau_2}$, the required response may be obtained by directly appealing to our earlier result in Eq. (12) for $\boldsymbol{\tau = 1}$ and ***K = 1***; what we have is the following result:

$$\boldsymbol{y_2(t) = (1 - e^{-t} - te^{-t})} \qquad (\mathbf{18})$$

2. In the interacting situation, introducing the given parameter values into Eq. (17), we find that the transfer function model is given by:

$$\boldsymbol{y_2(s) = \frac{1}{s^2 + 3s + 1} u(s)}$$

which, for a unit step input, becomes (after factorization of the denominator quadratic):

$$\boldsymbol{y_2(s) = \frac{1}{(2.618s + 1)(0.382s + 1)} \frac{1}{s}} \qquad (\mathbf{19})$$

Before carrying out the final step of Laplace inversion in Eq. (19), note first of all that this equation, when compared with Eq. (9), indicates that the interacting configuration of two first-order systems in series, each having a time constant of ***1 minute***, is equivalent to a noninteracting configuration of two first-order systems having unequal time constants ***2.618 min*** and ***0.382 min***. Thus the first effect of interaction is to alter the effective time constants of the individual contributing first-order systems. Laplace inversion of Eq. (19) yields:

$$\boldsymbol{y_2(t) = (1 - 1.171e^{-0.382t} - 0.171e^{-2.618t})}$$

A plot of the two responses (18) and (19) are shown in Figure 5. It is now clear that the main effect of interaction is to make the response more sluggish. It should be easy to see, from purely physical grounds, that this is indeed to be expected once we take cognizance of the fact that with the interacting arrangement, any increase in tank 2 level will reduce the flow from tank 1, further slowing down the rate at which this level will increase. This is not the case in the noninteracting arrangement.

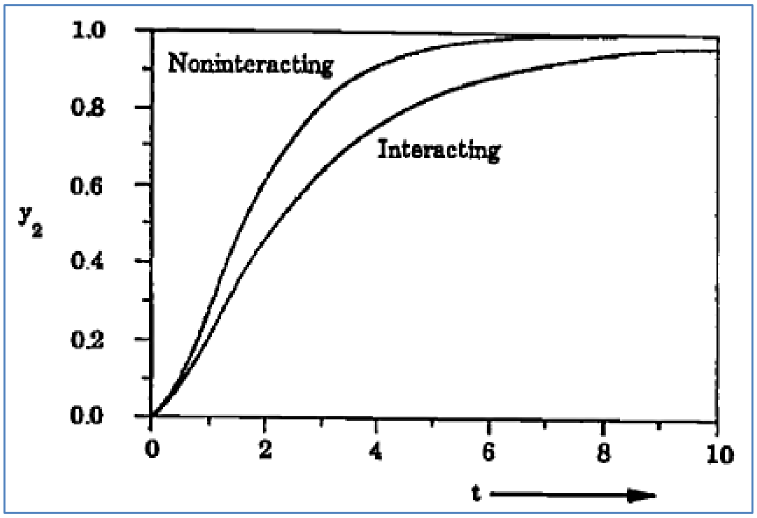


Fig. 5. Step response of interacting and noninteracting tanks in series

Processes Control:

The Lead/Lag System

The dynamic system whose transfer function is given by:

$$\boldsymbol{G(s) = K\frac{(\xi s + 1)}{(\tau s + 1)}} \qquad (\mathbf{1})$$

is known as a *lead/lag system* . It is important to note that the one - and very critical – difference between this transfer function and that for the first-order system is the presence of the additional first-order numerator term. Both transfer functions otherwise have denominator polynomials of identical order. Let us now recall that when considering the frequency response of a first-order system, we discovered that the ***UPR*** sine wave lags behind the input sine wave by $\boldsymbol{\phi}$ degrees. For this reason, first-order terms appearing in the denominator of a transfer function are sometimes referred to as first-order lags. For a system whose transfer function is of the form:

$$\boldsymbol{G(s) = K(\xi s + 1)} \qquad (\mathbf{2})$$

with no denominator polynomial, frequency response analysis results indicate that the ***UPR*** sine wave leads the input sine wave by:

$$\boldsymbol{\phi = \; tan^{-1}(\omega\xi)} \qquad (\mathbf{3})$$

For this reason, first-order numerator terms in a transfer function are referred to as first-order *leads*. Because the system currently under consideration has *both* a lead and a lag term, it is therefore naturally referred to as a lead/lag system. The characteristic parameters of the lead/lag system are: K, the system gain; $\boldsymbol{\xi}$ the lead time constant, $\boldsymbol{\tau}$, the lag time constant; and the "lead-to-lag" ratio, $\boldsymbol{\rho = \xi/\tau}$.

Although it is rare to find a chemical process whose behavior is characterized by the lead/lag transfer function of Eq. (1), it is important to understand them because these systems find important applications in control system design such as that in the implementation of feed-forward control systems.

Lead/lag systems are typically implemented in control systems as analog units, or electronic (microprocessor-based) systems that operate according to the transfer function form given in Eq. (1).

In investigating the response of the lead/lag system to various inputs, we find it more informative first to recast the somewhat strange transfer function in Eq. (1) into a more familiar form. Observe that by partial fraction expansion, we may break up the transfer function into two terms as follows:

$$G(s) = K\frac{(\xi s+1)}{(\tau s+1)} = K[A_0 + \frac{A_1}{(\tau s+1)}] \qquad (4)$$

where the constants A_0 and A_1 are easily found to be given by:

$$A_0 = \frac{\xi}{\tau} = \rho \qquad ; \quad A_1 = 1 - \frac{\xi}{\tau} = 1 - \rho$$

Thus the dynamic behavior of the lead/lag system may be determined using the following transfer function representation:

$$y(s) = G(s)\,u(s) = [\rho K + (1-\rho)(\frac{K}{\tau s+1})]\,u(s) \qquad (5)$$

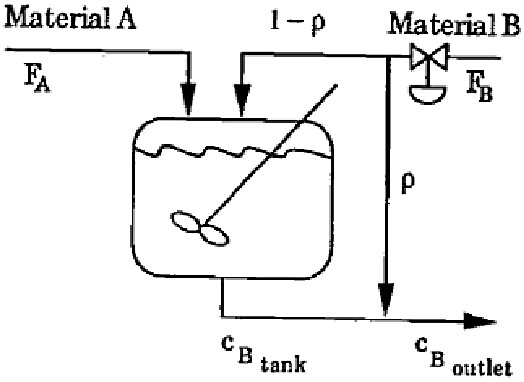
A careful observation of Eq. (5) now reveals a very important fact about lead/lag systems:

The dynamic behavior of the lead/lag system is a weighted average of the dynamic behavior of a pure gain system and that of a first-order system, the weighting factor being ρ*, the "lead-to-lag" ratio.*

This fact significantly simplifies the task of obtaining the dynamic response of lead/lag systems to various input functions. For example, observe that if $y_g(t)$ is the response of a pure gain system (Gain= K) to a certain input function, and $y_1(t)$ is the response of the first-order system (Gain K, time constant τ to the same input function, then the response of the lead/lag system (Gain K, lead ξ, lag τ , and lead-to-lag ratio ρ) is given by:

$$y(s) = \rho\, y_g(t) + (1-\rho)\, y_1(t) \qquad (6)$$

Example 17.1. Consider the mixing tank shown in the following Figure. A large flowrate F_A of material A is fed into a tank of volume V, where a tiny quantity F_B of material B (e.g., a dye, catalyst, etc.) is mixed with it. However, it has been found that in order to achieve a fast changeover from one mixture to another, it is desirable also to feed a fraction ρ of the B feed stream directly into the tank outlet.



***Ans.*:** In this case the outlet concentration of the B, c_B, would be given by:

$$V\frac{dc_{B_{tank}}}{dt} = (1-\rho)F_B c_{Bf} - [F_A + (1-\rho)F_B]c_{B_{tank}} \qquad (7)$$

$$c_{B_{outlet}} = \frac{[F_A + (1-\rho)F_B]c_{B_{tank}} + \rho F_B c_{Bf}}{F_A + F_B} \qquad (8)$$

If we note that $F_A \gg F_B$, and define:

$$\tau = \frac{V}{F_A}$$

then it is reasonable to make the following approximations:

$$F_A + F_B = F_A + (1-\rho)F_B \approx F_A$$

and the process model simplifies to:

$$\tau\frac{dc_{B_{tank}}}{dt} = [\frac{(1-\rho)F_B}{F_A}]c_{Bf} - c_{B_{tank}} \qquad (9)$$

$$c_{B_{outlet}} = c_{B_{tank}} + (\frac{\rho F_B}{F_A})\, c_{Bf} \qquad (10)$$

if we now define the following variables:

$$y = c_{B_{outlet}} - (c_{B_{outlet}})_s$$

$$u = \frac{F_B}{F_A}\, c_{Bf} - (\frac{F_B}{F_A}\, c_{Bf})_s$$

$$x = c_{B_{tank}} - (c_{B_{tank}})_s$$

Then in equations 9 and 10,

$$\tau \frac{dx(t)}{dt} = (1-\rho)u(t) - x(t) \qquad (11)$$

$$y(t) = x(t) + \rho\, u(t) \qquad (12)$$

and by taking the Laplace transform we obtain:

$$x(s) = \frac{(1-\rho)}{\tau s + 1} u(s)$$

$$y(s) = x(s) + \rho\, u(s) = \frac{(1-\rho)}{\tau s + 1} u(s) + \rho\, u(s) = \left[\frac{(1-\rho)}{\tau s + 1} + \rho\right] u(s)$$

Or if we let $\rho = \xi/\tau$

$$y(s) = \frac{(\xi s + 1)}{(\tau s + 1)} u(s)$$

We now note that the transfer function for a process will involve a lead term whenever the state-space model has an output depending directly on an input, as in this case (see Eq. (12).

Unit Step Response of the Lead/Lag System

The most important response of the lead/lag system is the *unit step response*, its response to a step input function of unit magnitude. Taking advantage of Eqs. (5) and (6), since from previous discussions in previous lectures we already know that the unit step responses of the pure gain and first-order components are respectively given by:

$$y_g(t) = K \;\; ; \; y_1(t) = K(1 - e^{-\frac{t}{\tau}})$$

the required unit step response of the lead/lag system is easily seen to be given by:

$$\boldsymbol{y(s) = K[\rho + (1 - \rho)\left(1 - e^{-\frac{t}{\tau}}\right)]} \qquad (\mathbf{13})$$

This may now be rearranged to give:

$$\boldsymbol{y(s) = K[1 + (1 - \rho)e^{-\frac{t}{\tau}}]} \qquad (\mathbf{14})$$

The unit step response of the lead/lag system is shown in Figure 1 drawn for the situation with $\boldsymbol{\tau}$ = 1 minute.

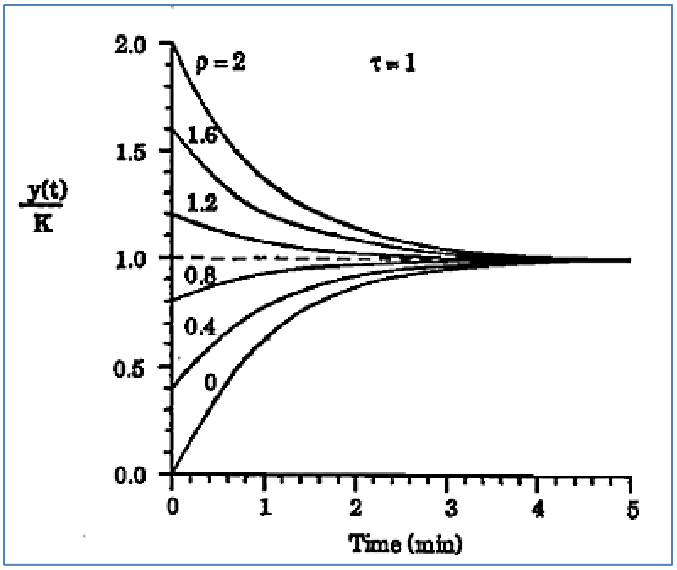


Fig. 1. The unit step response of the lead/lag system.

The important points to note about this response are:

1. At ***t = 0, y(t) = Kρ;*** i.e., the output experiences an immediate "jump" to the value ***Kρ***. The pure gain component is clearly responsible for this aspect of the response.

2. As indicated by Eqs. (13) or (14), when *t* →∞, ***y(t)*** attains to its final value ***K***, in exponential fashion, at a speed determined solely by $\boldsymbol{\tau}$, the lag time constant.

3. Observe that the initial value of ***y(t)*** is $K\rho$, while the final ultimate value is ***K;*** thus whether the initial value is higher or lower than the final value will be determined by whether ρ is greater or less than 1.

As we might therefore expect from note 3 above, three distinct types of responses are represented in this figure, and they depend on the parameter ρ.

CASE 1. $\rho < 1$ (i.e., $\xi < \tau$)

This case is characterized by an initial jump to a value lower than the ultimate value followed by an exponential approach to the final, higher steady-state value. To obtain this type of response requires $\xi < \tau$; i.e., a weaker lead (or a stronger lag). Such systems give the impression of being more reluctant to "move" by virtue of the dominant influence of the lag term. In the limit as $\xi \rightarrow 0$, the lead/lag system becomes a pure first-order system.

CASE 2. $\rho = 1$ (i.e., $\xi = \tau$)

Under conditions of identical lead and lag time constants, the lead/lag system becomes a pure gain system (by virtue of the cancellation of the identical numerator and denominator polynomials), attaining the ultimate value instantaneously; i.e., $y(t) = K$ for all times.

CASE 2. $\rho > 1$ (i.e., $\xi > \tau$)

The response in this case is characterized by an initial jump that overshoots the final value, followed by an exponential decay to the steady-state value. This type of response is obtained when $\xi > \tau$; i.e., when we have a stronger lead (or a weaker lag). The swiftness of this type of response- indicative of the dominant influence of the lead term - tends to give the impression of an impulse response of a first-order system.

It is worthwhile to remark once again that the approach to the final steady state is governed solely by the lag time constant; but the nature and value of the initial response is determined by the lead-to-lag ratio.

H.W. 17.1. *What relationship exists between the dynamic behavior of the lead/lag system and that of the pure gain system and the first-order system?*

Processes Control:

Degrees of freedom analysis

$$N_F = N_V - N_E \qquad (1)$$

1. $N_F = 0$: The process model is exactly specified.
2. $N_F > 0$: The process is underspecified.
3. $N_F < 0$: The process model is over-specified.

Laplace transforms of representative functions:

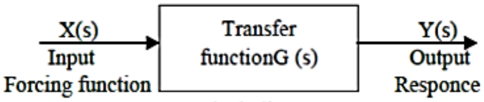
The Laplace transform of a function f(t) is defined as:

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt \qquad (2)$$

$$L\left(\frac{d^nf}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \qquad (3)$$

Transfer Function Models

$$\mathrm{T.F} = \frac{\text{Laplace transform of the output (responce)}}{\text{Laplace transform of the input (forcing function distubance)}}$$



Block diagram

$$\mathrm{T.F} = \mathrm{G(s)} = \frac{\mathrm{y(s)}}{\mathrm{x(s)}}$$

First-Order systems

$$\tau\frac{dy}{dx} + y = K\,x(t) \qquad (4)$$

$$G(s) = \frac{y(s)}{x(s)} = \left(\frac{K}{\tau s + 1}\right) \qquad (5)$$

Response of First-Order Systems to Various Inputs

$$\boldsymbol{y(s)=\left(\frac{K}{\tau s+1}\right)u(s)} \qquad \textit{where } \boldsymbol{y(s)}=? \quad \textit{for different input } \boldsymbol{x(s)} \qquad (6)$$

Step Response

$$\boldsymbol{y(t)=AK\left(1-e^{-t/\tau}\right)} \qquad (7)$$

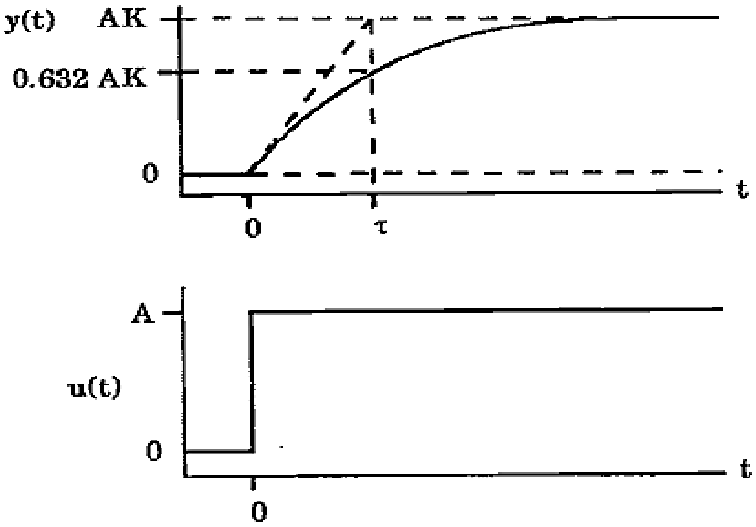


Fig. 1. Response of a first-order system to the step function

Rectangular Pulse Response

$$\boldsymbol{u(t)=\begin{cases}0 & t<0\\ A & 0\le t<b\\ 0 & t\ge b\end{cases}}$$

$$\boldsymbol{y(s)=\frac{AK}{s(\tau s+1)}\left(1-e^{-bs}\right)} \qquad (8)$$

$$\boldsymbol{y(t)=\begin{cases}AK\left(1-e^{-t/\tau}\right) & t<b\\ AK\left[\left(1-e^{-t/\tau}\right)-\left(1-e^{-(t-b)/\tau}\right)\right] & t>b\end{cases}}$$

$$\boldsymbol{y_{max}=y(b)=AK\left(1-e^{-b/\tau}\right).}$$

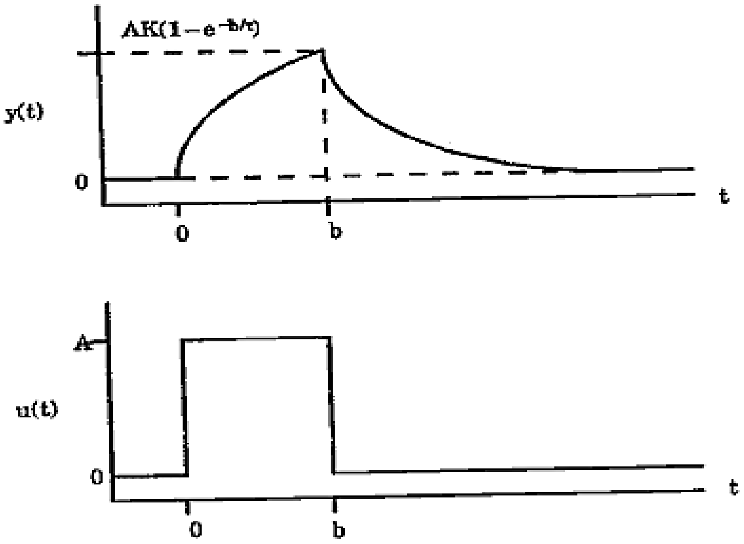


Fig. 2. Rectangular pulse response of a first-order system

Impulse Response

$$y(t) = \frac{KA}{\tau} e^{-t/\tau} \qquad (9)$$

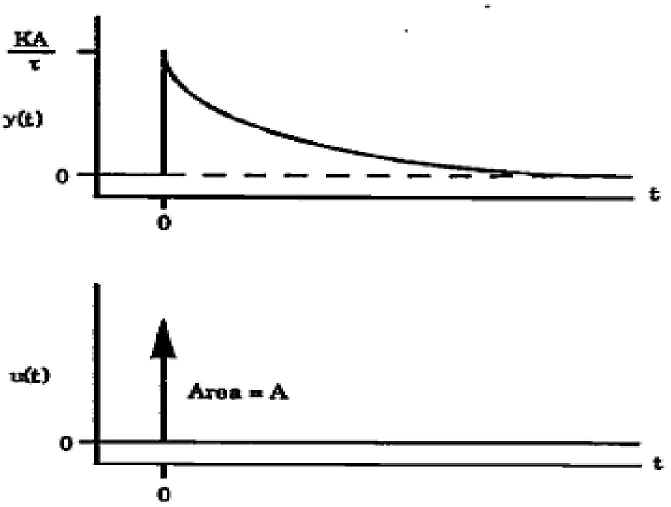


Fig. 3. Impulse response of a first-order system

Ramp Response

$$u(t) = \begin{cases} 0 & t < 0 \\ At & t \geq 0 \end{cases}$$

$$y(t) = AK\tau(e^{\frac{-t}{\tau}} + \frac{t}{\tau} - 1) \qquad (10)$$

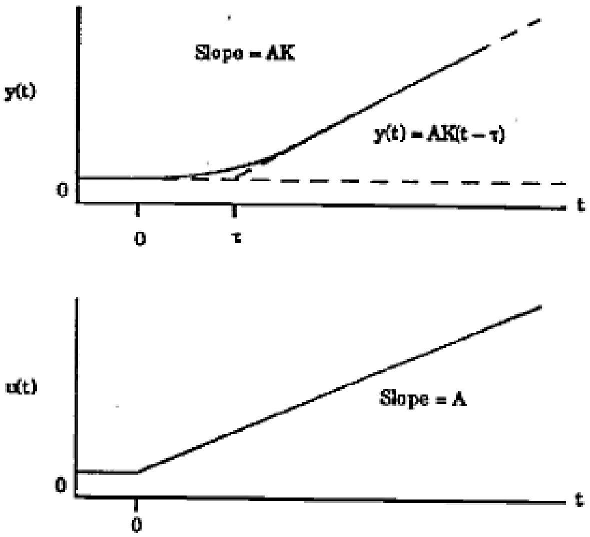


Fig. 4. Ramp response of a first-order system

Sinusoidal Response

$$\boldsymbol{y(t) = AK\left[\frac{\omega\tau}{(\omega\tau)^2+1}e^{-t/\tau} + \frac{1}{\sqrt{(\omega\tau)^2+1}}sin(\omega\tau+\phi)\right]} \tag{11}$$

$$\boldsymbol{y(t)|_{t\to\infty} = \frac{AK}{\sqrt{(\omega\tau)^2+1}}sin(\omega\tau+\phi)} \tag{12}$$

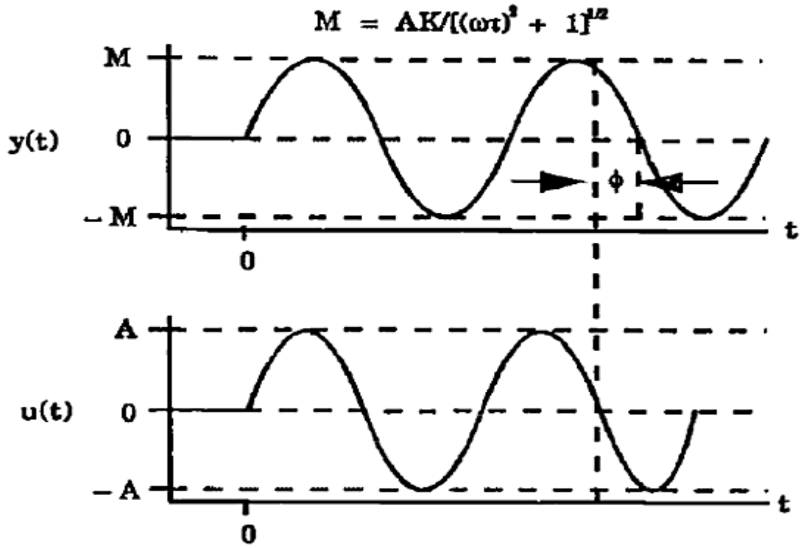


Fig. 5. Sinusoidal response of a first-order system

the **UPR**, can be characterized by the following two parameters:

$$AR = \frac{K}{\sqrt{(\omega\tau)^2 + 1}} \qquad (13)$$

$$\phi = tan^{-1}(-\omega\tau) \qquad (14)$$

$$phase\ lag = \frac{|\phi|}{360}\frac{1}{f} \qquad (15)$$

Pure Gain Systems

$$y(t) = K\,u(t) \qquad (16)$$

$$G(s) = K \qquad (17)$$

Response of Pure Gain System to Various Inputs

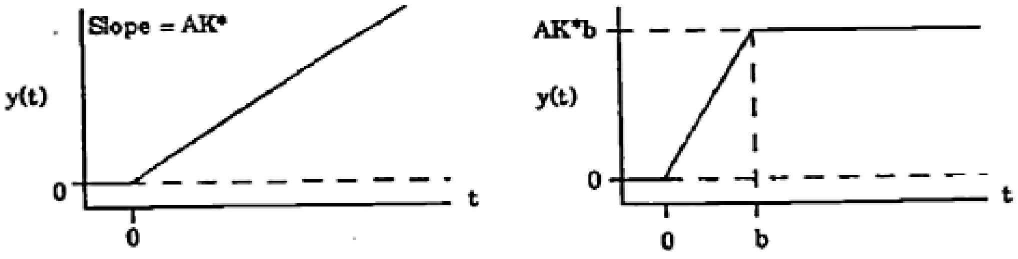
The output of a pure gain process is directly proportional to the input, the constant of proportionality being the process gain.

The input function is amplified if $K > 1$, attenuated if $K < 1$, and left unchanged if $K = 1$.

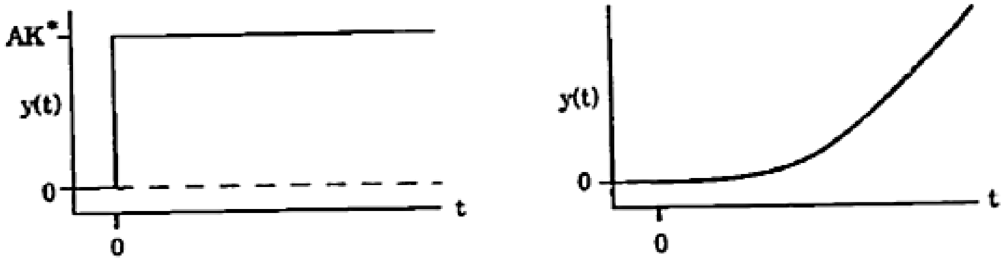
Pure Capacity Systems

$$G(s) = \frac{K^*}{s} \qquad (18)$$

Response of Pure Capacity System to Various Inputs



(a) Step response (b) Rectangular pulse response



(c) Impulse response d) Ramp response

Fig. 6. Pure capacity system responses

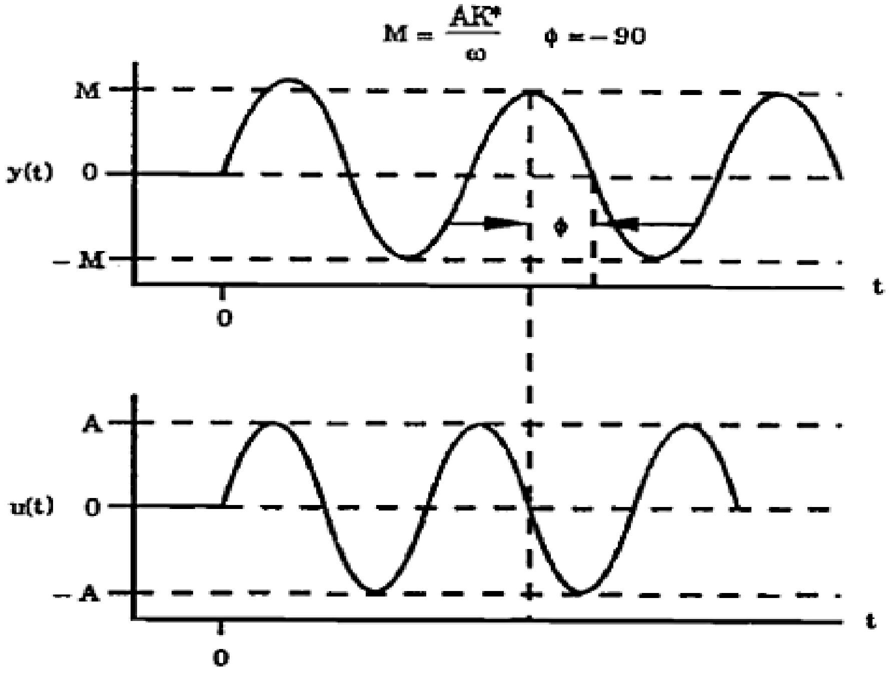


Fig. 7. Sinusoidal response of a pure capacity system

Processes Control:

Degrees of freedom

1- Distillation Column

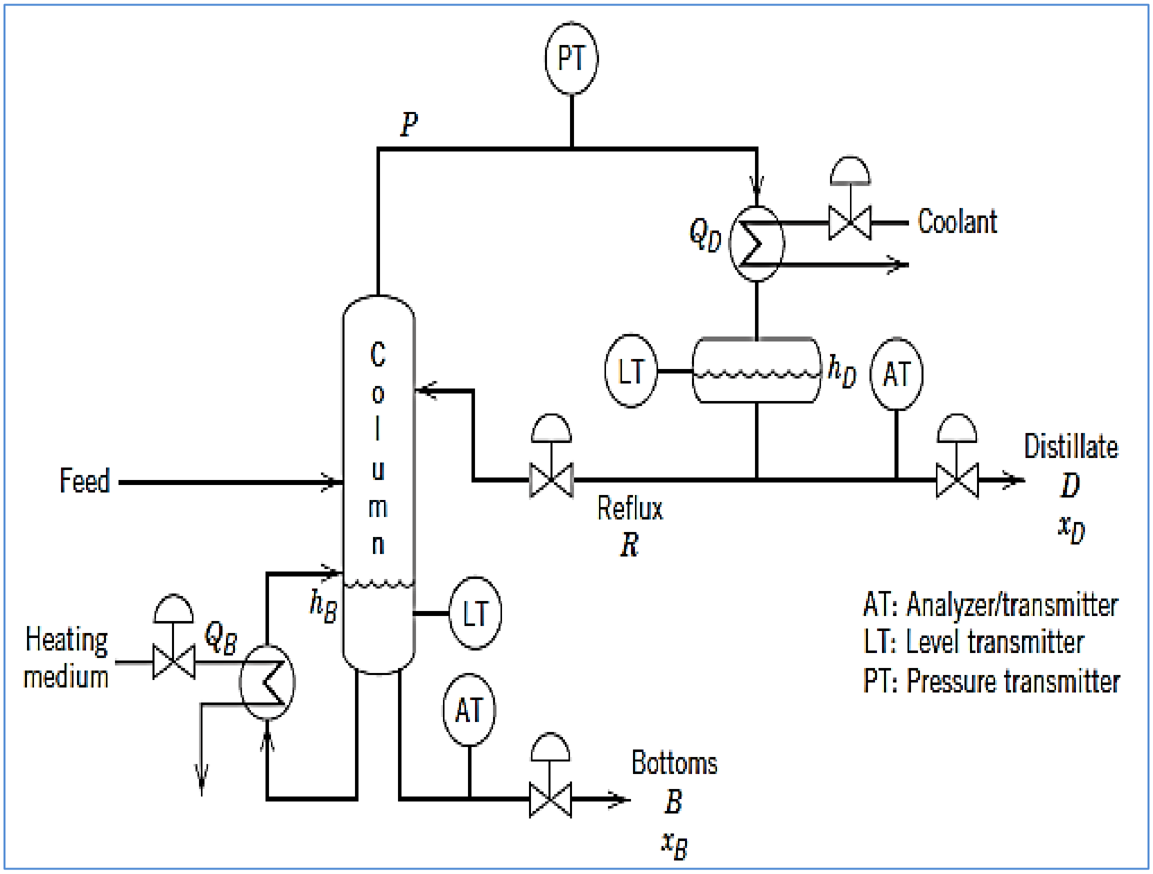


Fig. 1. Distillation column.

The controlled variables are product compositions, $\boldsymbol{x_D}$ and $\boldsymbol{x_B}$, column pressure, $\boldsymbol{P}$, and the liquid levels in the reflux drum and column base, $\boldsymbol{h_D}$ and $\boldsymbol{h_B}$. The five manipulated variables are product flow rates, **D** and **B**, reflux flow, **R**, and the heat duties for the condenser and reboiler, $\boldsymbol{Q_D}$ and $\boldsymbol{Q_B}$.

2- Blending Tank

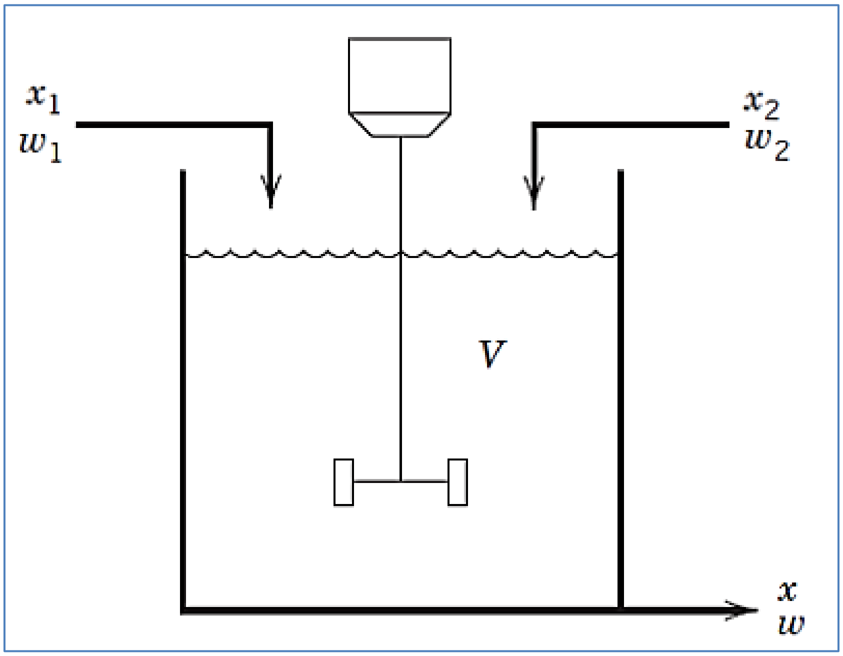


Fig. 2. The blending process

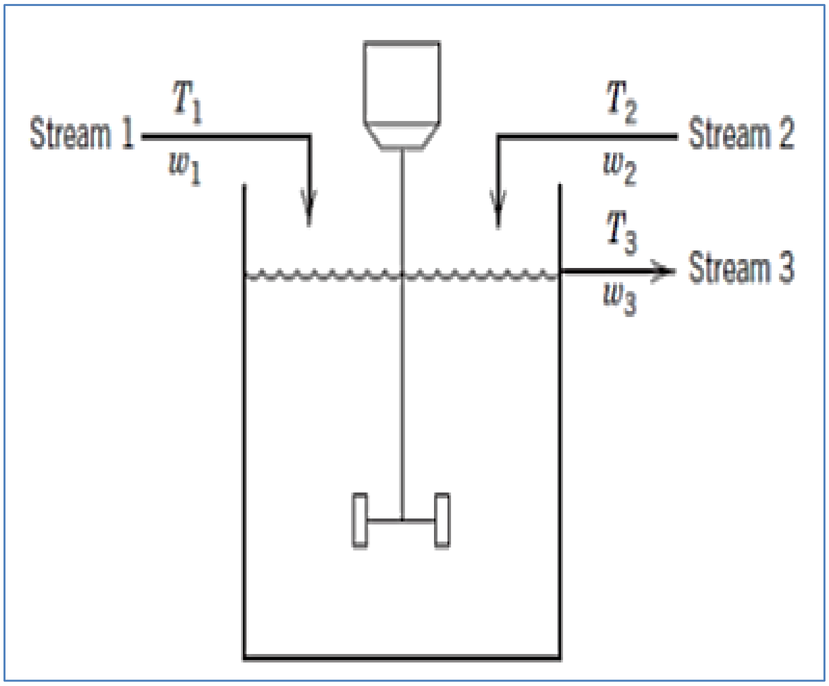


Fig. 3. A liquid-level storage process**.**

3- Stirred-Tank Heating Process

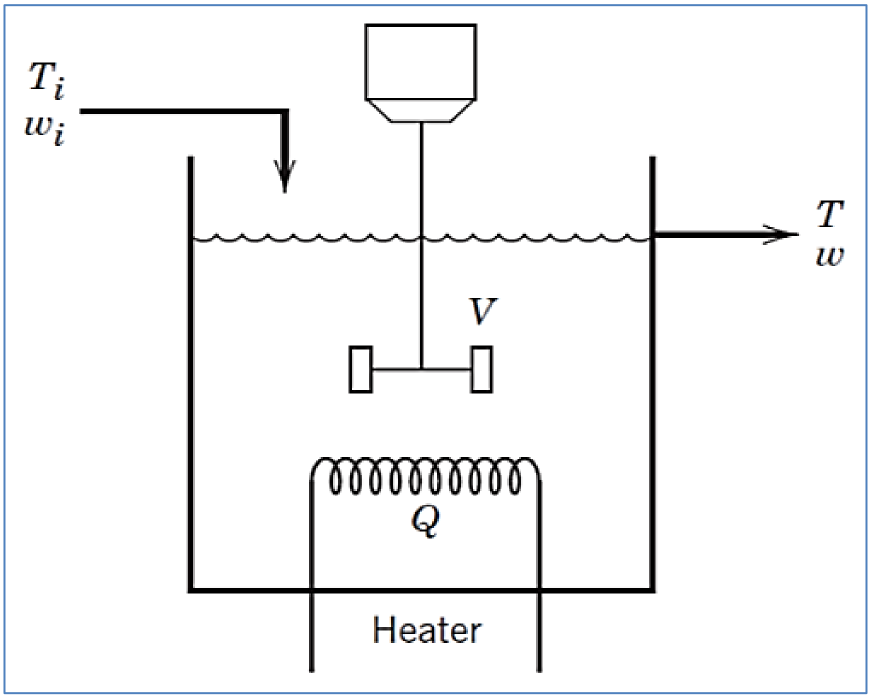


Fig. 4. Stirred-tank heating process with constant holdup, **V.**

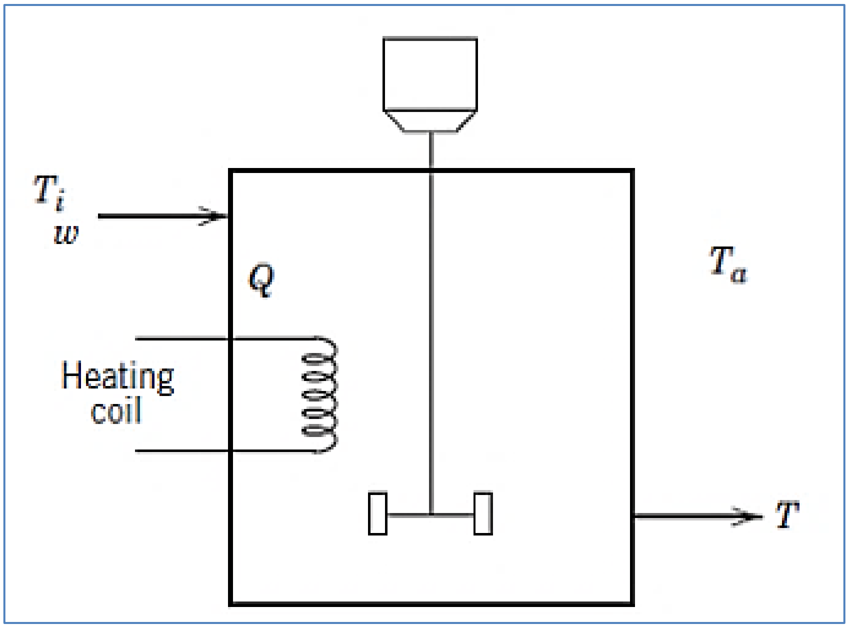


Fig. 5. Stirred-tank heating process with constant holdup, **V.**

4- Liquid Storage Systems

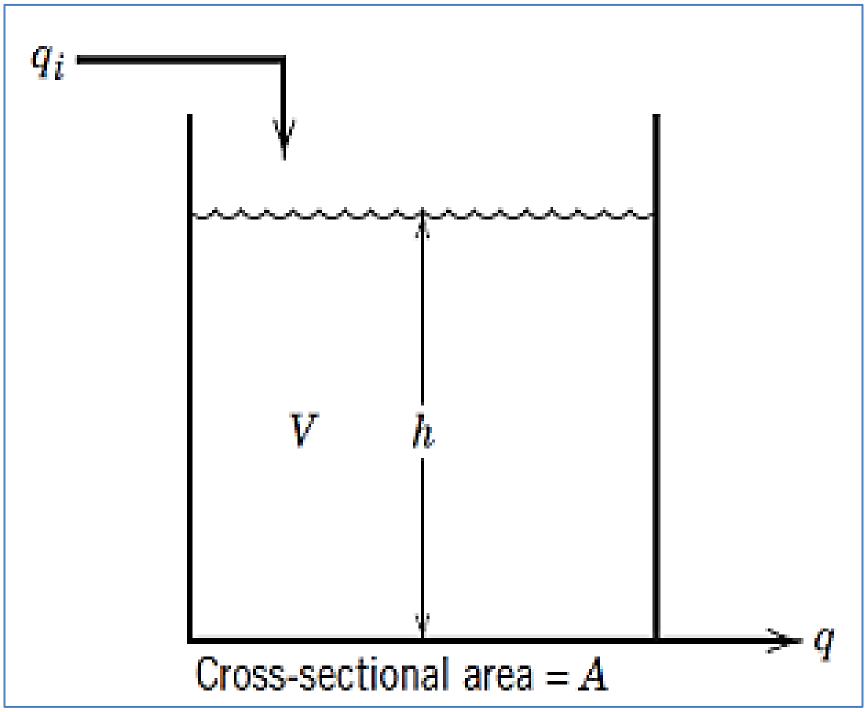


Fig. 6. A liquid-level storage process.

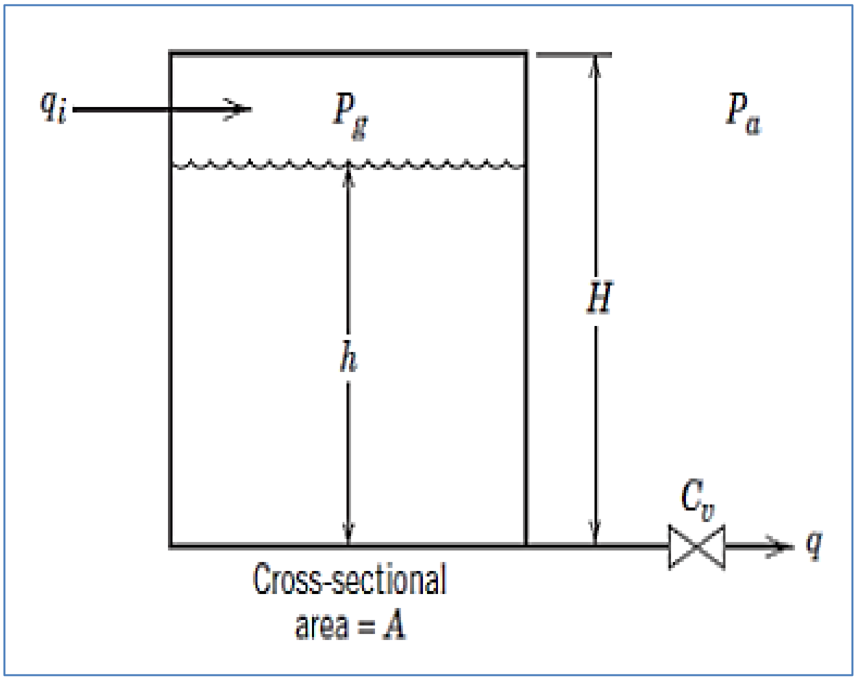


Fig. 7. A liquid-level storage process.

5- The Continuous Stirred-Tank Reactor (CSTR)

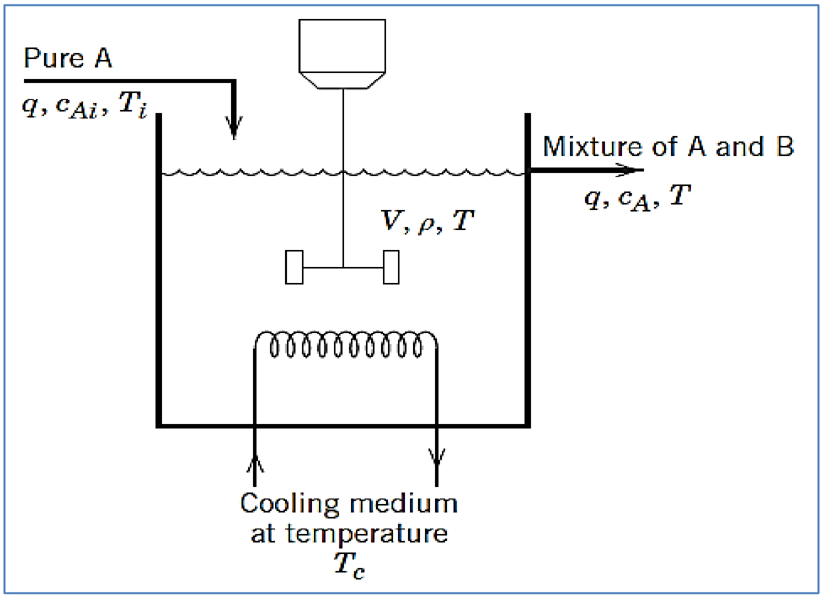


Fig. 8. A nonisothermal continuous stirred-tank reactor**.**

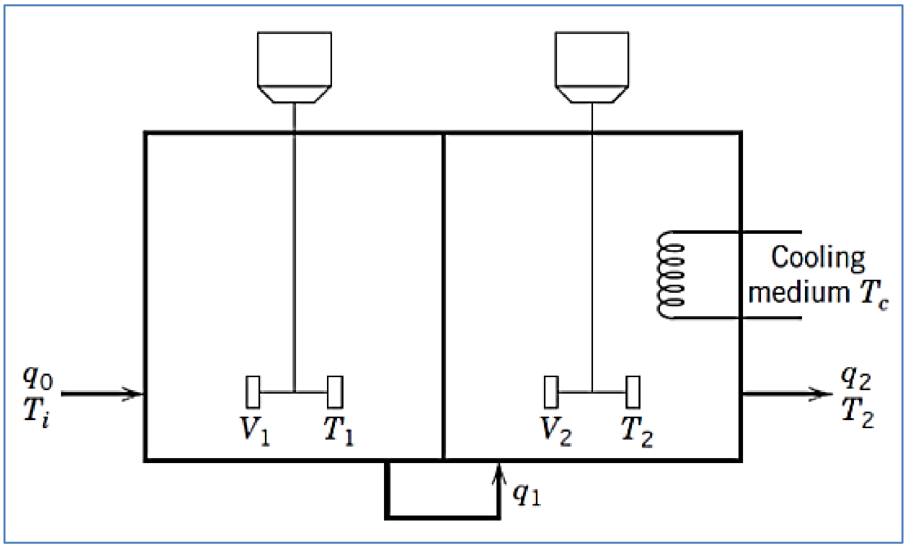


Fig. 9. A nonisothermal continuous stirred-tank reactor**.**

Processes Control:

Second-Order Systems

A second-order system is one whose dynamic behavior is represented by the second-order differential equation of the type:

$$\boldsymbol{a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu(t)} \qquad (\mathbf{1})$$

where ***y(t)*** and ***u(t)*** are respectively the system's output and input variables. Once again, it is customary to rearrange such an equation to a "standard form" in which the characteristic system parameters will be more obvious. In this case, the "standard form" is:

$$\boldsymbol{\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = Ku(t)} \qquad (\mathbf{2})$$

and the newly introduced parameters are given (for $a_0 \neq 0$) by:

$$\boldsymbol{\tau^2 = a_2/a_0} \qquad \boldsymbol{2\xi\tau = a_1/a_0} \qquad \boldsymbol{K = b/a_0}$$

Assuming, as usual, that the model in Eq. (1) is in terms of deviation variables, Laplace transformation and subsequent rearrangement gives the transfer function model:

$$\boldsymbol{y(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} u(s)}$$

so that the general transfer function for the second-order system is given by:

$$\boldsymbol{G(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}} \qquad (\mathbf{3})$$

Note that whereas the transfer function of the first-order system has a first-order denominator polynomial, the transfer function for the second-order system has a second-order denominator polynomial (i.e., a quadratic).

The second-order system has three characteristic parameters:

1. ***K***, the steady-state gain,

2. ξ, the damping coefficient, and
3. τ, the natural period (or the inverse natural frequency)

Physical Examples of Second-Order Systems

1. Two First-Order Systems in Series

Let us now recall the dynamic behavior of two first-order systems in series. We obtained the differential equation model for the non-interacting arrangement as the second-order differential equation in Eq. (7-Lecture 16). Comparing this now with the standard form in Eq. (2), we see that:

The system consisting of a noninteracting, series arrangement of two first-order systems (with time constants $\tau_1$and τ_2 and steady-state gains K_1 and K_2) is a second-order system with the following parameters:

$$\tau^2 = \tau_1\tau_2 \;\Rightarrow\; \tau = \sqrt{\tau_1\tau_2}$$

$$K=K_1\,K_2$$

$$2\xi\tau = (\tau_1+\tau_2)$$

$$\xi = \frac{1}{2}\frac{(\tau_1+\tau_2)}{\sqrt{\tau_1\tau_2}}$$

A comparison of the transfer function representation in Eq. (17-Lecture 16) with Eq. (3) shows that the interacting configuration gives rise to a second-order system in which τ and K are as given above for the noninteracting system, but with ξ given by:

$$\xi = \frac{1}{2}\frac{(\tau_1+\tau_2+K\tau_1)}{\sqrt{\tau_1\tau_2}}$$

2. The U-Tube Manometer

The U-tube manometer shown in Figure 1 is a device used for measuring pressure. The dynamic behavior of the liquid level in each leg of the manometer tube in response to pressure changes can be obtained by carrying out a force balance on this system. The resulting equation is:

$$\frac{d^2h}{dt^2}+\frac{6\mu}{R^2\rho}\frac{dh}{dt}+\frac{3}{2}\frac{g}{L}h=\frac{3}{4\rho L}\Delta P \qquad (4)$$

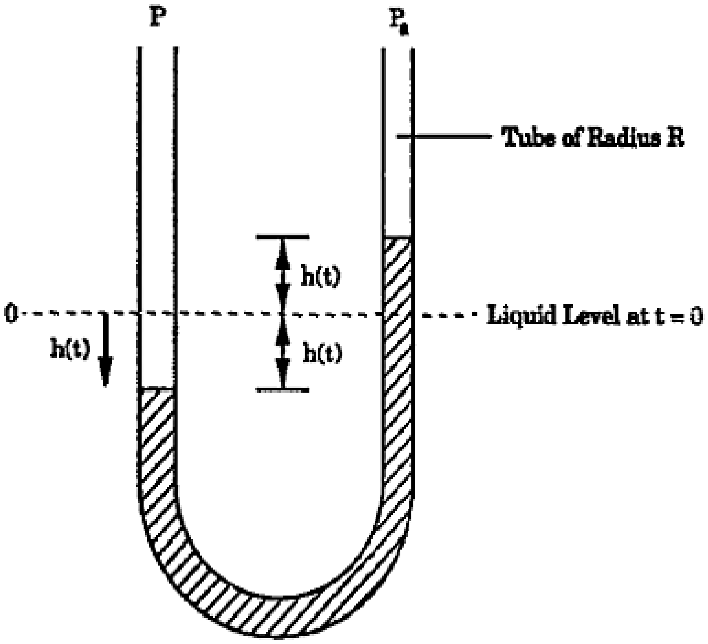


Fig. 1. The U-tube manometer

where $\boldsymbol{h}$ is the displacement of the liquid level from rest position, $\boldsymbol{L}$ is the total length of liquid in the manometer, $\boldsymbol{R}$ is the radius of the manometer tube; $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ are respectively the density and viscosity of the manometer liquid; $\Delta \boldsymbol{P}$ is the pressure difference across the tops of the two manometer legs; and **g** is the acceleration due to gravity. When Eq. (4) is arranged in the standard form, we observe that the dynamic behavior of this system is second order with:

$$K = \frac{1}{2\rho g}$$

$$\tau^2 = \frac{2}{3}\frac{L}{g} \quad \rightarrow \quad \tau = \sqrt{\frac{2}{3}\frac{L}{g}}$$

$$2\xi\tau = \frac{4\mu L}{\rho g R^2} \quad \rightarrow \quad \xi = \frac{\mu}{\rho R^2}\sqrt{\frac{6L}{g}}$$

Response of Second-Order Systems to Various Inputs

Step Response

The response of the second-order system to a step function of magnitude ***A*** may be obtained using Eq. (3). Since ***u(s) = A/s*** we have:

$$y(s)=\frac{K}{\tau^2 s^2+2\xi\tau s+1}\,\frac{A}{s} \qquad (5)$$

We now choose to rearrange this expression to read:

$$y(s)=\frac{AK/\tau^2}{s(s-r_1)(s-r_2)} \qquad (6)$$

where r_1 and r_2 are the roots of the denominator quadratic (the transfer function poles). Ordinarily, one would invert Eq. (6), after partial fraction expansion, and obtain the general result:

$$y(t)=A_0+A_1e^{r_1t}+A_2e^{r_2t} \qquad (7)$$

where, as we now know, A_0 , A_1 , and A_2 are the usual constants obtained during partial fraction expansion. However, the values of the roots r_1 and r_2, obtained using the quadratic formula, are:

$$r_1, r_2=-\frac{\xi}{\tau}\pm\frac{\sqrt{\xi^2-1}}{\tau} \qquad (8)$$

By examining the quantity under the radical sign we may now observe that these roots can be real or complex, depending on the value of the parameter ξ. Observe further that the type of response obtained in Eq. (7) depends on the nature of these roots. In particular when $0 < \xi < 1$, Eq. (8) indicates complex conjugate roots, and we will expect the response under these circumstances to be different from that obtained when $\xi > 1$, and the roots are real and distinct. When $\xi = 1$, we have a pair of repeated roots, giving rise to yet another type of response which is expected to be different from the other two. The case $\xi < 0$ can occur only in special circumstances and signals process instability.

There are thus three different possibilities for the response in Eq. (7), depending on the nature of the roots $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, that are dependent on the value of the parameter ξ. Let us now consider each case in turn.

H.W. 19.1. *Find Eq. 7 from Eq. 6.*

***CASE 1*: 0< ξ <1 (*r₁* and *r₂* Complex conjugates)**

It can be shown that in this case, Eq. (7) becomes, after some simplification:

$$\boldsymbol{y(t) = AK\left[1 - \frac{1}{\beta} e^{-\frac{\xi t}{\tau}} sin\left(\frac{\beta}{\tau}t + \phi\right)\right]} \qquad (\mathbf{9})$$

Where:

$$\boldsymbol{\beta = \left|\xi^2 - 1\right|^{1/2}}$$

$$\boldsymbol{\phi = tan^{-1}(\frac{\beta}{\xi})}$$

Note:

1. The time behavior of fuis response is that of a damped sinusoid with $\boldsymbol{\beta/\tau}$ as the frequency of oscillation. The damping is provided by the exponential term $\boldsymbol{e^{-\frac{\xi t}{\tau}}}$ which gets smaller in magnitude with time and eventually goes to zero as t → ∞.
2. The response ultimately settles, as t → ∞, to the value AK.

***CASE 2*: ξ = 1 (r_1 and r_2 Real and equal roots)**

In this case because we have a pair of repeated roots, we cannot use Eq. (7) directly. Laplace inversion of the appropriately modified version of Eq. (6):

$$\boldsymbol{y(s) = \frac{AK/\tau^2}{s(s-r)^2}} \qquad (\mathbf{10})$$

gives the required response:

$$\boldsymbol{y(t) = AK\left[1 - (1 + \frac{t}{\tau})\, e^{-\frac{t}{\tau}}\right]} \qquad (\mathbf{11})$$

since $\boldsymbol{r} = \mathbf{-1/\tau}$. The time behavior indicated by this equation is an exponential approach to the ultimate value of ***AK***. Note that just as in *Case 1*, as t → ∞, ***y(t)*** → ***AK***.

***CASE 2*: ξ > 1 (r$_1$ and r$_2$ Real and distinct roots)**

With β as defined before, the roots are now given by:

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\beta}{\tau} \qquad (12)$$

and the response is given by:

$$y(t) = AK\left[1 - e^{-\frac{\xi t}{\tau}}\left(\cosh\frac{\beta}{\tau}t + \frac{\xi}{\beta}\sinh\frac{\beta}{\tau}t\right)\right] \qquad (13)$$

Where the hyperbolic functions sinh, cosh are defined as::

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

$$\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

The indicated time response is another exponential approach to the ultimate value of AK. Although perhaps not immediately obvious from Eq. (13), it is true, however, that this particular exponential approach is somewhat slower than the one indicated in Eq. (11).

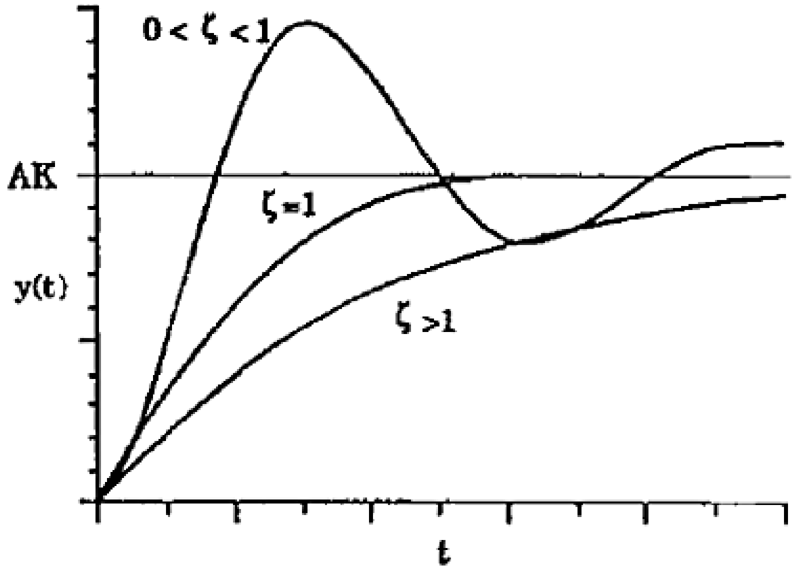


Fig. 2. Step responses of the second-order system

These three responses are sketched in Figure 2.

1. The Case 1 response (when $0< \xi <1$) is oscillatory and is said to be under-damped.
2. The Case 2 response (when $\xi = 1$) is said to be critically damped. It offers the most rapid approach to the final value without oscillation.
3. The Case 3 response (when $\xi > 1$) is sluggish and is said to be over-damped.

We thus see that whether the second-order response is under-damped, over-damped, or critically damped is determined solely by one parameter ξ. This is why it is referred to as the *damping coefficient*.

Let us now return to the under-damped response in Eq. (9) and consider what happens when (is set equal to zero. This represents the situation in which there is no damping at all. The resulting response in this case is:

$$\boldsymbol{y(t) = AK\left[1 - sin\left(\frac{1}{\tau}t + \frac{\pi}{2}\right)\right]} \qquad (\mathbf{14})$$

a pure, undamped sine wave, with frequency $1/\boldsymbol{\tau}$. This is referred to as the natural frequency of oscillation ω_n. Observe that its reciprocal, the natural period of oscillation, is $\boldsymbol{\tau}$, establishing the reason for the name given to this parameter.

Having established that there are three categories of second-order systems (under-damped, critically damped, and over-damped).