LECTURE (1)

DIFFERENTIAL EQUATIONS
(PART ONE)

A differential equation is an equation that involves one or more derivatives. They are classified by:

1- **Type** (ordinary, partial).

2- **Order** (the highest order derivative that occurs in the equation).
3- **Degree** (the highest power of the highest order derivative).

If y is a function of x, where y is called the dependent variable and x is called the independent variable, thus, differential equation is a relation between x and y which includes at least one derivative of y with respect to x.

If the differential equation involves only a single independent variable , this derivative is called ORDINARY DERIVATIVE & the equation is called ORDINARY DIFFERENTIAL EQUATION (ODE).

If the differential equation involves two or more independent variables, this derivative is called PARTIAL DERIVATIVE & the equation is called PARTIAL DIFFERENTIAL EQUATION (PDE).

❑ **y= f(x,t)**

$$\frac{\partial 2y}{\partial t^2} = c^2 \left(\frac{\partial 2y}{\partial x^2}\right)$$ (2nd order ; 1st degree)

❑ **y= f(x)**

$$\frac{dy}{dx} = 3x + 5$$ (1st order ; 1st degree)

$$\left(\frac{d3y}{dx^3}\right)^2 + \left(\frac{d2y}{dx^2}\right)^4 = 0$$ (3rd order ; 2nd degree)

$$5\frac{d3y}{dx^3} + \cos\frac{d2y}{dx^2} + 2xy = 0$$ (3rd order ; 1st degree)

SOLUTION OF DIFFERENTIAL EQUATIONS:

1- **GENERAL SOLUTION.**
2- **PARTICULAR SOLUTION.**

y = x + c   (**general solution**)
If y = 2 & x = 1, then
2 = 1+c ; c = 1
y= x+1   (**particular solution**)

The differential equation may be linear or non-linear depending on the presence of the dependent variable y and its derivatives in one term of the equation.

$\frac{d2y}{dx^2}$ + 4**x** $\frac{dy}{dx}$ + 2**y** = 0   (**linear equation**)

$\frac{d2y}{dx^2}$ + 4**y** $\frac{dy}{dx}$ + 2**y** = 0   (**non-linear equation**)

$\frac{d2y}{dx^2} + \sin y = 0$ (**non- linear equation** since it contains $\boldsymbol{\sin y}$ which is non-linear

The complexity of solving differential equations increases with the order.

1) SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS:

1. **Variable Separable Equation.**
2. **Homogenous Equation.**
3. **Exact Equation.**
4. **Linear Equation.**
5. **Bernoulli's Equation.**

1. Variable Separable Equation.

A first order Ordinary Differential Equation has the form:

$$F(x,y,,y`) = 0$$

In theory, at least, the method of algebra can be used to write it in the form:

$$y` = G(x,y).$$

If G(x,y) can be factored to give:

$$G(x,y)= Mx.Ny,$$

then the equation is called separable.
To solve the separable equation y` = Mx.Ny, we rewrite it in the form

$$f(y)y` = g(x).$$

Integrating both sides gives:

$$\int f(y)\ y`\ dx = \int g(x)dx$$

$$\int f(y)\ dy = \frac{dy}{dx}\ dx$$

Ex (1) Solve the equation $\mathbf{y\,\frac{dy}{dx} + x\,y^2 = x}$ **.**

Solution:

$$y\,\frac{dy}{dx} + x\,y^2 - x = 0$$

$$y\,\frac{dy}{dx} + (y^2 - 1)\,x = 0$$

$$(\frac{y}{y^2-1})\,\frac{dy}{dx} + (\frac{y^2-1}{y^2-1})\,x = 0$$

$$(\frac{y}{y^2-1})\,dy + x.dx = 0$$

$$\int \frac{y}{y^2-1})\,dy + \int x.dx = 0$$

$$\frac{1}{2}\ln(y^2 - 1) + \frac{x^2}{2} + c = 0$$

Ex (2) Solve the equation $\mathbf{\frac{dy}{dx} = (1+ y^2)\, e^x}$ **.**

Solution:

$$\frac{dy}{1+y^2} = e^x\ dx$$

$$e^x\ dx - \frac{dy}{1+y^2} = 0$$

$$\int e^x\,dx - \int \frac{dy}{1+y^2} = 0$$

$$e^x - tan^{-1}\,y = c$$

$$y = tan(e^x - c)$$

Ex (3) Solve the equation $\boldsymbol{\frac{dy}{dx} = \cos(x+y)}$.

Solution:

Let u = x+y

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

Sub. for $\frac{dy}{dx}$ in the main equation

$$\frac{du}{dx} - 1 = \cos(u)$$

$$\frac{du}{dx} = \cos(u) + 1 \quad \rightarrow \quad dx = \frac{du}{1+\cos(u)}$$

$$\int \frac{(1-\cos u)}{(1-\cos u)(1+\cos u)}\, du = \int dx$$

$$\int \frac{(1-\cos u)}{(1-\cos^2 u)}\, du = \int dx$$

$$\int \frac{1}{(\sin^2 u)}\, du - \int \frac{\cos u}{\sin^2 u}\, du = \int dx$$

$$\int \csc^2 u .\ du - \int \cot u .\ \csc u\, du = \int dx$$

$$-\cot u + \csc u = x + c \quad \rightarrow \quad -\cot(x+y) + \csc(x+y) = x + c$$

2- Homogenous Equation:

A(x,y) dx + B (x,y) dy = 0

where the functions A(x,y) & B(x,y) are of the same degree.

The equation can be put in the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Such equation is called homogenous

Let $v = \frac{y}{x}$ ……..…………….(2) ; y= v.x

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ ………………. (3)}$$

$$F(v) = v + x\frac{dv}{dx}$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

Examples:

1) $y.dx + x.dy = 0$ (homogenous/ same degree)
2) $y^2.dx + xy.dy = 0$ (homogenous/ same degree)
3) $y.dx + dy = 0$ (not homogenous)
4) $(y+1)\,dx + dy = 0$ (not homogenous)
5) $(y + \sin\frac{y}{x})\,dx + x.dy = 0$ (not homogenous)
6) $(y + x.\sin\frac{y}{x})\,dx + x.dy = 0$ (homogenous/ same degree)
7) $(x+y)dy + x.dx = 0$ (homogenous/same degree)
8) $x.dy + \sin y.dx = 0$ (not homogenous)

EX (1) Solve the equation $(x+y).dy - (x-y).dx = 0$.

Solution:

the equation is homogenous ; $v = \frac{y}{x}$

$$F(v) = \frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} = \frac{1-v}{1+v}$$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$; (homogenous) $\rightarrow\rightarrow$ $\frac{1-v}{1+v} = v + x\frac{dv}{dx}$

$$\frac{dx}{x} - \frac{dv}{\frac{1-v}{1+v}-v} = 0 \quad \rightarrow\rightarrow \quad \frac{dx}{x} - \frac{dv}{\frac{1-v-v-v^2}{1+v}} = 0$$

$$\int \frac{dx}{x} - \int \frac{(\text{١}+v)dv}{\text{١}-\text{٢}v-v^2} = 0 \rightarrow\rightarrow \quad \ln x + \frac{1}{2}\ln\left(\text{١} - \text{٢}v - v^2\right) = \ln c$$

$$\ln x^2 + \ln[1 - \frac{2y}{x} - (\frac{y}{x})^2] = \ln c`$$

$$x^2[1 - \frac{2y}{x} - (\frac{y}{x})^2] = c`$$

$$x^2 - 2yx - y^2 = c`$$

EX (2) Solve the equation $(x^2 - y^2)$.dx – 2xy.dy = 0.

Solution:

the equation is homogenous ; v = $\frac{y}{x}$; $\frac{dy}{dx} = \text{v} + x \frac{dv}{dx}$

$$\text{F (v)} = \frac{dy}{dx} = -\left(\frac{x^2 - y^2}{2xy} \right) = \left[\frac{1+(\frac{Y}{X})^2}{2(\frac{Y}{X})} \right]$$

$$\frac{dx}{x} + \frac{dv}{v + \frac{1+v^2}{2v}} = 0 \quad \rightarrow\rightarrow\rightarrow \quad \frac{dx}{x} + \frac{dv}{\frac{2v^2+1+v^2}{2v}} = 0$$

$$\frac{dx}{x} + \frac{2vdv}{1+3v^2} = 0 \quad \rightarrow\rightarrow\rightarrow \quad \int \frac{dx}{x} + \int \frac{2vdv}{1+3v^2} = 0$$

$$\ln x + \frac{1}{3} \ln\left(1 + 3v^2\right) = \ln c$$

$$\ln x^3 + \ln\left(1 + 3v^2\right) = \ln c`$$

$$x^3 (1 + 3v^2) = c`$$

$$x^3 (1 + 3\frac{y^2}{x^2}) = c`$$

$$x(x^2 + 3y^2) = c`$$

LECTURE (2)

DIFFERENTIAL EQUATIONS
PART TWO

3- Exact Equation

A (x,y).dx + B (x,y).dy = 0

on the condition that: $\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$

Method of solution:

First we assume the solution is $\emptyset(\mathrm{x,y}) = \mathrm{constant}$

$$\mathrm{A} = \frac{d\emptyset}{dx} \ \& \ \mathrm{B} = \frac{d\emptyset}{dy}$$

$$\int d\emptyset = \int Adx$$

$$\emptyset = \int Adx$$

$$\frac{d\emptyset}{dy} = \frac{d}{dy}\int Adx = \mathrm{B}$$

$$\mathrm{B} = \frac{d\emptyset}{dy} = \frac{d}{dy}\int Adx$$

EX (1) Solve the equation: $(x^3$ **-3**$x^2\ y + 2x\ y^2)$**.dx – (**x^3 **-2**$x^2\ y + y^3$**)dy = 0**

Solution:

First we must check if the equation is exact.

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

$$\mathrm{A} = x^3 \text{ -3}x^2\ y + 2x\ y^2;\ \ \mathrm{B} = -(x^3 \text{ -2}x^2\ y + y^3)$$

$\frac{\partial A}{\partial y}$ (with respect to x) = -3 x^2 + 4xy

$\frac{\partial B}{\partial x}$ (with respect to y) = -3 x^2 + 4xy

Thus the equation is exact

$$\emptyset = \int Adx = \int (x^3 \ - 3x^2\ y + 2x\ y^2).\mathrm{dx}$$

$$\emptyset = \frac{x^4}{4} - x^3\ y + x^2\ y^2 + \mathrm{Cy}\ \ldots\ldots\ldots\ldots\ldots..\ (1)$$

where **C** is constant that may be a function of **y**.

$$\frac{d\emptyset}{dy} = \ - x^3 \ + 2x^2 y + \frac{\partial c}{\partial y} \qquad ;\ \mathrm{B} = \frac{d\emptyset}{dy}$$

$$-\ (x^3 \ \text{-}2x^2\ y + y^3) = - x^3 \ + 2x^2 y + \frac{\partial c}{\partial y} \ \longrightarrow\longrightarrow\longrightarrow \ \frac{\partial c}{\partial y} = - y^3$$

$$\mathrm{Cy} = - \frac{y^4}{4} - \mathrm{D}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (2)$$

Sub. Cy in the main equation (1);

$$\emptyset = \frac{x^4}{4} - x^3\ y + x^2\ y^2 - \frac{y^4}{4} - \mathrm{D}$$

EX (2) Solve the equation: $\sin x.\mathrm{dy} + \mathrm{y}\cos x.dx = 0$

Solution:

First we must check if the equation is exact.

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

$$\mathrm{A} = y\cos x;\ \mathrm{B} = \sin x$$

$$\frac{\partial A}{\partial y}\ \text{(with respect to x)} = \cos x$$

$$\frac{\partial B}{\partial x}\ \text{(with respect to y)} = \cos x$$

Thus the equation is exact

$$\emptyset = \int Adx = \int y\cos x.\mathrm{dx}$$

$$\emptyset = y\sin x + \mathrm{Cy}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (1)$$

$$\frac{d\emptyset}{dy} = \sin x + \frac{\partial c}{\partial y}$$

$$\text{B} = \frac{d\emptyset}{dy}\text{ ; B} = \sin x$$

thus, $\sin x = \sin x + \frac{\partial c}{\partial y} \longrightarrow\!\longrightarrow\!\longrightarrow \quad \frac{\partial c}{\partial y} = 0$

$$\text{Cy} = \text{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

sub. in (1); $\emptyset = y\sin x + \text{D}$

4- Linear Equation

This type of equation has the general form**:**

$$\frac{dy}{dx} + P_x.\text{y} = \text{Q}_x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and solved by an integration factor (R), given by:

$$\text{R} = e^{\int P_x.\text{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

and the solution is:

$$\text{R.y} = \int \text{R. Q}_x.\text{dx} \ + \text{C} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3)$$

EX (1) Solve the equation: $\boldsymbol{x.\frac{dy}{dx} - \text{y} = x^3}$

Solution:

$$\frac{dy}{dx} - \frac{1}{x}\text{y} = x^2$$

$$Q_x = x^2 \ ; \ P_x = -\frac{1}{x}$$

$$\text{R} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$$

$$\text{R}.y = \int R.Q_x.dx \ + \text{C}$$

$$\frac{1}{x}\text{ y} = \int \frac{1}{x}x^2.dx \ + \text{C} = \int x.dx \ + \text{C}$$

$$\frac{1}{x}\text{ y} = \frac{x^2}{2} + \text{C} \longrightarrow\!\longrightarrow\!\longrightarrow \quad \text{y} = \frac{x^3}{3} + x\text{ C}$$

EX (2) Solve the equation: $\boldsymbol{x.\frac{dy}{dx} + 3\,y = \frac{\sin x}{x^2}}$

Solution:

$$\frac{dy}{dx} + \frac{3}{x}\,y = \frac{sin\,x}{x^3} \quad \longrightarrow Q_x = \frac{sin\,x}{x^3};\ P_x = \frac{3}{x}$$

$$R = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ \ln x^3} = x^3$$

$$R.y = \int R.Q_x.dx \ + C$$

$$x^3.y = \int x^3.\frac{sin\,x}{x^3}dx \quad + \ C = \int sin\,x\,.dx + C$$

$$x^3.y = - \cos x + C$$

$$y = \frac{-\cos x}{x^3} + \frac{c}{x^3}$$

5- Bernoulli's Equation

This type of equations has a general form:

$$\frac{dy}{dx} + P_x.y = Q_x\ y^n\ ;\ [n > 1]$$

The solution starts by putting the equation as:

$$\boldsymbol{y^{-n}}\ (\boldsymbol{\tfrac{dy}{dx}}) + P_x.\,\boldsymbol{y^{1-n}} = Q_x \ \dots\dots\dots\dots\ (1)$$

assume; $\boldsymbol{y^{1-n}} = w$

Differentiate with respect to $\boldsymbol{x}$

$$(1\text{-}n)\ y^{-n}\ (\tfrac{dy}{dx}) = \frac{dw}{dx}$$

$$\boldsymbol{y^{-n}}\ (\boldsymbol{\tfrac{dy}{dx}}) = \frac{1}{(1-n)}\ \frac{dw}{dx}$$

sub. in (1) ; $\frac{1}{(1-n)}\ \frac{dw}{dx} + P_x.w = Q_x$

$$\frac{dw}{dx} + (1\text{-}n)\ P_x.w = (1\text{-}n)\ Q_x \ \dots\dots\dots\dots\dots\dots\ (2) \qquad \text{(Linear Equation)}$$

EX (1) Solve the equation: $\mathbf{y\,(6y^2 - x - 1)dx + 2x.\,dy = 0}$

Solution:

$$\frac{dy}{dx} + \frac{y\,(6y^2-x-1)}{\mathbf{2x}} = 0$$

$$\frac{dy}{dx} + \frac{-\,(x+1)}{\mathbf{2x}}\,y + \frac{6y^3}{2x} = 0$$

$$\frac{dy}{dx} - \frac{x+1}{\mathbf{2x}}\,y + \frac{3}{x}\,y^3 = 0$$

$$\frac{dy}{dx} - \frac{x+1}{\mathbf{2x}}\,y = -\frac{3}{x}\,y^3$$

$$y^{-3}\,\frac{dy}{dx} - \frac{\boldsymbol{x+1}}{\mathbf{2x}}\,\boldsymbol{y^{-2}} = -\frac{3}{x}$$

$$\frac{\boldsymbol{dw}}{\boldsymbol{dx}} + \textbf{(1-n)}\ \boldsymbol{P_x.\,w} = \text{(1-n) } Q_x \text{ (main equation)}$$

$$\mathbf{w} = \boldsymbol{y^{-2}}$$

$$\frac{\boldsymbol{dw}}{\boldsymbol{dx}} = \mathbf{-2}\ \boldsymbol{y^{-3}}\,\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$$

Sub. in the main equation:

$$-\frac{1}{2}\frac{dw}{dx} - \frac{x+1}{2x}\,w = -\,\frac{3}{x}$$

$$\frac{dw}{dx} + \frac{x+1}{x}\,w = \frac{6}{x} \quad \text{Linear Equation}$$

Solving as linear equation;

$$\text{R} = e^{\int P_x.\text{dx}} = e^{\int\frac{x+1}{x}.\text{dx}} = e^{\int \text{dx} + \int\frac{dx}{x}}$$

$$= e^{x+\,\ln x} = e^{x}.\ e^{\ln x} = xe^{x}$$

$$\text{R} = xe^{x}$$

$$\text{R}.y = \int R.\,Q_x.\,dx \quad + \text{C}$$

$$\text{w} = y^{-2};\ Q_x = \frac{6}{x}$$

$$xe^{x}.\,w = \int xe^{x}.\frac{6}{x}\ .\,dx \quad + \text{C}$$

$$xe^{x}.w = 6\int e^{x}.\ dx \quad + \mathrm{C}$$

$$xe^{x}.w = 6\ e^{x} + \mathrm{C}$$

$$xe^{x}.y^{-2} = 6\ e^{x} + \mathrm{C}$$

EX (2) Solve the equation: $\boldsymbol{6y^2dx - \mathrm{x}(2x^3 + y)dy = 0}$

Solution:

$$6y^2dx = \ x(2x^3 + y)dy$$

$$\frac{dx}{dy} = \frac{x(2x^3+y)}{6y^2} = \frac{(2x^4+xy)}{6y^2} = \frac{2x^4}{6y^2} + \frac{xy}{6y^2}$$

$$\frac{dx}{dy} - \frac{x}{6y} = \frac{x^4}{3y^2} \qquad \text{(Bernoulli's Equation)}$$

$$x^{-4}.\frac{dx}{dy} - \frac{x^{-3}}{6y} = \frac{1}{3y^2} \ \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$\mathrm{W} = \ x^{-3} \longrightarrow\longrightarrow \frac{dw}{dy} = -3\ x^{-4}\,\frac{dx}{dy}$$

$$\frac{dx}{dy} = -\ \frac{1}{3x^{-4}}\ \frac{dw}{dy}$$

$$\text{sub. in (1)} \qquad -\frac{1}{3}\ \frac{dw}{dy} - \ \frac{w}{6y} = \frac{1}{3y^2}$$

$$\frac{dw}{dy} - \ \frac{3w}{6y} = \frac{-3}{3y^2}$$

$$\frac{dw}{dy} + \ \frac{\boldsymbol{w}}{2y} \mathbf{=} \frac{-1}{y^2} \qquad \text{(Linear Equatio}\mathbf{n)}$$

$$\mathrm{R} = e^{\int \frac{\mathbf{1}}{\mathbf{2y}}.\mathrm{dx}} = e^{\frac{1}{2}\ln y} = y^{\frac{1}{2}}$$

$$\boldsymbol{w = \ x^{-3}; }\ Q_y \mathbf{=} \frac{\mathbf{-1}}{\boldsymbol{y^2}}$$

$$y^{\frac{1}{2}}\ .\ w = \int y^{\frac{1}{2}}.\frac{\mathbf{-1}}{\boldsymbol{y^2}}.\,dy \quad + \mathrm{C} = -\ \frac{\boldsymbol{y^{-\frac{1}{2}}}}{\mathbf{-\frac{1}{2}}} \ + \mathrm{C} = 2y^{\frac{-1}{2}} + \mathrm{C}$$

$$y^{\frac{1}{2}}\ .\ x^{-3} = 2y^{\frac{-1}{2}} + \mathrm{C}$$

Partial Differential Equations :

Partial differential equations are differential equations containing one dependent variable and two or more independent variables. There are many methods of solution for these equations.

1. Method of Direct Integration.
2. Separation of Variables (Forier Transforms).
3. Combination of Variables (Variation of Parameters).
4. Caplace Transforms.

Method of Direct Integration :

Ex: Solve the partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} = x e^y$$

For the boundary conditions,

$$u(0,y) = y^2$$
$$u(1,y) = \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = x e^y \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x e^y$$

Integrating with respect to x

$$\frac{\partial u}{\partial x} = e^y \frac{x^2}{2} + F_1(y)$$

Integrating again,

$$u(x,y) = e^y \frac{x^3}{6} + x F_1(y) + F_2(y)$$

$F_1(y)$ and $F_2(y)$ are constants of integration with respect to x, but may be functions of y.

$$x=0 \Rightarrow u=y^2$$

$$u(0,y)=F_2(y)=y^2$$

$$u(x,y)=\frac{x^3e^y}{6}+xF_1(y)+y^2$$

$$x=1 \Rightarrow u=\sin y$$

$$u(1,y)=\frac{e^y}{6}+F_1(y)+y^2=\sin y$$

$$\therefore \; F_1(y)=\sin y-y^2-\frac{e^y}{6}$$

$$u(x,y)=\frac{x^3e^y}{6}+x\left(\sin y-y^2-\frac{e^y}{6}\right)+y^2$$

Separation of Variables :

The solution starts by assuming the solution is a product of functions of the independent variables.

Ex: Find the general solutions for the equation :

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}$$

Assume : $C(x,t)=X_{(x)}\,.\,T_{(t)}$

$$\frac{\partial c}{\partial t}=X.T'$$

$$\frac{\partial c}{\partial x} = T . X' \quad \& \quad \frac{\partial^2 c}{\partial x^2} = T . X''$$

$$X T' = D X'' T \Rightarrow \frac{T'}{T} = D \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = D \frac{X''}{X} = k$$

There are three cases for k

Case (1): $K > 0 \Rightarrow K = \alpha^2$

$$\frac{T'}{T} = \alpha^2 \Rightarrow \ln T = \alpha^2 t + \ln \bar{C} \Rightarrow \ln T - \ln \bar{C} = \alpha^2 t$$

$$\ln \frac{T}{\bar{C}} = \alpha^2 t \Rightarrow \frac{T}{\bar{C}} = e^{\alpha^2 t} \Rightarrow T = \bar{C} e^{\alpha^2 . t}$$

$$D \frac{X''}{X} = \alpha^2 \Rightarrow X'' = \frac{\alpha^2}{D} X \Rightarrow X'' - \frac{\alpha^2}{D} X = 0$$

$$m^2 - \frac{\alpha^2}{D} = 0 \Rightarrow m = \mp \frac{\alpha}{\sqrt{D}}$$

$$X = \bar{A} e^{\frac{\alpha}{\sqrt{D}} X} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} X}$$

$$C(x,t) = \bar{C} e^{\alpha^2 . t} \left(\bar{A} e^{\frac{\alpha}{\sqrt{D}} x} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

$$C(x,t) = e^{\alpha^2 . t} \left(A e^{\frac{\alpha}{\sqrt{D}} x} + B e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

Case (2): $K = 0$

$$\frac{T'}{T} = 0 \Rightarrow T' = 0 \Rightarrow T = \bar{A}$$

$$D \frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X' = \bar{B} \Rightarrow X = \bar{B} X + \bar{C}$$

$$c(x,t) = \bar{A} (\bar{B} X + \bar{C}) \Rightarrow C(x,t) = A X + B$$

case (3): $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow \ln T = -\beta^2 t + \ln \bar{C}$$

$$\ln \frac{T}{\bar{C}} = -\beta^2 . t \Rightarrow T = \bar{C}\, e^{-\beta^2 . t}$$

$$D\frac{X''}{X} = -\beta^2 \Rightarrow X'' = \frac{-\beta^2}{D} X \Rightarrow X'' + \frac{\beta^2}{D} X = 0$$

$$m^2 + \frac{\beta^2}{D} = 0 \Rightarrow m^2 = -\frac{\beta^2}{D} \Rightarrow m = \mp i \frac{\beta}{\sqrt{D}}$$

$$X = \bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x$$

$$C(x,t) = \bar{C}\, e^{-\beta^2 . t} \left(\bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x \right)$$

$$C(x,t) = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x \right)$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the following conditions,

i) $t = 0$ $\theta = 100$ °C

ii) $x = 0$ $\theta = 0$ °C

iii) $x = 1$ $\theta = 0$ °C

Assume : $\theta(x,t) = X(x) . T(t)$

$$\frac{\partial \theta}{\partial t} = X . T'$$

$$\frac{\partial \theta}{\partial x} = T . X' \quad \& \quad \frac{\partial^2 \theta}{\partial x^2} = T . X''$$

$$XT' = h^2 X'' T \Rightarrow \frac{T'}{T} = h^2 \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = h^2 \frac{X''}{X} = k$$

Case (3) : $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow T = \bar{C} \, e^{-\beta^2 . t}$$

$$h^2 \frac{X''}{X} = -\beta^2 \Rightarrow X = \bar{A} \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x$$

$$\theta(x, t) = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x \right)$$

To find the constants $A, B, \& \beta$:

B.C.1 $\quad x = 0 \quad \theta = 0$

$$0 = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{h} (0) + B \sin \frac{\beta}{h} (0) \right)$$

$$0 = e^{-\beta^2 . t} \left(A(1) + B(0) \right) \Rightarrow 0 = A e^{-\beta^2 . t}$$

$$e^{-\beta^2 . t} \neq 0 \Rightarrow A = 0$$

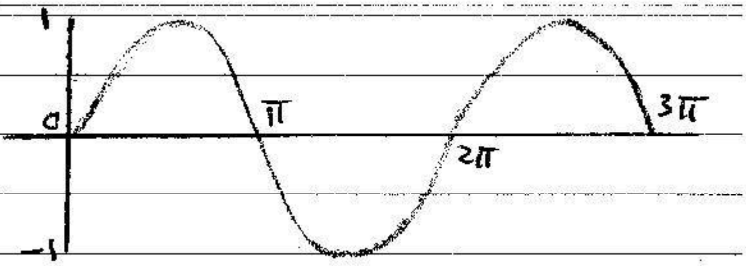
$$\theta = e^{-\beta^2 . t} B \sin \frac{\beta}{h} x$$

B.C.2 $\quad x = 1 \quad \theta = 0$

$$0 = e^{-\beta^2 . t} \, B \sin \frac{\beta}{h} (1)$$

$$e^{-\beta^2 \cdot t} \neq 0 \quad , \quad B \neq 0$$

$$\therefore \sin \frac{\beta}{h} = 0$$



$$\frac{\beta}{h} = n\pi \Rightarrow \beta = n\pi h \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$\theta = e^{-(n\pi h)^2 \cdot t} \cdot B \sin n\pi x$$

n 11

Ex: Solve the partial differential equation,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$$

for the following conditions:

i) $C(x,0) = C_o$
ii) $C(0,t) = C_i$
iii) $C(L,t) = C_i$

Case (3): $K < 0 \Rightarrow K = -\beta^2$

$$C(x,t) = e^{-\beta^2 t}\left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x\right)$$

Let $\bar{C} = C - C_i$

$$\frac{\partial \bar{C}}{\partial t} = D \frac{\partial^2 \bar{C}}{\partial x^2}$$

i) $\bar{C}(x,0) = C_o - C_i$

ii) $\bar{C}(0,t) = C_i - C_i = 0$

iii) $\bar{C}(L,t) = C_i - C_i = 0$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t} \left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x \right)$$

B.C. 1 : $x = 0 \quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t} \left(A(1) + B(0) \right) \Rightarrow 0 = A\, e^{-\beta^2 \cdot t}$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} x$$

B.C. 2 : $x = L \quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} L$$

$$e^{-\beta^2 \cdot t} \neq 0 \,,\; B \neq 0 \Rightarrow \sin \frac{\beta}{\sqrt{D}} L = 0$$

$$\frac{\beta}{\sqrt{D}} L = n\pi \Rightarrow \beta = \frac{n \cdot \pi \cdot \sqrt{D}}{L}$$

$$\bar{C}(x,t) = e^{-\left(\frac{n\pi\sqrt{D}}{L}\right)^2 \cdot t} \left(B \sin \frac{n\pi}{L} x \right)$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the conditions,

i) $\theta(0,t) = 20$

ii) $\theta(20,t) = 20$

iii) $$\theta(x,0) = \begin{cases} 120 & 0 \leqslant x \leqslant 15 \\ 30 & 15 \leqslant x \leqslant 20 \end{cases}$$

$\bar{\theta} = \theta - 20$, $\bar{\theta}(x,t) = \theta(x,t) - 20$

i) $\bar{\theta}(0,t) = 20 - 20 = 0$

ii) $\bar{\theta}(20,t) = 20 - 20 = 0$

iii) $$\bar{\theta}(x,0) = \begin{cases} 120 - 20 = 100 & 0 \leqslant x \leqslant 15 \\ 30 - 20 = 10 & 15 \leqslant x \leqslant 20 \end{cases}$$

$$\frac{\partial \bar{\theta}}{\partial t} = h^2 \frac{\partial^2 \bar{\theta}}{\partial x^2}$$

The general solution

$$\bar{\theta}(x,t) = e^{-\beta^2 \cdot t}\left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x\right)$$

B.C. 1 $\quad x = 0 \quad \bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} \left(A(1) + B(0) \right) \Rightarrow 0 = e^{-\beta^2 \cdot t} A$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{\theta}(x,t) = e^{-\beta^2 \cdot t} \left(B \cdot \sin \frac{\beta}{h} x \right)$$

B.C. 2 : $x = 20 \quad \bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} \cdot B \cdot \sin \frac{\beta}{h}(20)$$

$$e^{-\beta^2 \cdot t} \neq 0 \; , \; B \neq 0 \Rightarrow \sin \frac{\beta}{h}(20) = 0$$

$$\frac{\beta}{h} 20 = n\pi \Rightarrow \beta = \frac{n\pi h}{20}$$

$$\bar{\theta}(x,t) = B \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 \cdot t}$$

Partial Differential Equations :

Partial differential equations are differential equations containing one dependent variable and two or more independent variables. There are many methods of solution for these equations.

1. Method of Direct Integration.
2. Separation of Variables (Forier Transforms).
3. Combination of Variables (Variation of Parameters).
4. Caplace Transforms.

Method of Direct Integration :

Ex: Solve the partial differential equation,

$$\frac{\partial^2 u}{\partial x^2} = x e^y$$

For the boundary conditions,

$$u(0,y) = y^2$$
$$u(1,y) = \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = x e^y \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x e^y$$

Integrating with respect to x

$$\frac{\partial u}{\partial x} = e^y \frac{x^2}{2} + F_1(y)$$

Integrating again,

$$u(x,y) = e^y \frac{x^3}{6} + x F_1(y) + F_2(y)$$

$F_1(y)$ and $F_2(y)$ are constants of integration with respect to x, but may be functions of y.

$$x=0 \Rightarrow u=y^2$$

$$u(0,y)=F_2(y)=y^2$$

$$u(x,y)=\frac{x^3e^y}{6}+xF_1(y)+y^2$$

$$x=1 \Rightarrow u=\sin y$$

$$u(1,y)=\frac{e^y}{6}+F_1(y)+y^2=\sin y$$

$$\therefore \quad F_1(y)=\sin y-y^2-\frac{e^y}{6}$$

$$u(x,y)=\frac{x^3e^y}{6}+x\left(\sin y-y^2-\frac{e^y}{6}\right)+y^2$$

Separation of Variables :

The solution starts by assuming the solution is a product of functions of the independent variables.

Ex: Find the general solutions for the equation :

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}$$

Assume : $C(x,t)=X_{(x)}\,.\,T_{(t)}$

$$\frac{\partial c}{\partial t}=X\,.\,T'$$

$$\frac{\partial c}{\partial x} = T . X' \qquad \& \qquad \frac{\partial^2 c}{\partial x^2} = T . X''$$

$$X T' = D X'' T \Rightarrow \frac{T'}{T} = D \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = D \frac{X''}{X} = k$$

There are three cases for k

Case (1): $K > 0 \Rightarrow K = \alpha^2$

$$\frac{T'}{T} = \alpha^2 \Rightarrow \ln T = \alpha^2 t + \ln \bar{C} \Rightarrow \ln T - \ln \bar{C} = \alpha^2 t$$

$$\ln \frac{T}{\bar{C}} = \alpha^2 t \Rightarrow \frac{T}{\bar{C}} = e^{\alpha^2 t} \Rightarrow T = \bar{C} e^{\alpha^2 . t}$$

$$D \frac{X''}{X} = \alpha^2 \Rightarrow X'' = \frac{\alpha^2}{D} X \Rightarrow X'' - \frac{\alpha^2}{D} X = 0$$

$$m^2 - \frac{\alpha^2}{D} = 0 \Rightarrow m = \mp \frac{\alpha}{\sqrt{D}}$$

$$X = \bar{A} e^{\frac{\alpha}{\sqrt{D}} x} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} x}$$

$$C(x,t) = \bar{C} e^{\alpha^2 . t} \left(\bar{A} e^{\frac{\alpha}{\sqrt{D}} x} + \bar{B} e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

$$C(x,t) = e^{\alpha^2 . t} \left(A e^{\frac{\alpha}{\sqrt{D}} x} + B e^{-\frac{\alpha}{\sqrt{D}} x} \right)$$

Case (2): $K = 0$

$$\frac{T'}{T} = 0 \Rightarrow T' = 0 \Rightarrow T = \bar{A}$$

$$D \frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X' = \bar{B} \Rightarrow X = \bar{B} X + \bar{C}$$

$$c(x,t) = \bar{A} (\bar{B} x + \bar{c}) \Rightarrow C(x,t) = A x + B$$

case (3): $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow \ln T = -\beta^2 t + \ln \bar{C}$$

$$\ln \frac{T}{\bar{C}} = -\beta^2 . t \Rightarrow T = \bar{C}\, e^{-\beta^2 . t}$$

$$D \frac{X''}{X} = -\beta^2 \Rightarrow X'' = \frac{-\beta^2}{D} X \Rightarrow X'' + \frac{\beta^2}{D} X = 0$$

$$m^2 + \frac{\beta^2}{D} = 0 \Rightarrow m^2 = -\frac{\beta^2}{D} \Rightarrow m = \mp i \frac{\beta}{\sqrt{D}}$$

$$X = \bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x$$

$$C(x,t) = \bar{C}\, e^{-\beta^2 . t} \left(\bar{A} \cos \frac{\beta}{\sqrt{D}} x + \bar{B} \sin \frac{\beta}{\sqrt{D}} x \right)$$

$$C(x,t) = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x \right)$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the following conditions,

i) $t = 0$ $\theta = 100$ °C
ii) $x = 0$ $\theta = 0$ °C
iii) $x = 1$ $\theta = 0$ °C

Assume : $\theta(x,t) = X_{(x)} . T_{(t)}$

$$\frac{\partial \theta}{\partial t} = X . T'$$

$$\frac{\partial \theta}{\partial x} = T . X' \quad \& \quad \frac{\partial^2 \theta}{\partial x^2} = T . X''$$

$$X T' = h^2 X'' T \Rightarrow \frac{T'}{T} = h^2 \frac{X''}{X} = \text{constant}$$

$$\frac{T'}{T} = h^2 \frac{X''}{X} = k$$

Case (3) : $K < 0 \Rightarrow K = -\beta^2$

$$\frac{T'}{T} = -\beta^2 \Rightarrow T = \bar{C} e^{-\beta^2 . t}$$

$$h^2 \frac{X''}{X} = -\beta^2 \Rightarrow X = \bar{A} \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x$$

$$\theta(x, t) = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x \right)$$

To find the constants $A, B, \& \beta$:

B.C.1 $\quad x = 0 \quad \theta = 0$

$$0 = e^{-\beta^2 . t} \left(A \cos \frac{\beta}{h} (0) + B \sin \frac{\beta}{h} (0) \right)$$

$$0 = e^{-\beta^2 . t} \left(A(1) + B(0) \right) \Rightarrow 0 = A e^{-\beta^2 . t}$$

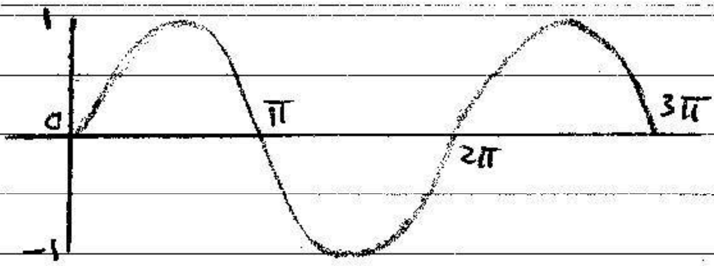
$$e^{-\beta^2 . t} \neq 0 \Rightarrow A = 0$$

$$\theta = e^{-\beta^2 . t} B \sin \frac{\beta}{h} x$$

B.C.2 $\quad x = 1 \quad \theta = 0$

$$0 = e^{-\beta^2 . t} B \sin \frac{\beta}{h} (1)$$

$e^{-\beta^2 \cdot t} \neq 0$, $B \neq 0$



$\therefore \sin \frac{\beta}{h} = 0$

$\frac{\beta}{h} = n\pi \Rightarrow \beta = n\pi h$, $n = 0, 1, 2, 3, \ldots$

$\theta = e^{-(n\pi h)^2 \cdot t} \cdot B \sin n\pi x$

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Ex: Solve the partial differential equation,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$$

for the following conditions:

i) $C(x,0) = C_o$
ii) $C(0,t) = C_i$
iii) $C(L,t) = C_i$

Case (3): $K < 0 \Rightarrow K = -\beta^2$

$$C(x,t) = e^{-\beta^2 t}\left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x\right)$$

Let $\bar{C} = C - C_i$

$$\frac{\partial \bar{C}}{\partial t} = D \frac{\partial^2 \bar{C}}{\partial x^2}$$

i) $\bar{C}(x,0) = C_o - C_i$

ii) $\bar{C}(0,t) = C_i - C_i = 0$

iii) $\bar{C}(L,t) = C_i - C_i = 0$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t}\left(A \cos \frac{\beta}{\sqrt{D}} x + B \sin \frac{\beta}{\sqrt{D}} x\right)$$

B.C. 1 : $x = 0 \quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t}\left(A(1) + B(0)\right) \Rightarrow 0 = A\, e^{-\beta^2 \cdot t}$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{C}(x,t) = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} x$$

B.C. 2 : $x = L \quad \bar{C} = 0$

$$0 = e^{-\beta^2 \cdot t} \cdot B \sin \frac{\beta}{\sqrt{D}} L$$

$$e^{-\beta^2 \cdot t} \neq 0 \,,\; B \neq 0 \Rightarrow \sin \frac{\beta}{\sqrt{D}} L = 0$$

$$\frac{\beta}{\sqrt{D}} L = n\pi \Rightarrow \beta = \frac{n \cdot \pi \cdot \sqrt{D}}{L}$$

$$\bar{C}(x,t) = e^{-\left(\frac{n\pi\sqrt{D}}{L}\right)^2 \cdot t}\left(B \sin \frac{n\pi}{L} x\right)$$

Ex: Solve the partial differential equation,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

for the conditions,

i) $\theta(0,t) = 20$

ii) $\theta(20,t) = 20$

iii) $$\theta(x,0) = \begin{cases} 120 & 0 \leqslant x \leqslant 15 \\ 30 & 15 \leqslant x \leqslant 20 \end{cases}$$

$\bar{\theta} = \theta - 20$, $\bar{\theta}(x,t) = \theta(x,t) - 20$

i) $\bar{\theta}(0,t) = 20 - 20 = 0$

ii) $\bar{\theta}(20,t) = 20 - 20 = 0$

iii) $$\bar{\theta}(x,0) = \begin{cases} 120 - 20 = 100 & 0 \leqslant x \leqslant 15 \\ 30 - 20 = 10 & 15 \leqslant x \leqslant 20 \end{cases}$$

$$\frac{\partial \bar{\theta}}{\partial t} = h^2 \frac{\partial^2 \bar{\theta}}{\partial x^2}$$

The general solution

$$\bar{\theta}(x,t) = e^{-\beta^2 \cdot t}\left(A \cos \frac{\beta}{h} x + B \sin \frac{\beta}{h} x\right)$$

B.C. 1 $x = 0$ $\bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} \left(A(1) + B(0) \right) \Rightarrow 0 = e^{-\beta^2 \cdot t} A$$

$$e^{-\beta^2 \cdot t} \neq 0 \Rightarrow A = 0$$

$$\bar{\theta}(x,t) = e^{-\beta^2 \cdot t} \left(B \cdot \sin \frac{\beta}{h} x \right)$$

B.C. 2 : $x = 20 \quad \bar{\theta} = 0$

$$0 = e^{-\beta^2 \cdot t} \cdot B \cdot \sin \frac{\beta}{h} (20)$$

$$e^{-\beta^2 \cdot t} \neq 0 \,,\ B \neq 0 \Rightarrow \sin \frac{\beta}{h} (20) = 0$$

$$\frac{\beta}{h} 20 = n\pi \Rightarrow \beta = \frac{n\pi h}{20}$$

$$\bar{\theta}(x,t) = B \sin \frac{n\pi}{20} x \cdot e^{-\left(\frac{n\pi h}{20}\right)^2 \cdot t}$$

Combination of Variables :

In this method we introduce a dummy variable, η, where the choice of η is given in the table below. We see that the bounded variable, e.g., distance (x, y or r) appears in the numerator raised to the power 1, while the unbounded variable such as time (t) appears in the denominator raised to the power ($1/n$), where n equals the sum of powers of the bounded variable appearing in the equation.

$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$	$\eta = \frac{x}{\sqrt{t}}$
$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$	$\eta = \frac{x + y}{\sqrt{t}}$
$\frac{\partial c}{\partial t} = D \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right)$	$\eta = \frac{x + y + z}{\sqrt{t}}$
$y \frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial y^2}$	$\eta = \frac{y}{(t)^{1/3}}$
$x^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	$\eta = \frac{x}{(t)^{1/4}}$

Ex: Solve the partial differential equation

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$$

i) $C(x,0) = 0$
ii) $C(0,t) = C_i$
iii) $C(\infty,t) = 0$

We start by putting, $\eta = \dfrac{x}{\sqrt{t}}$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \cdot \left[-\frac{1}{2} x \, t^{-3/2} \right]$$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \left[-\frac{1}{2} x \frac{1}{\sqrt{t} \cdot t} \right] \Rightarrow \frac{\partial c}{\partial t} = \frac{\partial c}{\partial \eta} \left[-\frac{1}{2} \frac{\eta}{t} \right]$$

$$\therefore \frac{\partial c}{\partial t} = -\frac{1}{2} \frac{\eta}{t} \frac{\partial c}{\partial \eta}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \left[\frac{1}{\sqrt{t}} \right] \Rightarrow \frac{\partial c}{\partial x} = \frac{1}{\sqrt{t}} \frac{\partial c}{\partial \eta}$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial c}{\partial x} \right) \Rightarrow \frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{t}} \frac{\partial c}{\partial \eta} \right)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \left(\frac{1}{\sqrt{t}} \cdot \frac{\partial c}{\partial \eta} \right) \Rightarrow \frac{\partial^2 c}{\partial x^2} = \frac{1}{\sqrt{t}} \frac{\partial}{\partial \eta} \left(\frac{1}{\sqrt{t}} \cdot \frac{\partial c}{\partial \eta} \right)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{1}{t} \frac{\partial^2 c}{\partial \eta^2}$$

Sub. in equation :

$$-\frac{1}{2} \frac{\eta}{t} \frac{\partial c}{\partial \eta} = D \frac{1}{t} \frac{\partial^2 c}{\partial \eta^2}$$

$$D \frac{\partial^2 c}{\partial \eta^2} + \frac{1}{2} \eta \frac{\partial c}{\partial \eta} = 0 \Rightarrow D \frac{d^2 c}{d \eta^2} + \frac{1}{2} \eta \frac{dc}{d\eta} = 0$$

This is a second order ordinary differential equation where the dependent variable is not explicit.

$$P = A e^{-\frac{1}{4}\frac{\eta^2}{D}} \Rightarrow \frac{dc}{d\eta} = A e^{-\frac{1}{4}\frac{\eta^2}{D}}$$

$$dc = A e^{-\frac{1}{4}\frac{\eta^2}{D}} . d\eta \Rightarrow C = A \int e^{-\frac{1}{4}\frac{\eta^2}{D}} d\eta$$

$$C = A \, erf \sqrt{\frac{\eta^2}{4D}} + B \Rightarrow C = A \, erf \frac{\eta}{\sqrt{4D}} + B$$

$$C = A \, erf \frac{x}{\sqrt{4Dt}} + B$$ general solution

B.C. 1 $x = 0$ $C = C_i$

$C_i = A \, erf(0) + B$, $erf(0) = 0 \Rightarrow B = C_i$

B.C. 2 $x = \infty$ $C = 0$

$0 = A \, erf(\infty) + B$, $erf(\infty) = 1 \Rightarrow A = -B$

$\therefore A = -C_i$

$$C = -C_i \, erf \frac{x}{\sqrt{4Dt}} + C_i$$

$$C = C_i \left(1 - erf \frac{x}{\sqrt{4Dt}} \right)$$

$$C = C_i \, erfc \frac{x}{\sqrt{4Dt}}$$

Ex: Solve the partial differential equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}$$

For the conditions

i) $\theta(x,0) = 0$

ii) $\theta(0,t) = 100$

iii) $\frac{\partial \theta}{\partial x}(1,t) = 0$

We start by putting $\eta = \frac{x}{2h\sqrt{t}}$

$$\frac{\partial \eta}{\partial t} = \frac{-x}{2h \cdot 2\, t^{3/2}} = \frac{-x}{4ht\sqrt{t}} = \frac{-\eta}{2t}.$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{2h\sqrt{t}}$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial \theta}{\partial t} = \frac{-\eta}{2t}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) \Rightarrow \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}\left(\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{2h\sqrt{t}}\frac{\partial}{\partial \eta}\left(\frac{1}{2h\sqrt{t}}\frac{\partial \theta}{\partial \eta}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{4h^2 t}\frac{\partial^2 \theta}{\partial \eta^2}$$

Sub. in equation

$$\frac{-\eta}{2t}\frac{\partial \theta}{\partial \eta} = h^2 \cdot \frac{1}{4h^2 t}\frac{\partial^2 \theta}{\partial \eta^2}$$

$$\frac{\partial^2 \theta}{\partial \eta^2} + 2\eta\frac{\partial \theta}{\partial \eta} = 0 \Rightarrow \frac{d^2 \theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0$$

This is a second order ordinary differential equation where the dependent variable is not explicit.

$$P = \frac{d\theta}{d\eta} \quad , \quad \frac{dP}{d\eta} = \frac{d^2\theta}{d\eta^2}$$

$$\frac{dP}{d\eta} + 2\eta P = 0 \Rightarrow \frac{dP}{P} + 2\eta \, d\eta = 0$$

$$\ln \frac{P}{A} = -\eta^2 \Rightarrow P = A e^{-\eta^2} \Rightarrow \frac{d\theta}{d\eta} = A e^{-\eta^2}$$

$$d\theta = A e^{-\eta^2} d\eta$$

$$\theta(x,t) = A \operatorname{erf} \eta + B$$

$$\theta(x,t) = A \operatorname{erf} \frac{x}{2h\sqrt{t}} + B$$ general solution

B.C. 1 $\qquad x = 0 \qquad \theta = 100$

$$\theta(0,t) = A \operatorname{erf}(0) + B = 100 \qquad , \operatorname{erf}(0) = 0$$
$$\therefore B = 100$$

$$\theta(x,t) = A \operatorname{erf} \frac{x}{2h\sqrt{t}} + 100$$

I.C. $\qquad t = 0 \qquad \theta = 0$

$$\theta(x,0) = A \operatorname{erf} \frac{x}{2h\sqrt{0}} + 100 = 0$$

$$A \operatorname{erf}(\infty) + 100 = 0 \qquad , \quad \operatorname{erf}(\infty) = 1$$
$$A = -100$$

$$\theta(x,t) = -100 \operatorname{erf} \frac{x}{2h\sqrt{t}} + 100$$

$$\theta(x,t) = 100\left(1 - \operatorname{erf}\frac{x}{2h\sqrt{t}}\right)$$

$$\theta(x,t) = 100 \operatorname{erfc}\frac{x}{2h\sqrt{t}} .$$

Ex: Solve the partial differential equation

$$y\frac{\partial C_A}{\partial z} = \frac{\partial^2 C_A}{\partial y^2}$$

For the following conditions

$z = 0 \qquad C_A = 0$
$y = 0 \qquad C_A = C_{A0}$
$y = \infty \qquad C_A = 0$

We start by putting $\qquad \eta = \dfrac{y}{z^{1/3}}$

$$\frac{\partial \eta}{\partial y} = \frac{1}{z^{1/3}}$$

$$\frac{\partial \eta}{\partial z} = -\frac{1}{3} y z^{-4/3} \Rightarrow \frac{\partial \eta}{\partial z} = -\frac{y}{z^{1/3}}\,\frac{1}{3z} \Rightarrow \frac{\partial \eta}{\partial z} = -\frac{\eta}{3z}$$

$$\frac{\partial C_A}{\partial z} = \frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial z} = -\frac{\eta}{3z}\,\frac{\partial C_A}{\partial \eta}$$

$$\frac{\partial C_A}{\partial y} = \frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{\partial}{\partial \eta}\,\frac{\partial \eta}{\partial y}\left(\frac{\partial C_A}{\partial \eta}\,\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{1}{z^{1/3}}\right)\left(\frac{\partial C_A}{\partial \eta}\left(\frac{1}{z^{1/3}}\right)\right)$$

$$\frac{\partial^2 C_A}{\partial y^2} = \frac{1}{z^{2/3}}\,\frac{\partial^2 C_A}{\partial \eta^2}$$

$$y\left(-\frac{\eta}{3z}\frac{\partial C_A}{\partial \eta}\right)=\frac{1}{z^{2/3}}\frac{\partial^2 C_A}{\partial \eta^2}$$

$$\frac{\partial^2 C_A}{\partial \eta^2}+\frac{1}{3}\eta^2\frac{\partial C_A}{\partial \eta}=0 \Rightarrow \frac{d^2 C_A}{d\eta^2}+\frac{1}{3}\eta^2\frac{dC_A}{d\eta}=0$$

$$P=\frac{dC_A}{d\eta} \quad , \quad \frac{dP}{d\eta}=\frac{d^2 C_A}{d\eta^2}$$

$$\frac{dP}{d\eta}+\frac{1}{3}\eta^2 P=0 \Rightarrow \frac{dP}{P}+\frac{1}{3}\eta^2 d\eta=0$$

$$\ln P+\frac{1}{9}\eta^3-\ln A=0 \Rightarrow P=A\,e^{-\eta^3/9}$$

$$\frac{dC_A}{d\eta}=A\,e^{-\eta^3/9} \Rightarrow \int_{C_A}^{0} dC_A = A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta$$

$$-C_A = A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta + B$$

B.C.1 $y=0$ $C_A=C_{A0}$ $\eta=0$

B.C.2 $y=\infty$ $C_A=0$ $\eta=\infty$

Apply B.C.2 $\eta=\infty$ $C_A=0$

$$0=A\int_{\infty}^{\infty} e^{-\eta^3/9}\,d\eta + B \Rightarrow B=0$$

$$\therefore\ C_A=-A\int_{\eta}^{\infty} e^{-\eta^3/9}\,d\eta$$

Apply B.C.1 $\eta=0$ $C_A=C_{A0}$

$$C_{A0}=-A\int_{0}^{\infty} e^{-\eta^3/9}\,d\eta$$

This integration is the Gamma function (Γ).

Let $\beta = \frac{\eta^3}{9}$

$d\beta = 3\left(\frac{\eta^2}{9}\right) d\eta \Rightarrow d\eta = \frac{1}{3}\left(\frac{\eta^2}{9}\right)^{-1} d\beta$

$C_{A0} = -A \int_0^\infty e^{-\beta} \frac{1}{3}\left(\frac{\eta^2}{9}\right)^{-1} d\beta$

Partial Differential Equations by Caplace Transformation:

Example: Solve the PDE by using:

1. Separation of variable method.
2. Caplace transform method.

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

For the boundary condition

i) $\theta(0,t) = 0$

ii) $\theta(1,t) = 0$

iii) $\theta(x,0) = 3\sin 2\pi x$

1. Separation of variable:

$$\theta(x,t) = e^{-\beta^2 t}(A\cos\beta x + B\sin\beta x)$$

B.C. 1 $\quad x = 0 \;,\; \theta = 0$

$$0 = e^{-\beta^2 t}(A(1) + B(0)) \Rightarrow A = 0$$

$$\theta(x,t) = B\, e^{-\beta^2 t} \sin\beta x$$

B.C. 2 $\quad x = 1 \;,\; \theta = 0$

$$0 = B\, e^{-\beta^2 t} \sin\beta(1) \Rightarrow \beta = n\pi$$

$$\theta(x,t) = B\, e^{-(n\pi)^2 t} \sin n\pi x$$

I.C. $\quad t = 0 \;,\; \theta = 3\sin 2\pi x$

$$3\sin 2\pi x = B\sin n\pi x$$

$$\therefore \quad B = 3 \quad \& \quad n = 2$$

$$\vartheta(x,t) = 3\,e^{-4\pi^2 t}\,\sin 2\pi x$$

2. Laplace transform:

$$\mathcal{L}\frac{\partial\vartheta}{\partial t} = \mathcal{L}\frac{\partial^2\vartheta}{\partial x^2}$$

$$s\bar{\vartheta}(s) - \vartheta(0) = \frac{d^2\bar{\vartheta}(s)}{dx^2}$$

$$s\bar{\vartheta}(s) - 3\sin 2\pi x = \frac{d^2\bar{\vartheta}(s)}{dx^2}$$

$$\frac{d^2\bar{\vartheta}(s)}{dx^2} - s\bar{\vartheta}(s) = -3\sin 2\pi x$$

$$(D^2 - s)\,\bar{\vartheta}(s) = -3\sin 2\pi x$$

$$m^2 - s = 0 \Rightarrow m = \mp\sqrt{s}$$

$$\bar{\vartheta}(s)_c = C_1 e^{\sqrt{s}\,x} + C_2 e^{-\sqrt{s}\,x}$$

$$\bar{\vartheta}(s)_p = \frac{-3\sin 2\pi x}{D^2 - s}$$

$$y_p = \frac{1}{F(D^2)}\sin(ax+b) = \frac{1}{F(-a^2)}\sin(ax+b)$$

$$\bar{\vartheta}(s)_p = \frac{-3\sin 2\pi x}{-(2\pi)^2 - s}$$

$$\bar{\vartheta}(s)_p = \frac{3\sin 2\pi x}{4\pi^2 + s}$$

$$\bar{\theta}(s) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} + \frac{3\sin 2\pi x}{s+4\pi^2}$$

B.C.1 $x=0$, $\theta=0 \Rightarrow \bar{\theta}(s)=0$

$0 = C_1(1) + C_2(1) + 0$, $\sin(0)=0$

$\therefore C_1 = -C_2$

B.C.2 $x=1$, $\theta=0 \Rightarrow \bar{\theta}(s)=0$

$0 = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} + 0$, $\sin(2\pi)=0$

$0 = -C_2 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}}$

$0 = C_2(-e^{\sqrt{s}} + e^{-\sqrt{s}}) \Rightarrow C_2 = 0$

$C_1 = -C_2 \Rightarrow C_1 = 0$

$$\therefore \bar{\theta}(s) = \frac{3\sin 2\pi x}{s+4\pi^2}$$

$$\mathcal{L}^{-1}\bar{\theta}(s) = 3\sin 2\pi x\, \mathcal{L}^{-1}\frac{1}{s+4\pi^2}$$

$$\theta(x,t) = 3\sin 2\pi x\; e^{-4\pi^2 t}$$

Example: Solve the PDE

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$$

For the boundary condition

i) $C(x,0)=0$

ii) $C(0,t)=C_i$

iii) $C(\infty,t)=0$

$$\mathcal{L}\frac{\partial C}{\partial t} = D\,\mathcal{L}\frac{\partial^2 C}{\partial x^2}$$

$$s\bar{C}(s) - C(0) = D\frac{d^2\bar{C}(s)}{dx^2}$$

$$s\bar{C}(s) = D\frac{d^2\bar{C}(s)}{dx^2}$$

$$\frac{d^2\bar{C}(s)}{dx^2} - \frac{s}{D}\bar{C}(s) = 0$$

$$m^2 - \frac{s}{D} = 0 \Rightarrow m = \mp\sqrt{\frac{s}{D}}$$

$$\bar{C}(s) = C_1 e^{\sqrt{\frac{s}{D}}\,x} + C_2 e^{-\sqrt{\frac{s}{D}}\,x}$$

B.C.1 $x=0$, $C=C_i \Rightarrow \bar{C}(s) = \frac{C_i}{s}$

$$\frac{C_i}{s} = C_1 + C_2$$

B.C.2 $x=\infty$, $C=0 \Rightarrow \bar{C}(s)=0$

$$0 = C_1(\infty) + C_2(0) \quad , \quad e^{\infty}=\infty \;\&\; e^{-\infty}=0$$

$$C_1=0 \quad , \quad \frac{C_i}{s} = C_1 + C_2 \Rightarrow C_2 = \frac{C_i}{s}$$

$$\bar{C}(s) = \frac{C_i}{s} e^{-\sqrt{\frac{s}{D}}x}$$

$$\boxed{\mathcal{L}^{-1} \frac{1}{s} e^{-K\sqrt{s}} = \text{erfc} \frac{K}{2\sqrt{t}}} \quad , K = \frac{x}{\sqrt{D}}$$

$$C = C_i \ \text{erfc} \frac{x}{2\sqrt{D}\sqrt{t}}$$

$$C = C_i \ \text{erfc} \frac{x}{\sqrt{4Dt}}$$

LECTURE (2)

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

PART ONE

Second order differential equations may be classified as:

❶ Non-Linear Differential Equations:

1- Equations with dependent variable missing.

2- Equations with independent variable missing.

3- Homogenous equations.

❷ Linear Differential Equations:

1- Equations with constant coefficient.

2- Equations with constant coefficients as a function of the independent variable.

Examples:

1- $\frac{d2y}{dx^2} + 2\sin y$ Non-linear D.E. because the dependent variable appears as $\sin y, \cos y, \tan y, e^y, y^2$, ...

2- $\frac{d2y}{dx^2} - x\,(\frac{dy}{dx})^2 + y = 0$ Non-linear D.E. because $(\frac{dy}{dx})^2$

3- $\frac{d2y}{dx^2} + 4y\,\frac{dy}{dx} + 2y = \cos x$ Non-linear D.E. because $y\,\frac{dy}{dx}$

4- $\frac{d2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \cos x$ Linear D.E.

5- $\frac{d2y}{dx^2} + x\frac{dy}{dx} = e^{3x}$ Linear D.E.

6- $\frac{d2y}{dx^2} + y = x^2$ Linear D.E.

❶ Non-Linear Differential Equations:

1- Equations with dependent variable (y) missing.

$$P = \frac{dy}{dx} \rightarrow\rightarrow \frac{dp}{dx} = \frac{d2y}{dx^2}$$

EX(1) Solve the equation: $x\frac{d2y}{dx^2} - \frac{dy}{dx} = 0$

Solution:

$$P = \frac{dy}{dx} \ \& \ \frac{dp}{dx} = \frac{d2y}{dx^2}$$

Sub in the main equation: $x\frac{dp}{dx} - P = 0 \rightarrow\rightarrow \int\frac{dp}{p} - \int\frac{dx}{x} = 0$

$$\ln p - \ln x - \ln c_1 = 0$$

$$P = c_1\,x = \frac{dy}{dx} \rightarrow\rightarrow \int c_1\,x.\,dx = \int dy$$

$$y = \frac{c_1}{2}x^2 + c_2$$

EX(2) Solve the equation: $\frac{d2y}{dx^2} - x\frac{dy}{dx} = 0$

Solution:

$$P = \frac{dy}{dx} \ \& \ \frac{dp}{dx} = \frac{d2y}{dx^2}$$

$$\frac{dp}{dx} - xP = 0 \rightarrow\rightarrow \int\frac{dp}{p} - \int x\,dx = 0$$

$$\ln p = \frac{-x^2}{2} + \ln c_1 \rightarrow\rightarrow \ln p - \ln c_1 = \frac{-x^2}{2} \rightarrow\rightarrow \ln\frac{p}{c_1} = \frac{-x^2}{2}$$

$$P = c_1 e^{\frac{-x^2}{2}} = \frac{dy}{dx} \rightarrow\rightarrow dy = c_1 e^{\frac{-x^2}{2}}.\,dx$$

$$\int dy = c_1 \int e^{\frac{-x^2}{2}}.dx \;\; \rightarrow\rightarrow \;\; y = c_1 \int e^{\frac{-x^2}{2}}.dx$$

2- EQUATIONS WITH INDEPENDENT VARIABLE (x)MISSING.

$$P = \frac{dy}{dx} \rightarrow\rightarrow \frac{dp}{dx} = \frac{d2y}{dx^2}$$

$$\frac{dp}{dx} = \frac{dp}{dy} . \frac{dy}{dx} = \frac{dp}{dy} . P \quad \text{(chain Rule)}$$

EX(1) Solve the equation: $y \frac{d2y}{dx^2} + 1 = (\frac{dy}{dx})^2$

Solution:

$$P = \frac{dy}{dx} \;\&\; \frac{dp}{dx} = \frac{d2y}{dx^2}$$

$$\frac{dp}{dx} = \frac{dp}{dy} . \frac{dy}{dx} = \frac{dp}{dy} . P$$

Sub. in the main equation: $y.p. \frac{dp}{dy} + 1 = p^2$

$$\int \frac{dy}{y} = \int \frac{p.dp}{p^2 - 1}$$

$$\ln y + \ln c_1 = \frac{1}{2} \ln(p^2 - 1)$$

$$c_1{}^2 . y^2 = p^2 - 1 \rightarrow\rightarrow (c_1{}^2 . y^2) + 1 = p^2$$

$$P = \sqrt{(c_1{}^2 . y^2) + 1} = \frac{dy}{dx}$$

$$\int dx = \int \frac{dy}{\sqrt{(c_1{}^2 . y^2) + 1}}$$

$$x = \frac{1}{c_1} \sinh^{-1}(c_1 y) + c_2$$

$$y = \frac{1}{c_1} \sinh(c_1 x + c_1 c_2)$$

3- HOMOGENOUS EQUATIONS.

This is defined and recognized by the form:

$$\boldsymbol{x . \frac{d2y}{dx^2} = f\left(\frac{dy}{dx}, \frac{y}{x}, \frac{x}{y}\right)}$$

$$y = \mathrm{v}. x \quad \ldots\ldots\ldots\ldots (1)$$

$$\frac{dy}{dx} = v + x \ \frac{dv}{dx} \quad \ldots\ldots\ldots\ldots (2)$$

$$\frac{d2y}{dx^2} = \frac{dv}{dx} + x . \frac{d2v}{dx^2} + \frac{dv}{dx}$$

$$\frac{d2y}{dx^2} = 2\frac{dv}{dx} + x . \frac{d2v}{dx^2} \quad \ldots\ldots\ldots\ldots (3)$$

Substitute (1), (2), and (3) in D.E. , this leads to Euler's equation:

$$x^2 . \frac{d2y}{dx^2} = f\left(x \ \frac{dy}{dx}, y\right) \ \text{ or } \ x^2 . \frac{d2y}{dx^2} = f\left(x \ \frac{dv}{dx}, v\right)$$

Which is solved by substitute:

$$x = e^t \rightarrow\rightarrow \mathrm{t} = \ln x \ ; \ \ \frac{dt}{dx} = \frac{1}{x} \ \ldots\ldots\ldots (4)$$

$$\frac{dv}{dx} = \frac{dv}{dt} . \frac{dt}{dx} = \frac{1}{x} . \frac{dv}{dt} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{d2v}{dx^2} = -\frac{1}{x^2} \frac{dv}{dt} + \frac{1}{x} . \frac{d}{dx}\left(\frac{dv}{dt}\right) = -\frac{1}{x^2} . \frac{dv}{dt} + \frac{1}{x} . \frac{dt}{dx} \ \frac{d}{dt}\left(\frac{dv}{dt}\right) = -\frac{1}{x^2} . \frac{dv}{dt} + \frac{1}{x^2} . \frac{d2v}{dt^2}$$

$$x^2 . \frac{d2v}{dx^2} = \frac{d2v}{dt^2} - \frac{dv}{dt} \quad \ldots\ldots\ldots\ldots (6)$$

EX(1) Solve the equation: $\boldsymbol{2x^2 y . \frac{d2y}{dx^2} + y^2 = x^2 \left(\frac{dy}{dx}\right)^2}$

Solution: divided by $(2x\boldsymbol{y})$

$$x.\frac{d2y}{dx^2}+\frac{1}{2}\frac{y}{x} = \frac{1}{2}\frac{x}{y}(\frac{dy}{dx})^2$$ **(Homogenous D.E.)**

$$\boldsymbol{y = v.x} \quad .... (1)$$

$$\boldsymbol{\frac{dy}{dx} = v + x\ \frac{dv}{dx}} \quad .... (2)$$

$$\frac{d2y}{dx^2} = \frac{dv}{dx} + x.\frac{d2v}{dx^2} + \frac{dv}{dx}$$

$$\frac{d2y}{dx^2} = 2\frac{dv}{dx} + x.\frac{d2v}{dx^2} \quad .... (3)$$

Substitute (1), (2), and (3) in the main equation:

$$2x^3v.[2\frac{dv}{dx} + x.\frac{d2v}{dx^2}] + v^2x^2 = x^2\ [v\ +\ x\ \frac{dv}{dx}]^2$$

$$4x^3v.\frac{dv}{dx} + 2x^4v.\frac{d2v}{dx^2} + v^2x^2 = x^2\ [v^2 + 2\ vx\frac{dv}{dx} + x^2(\frac{dv}{dx})^2]$$

$$4x^3v.\frac{dv}{dx} + 2x^4v.\frac{d2v}{dx^2} + v^2x^2 = v^2x^2\ + 2\ vx^3\ \frac{dv}{dx} + x^4(\frac{dv}{dx})^2$$

$$2\ vx^3\ \frac{dv}{dx} + 2x^4v.\frac{d2v}{dx^2} = x^4(\frac{dv}{dx})^2$$

Divided by $\boldsymbol{x^2}$:

$$2xv.\frac{dv}{dx} + 2x^2v.\frac{d2v}{dx^2} = x^2(\frac{dv}{dx})^2$$ **Euler equation**

$$x = e^t \rightarrow\rightarrow \text{t} = \ln x\ ;\ \boldsymbol{\frac{dt}{dx} = \frac{1}{x}} \ ...... (4)$$

$$\boldsymbol{\frac{dv}{dx}} = \frac{dv}{dt}\ .\ \frac{dt}{dx} = \boldsymbol{\frac{1}{x}.\ \frac{dv}{dt}} \quad .... (5)$$

$$x^2.\frac{d2v}{dx^2} = \frac{d2v}{dt^2} - \frac{dv}{dt} \quad .... (6)$$

Sub. in the above Euler Equation :

$$2\ v\ .\ \frac{dv}{dt} + 2\ v\ (\frac{d2v}{dt^2} - \frac{dv}{dt}) = (\frac{dv}{dt})^2$$

$$2\ v\ .\ \frac{dv}{dt} + 2\ v\ .\frac{d2v}{dt^2} - 2\ v\ .\frac{dv}{dt}\ = (\frac{dv}{dt})^2$$

$$2\,v\,.\frac{d2v}{dt^2} - \left(\frac{dv}{dt}\right)^2 = 0 \text{ ..... (7)}$$

$$p = \frac{dv}{dt} \;\&\; \frac{dp}{dt} = \frac{d2v}{dt^2}$$

$$\frac{dp}{dt} = \frac{dp}{dv}.\frac{dv}{dt} = \text{p}.\frac{dp}{dv} \quad \textbf{sub.in (7)}$$

$$2\,vp.\frac{dp}{dv} - p^2 = 0 \;\to\to\to\; \frac{dp}{p} = \frac{dv}{2v}$$

$$\ln p = \frac{1}{2}\ln v + \ln c_1 \to\to\to p = c_1 v^{\frac{1}{2}} = \frac{dv}{dt} \;; \qquad \left(p = \frac{dv}{dt}\right)$$

$$v^{\frac{-1}{2}}\,.\,dv = c_1\,dt \to\to\to 2v^{\frac{1}{2}} = c_1\text{t} + c_2 \to\to\to v^{\frac{1}{2}} = \frac{c_1}{2}.\text{t} + \frac{c_2}{2}$$

$$v^{\frac{1}{2}} = c_1`\,.\text{t} + c_2` \to\to\to v = (\,c_1`\,.\,\text{t} + c_2`)^2 \to\to\to \frac{y}{x} = (\,c_1`\,.\ln x + c_2`)^2$$

EX(2) Solve the equation: $x^2.\frac{d2y}{dx^2} + y = x\,\frac{dy}{dx}$

Solution:

$$x.\frac{d2y}{dx^2} + \frac{y}{x} = \frac{dy}{dx} \qquad \textbf{(Homogenous D.E.)}$$

$$y = v.x \text{ .... (1)}$$

$$\frac{dy}{dx} = v + x\,\frac{dv}{dx} \text{ .... (2)}$$

$$\frac{d2y}{dx^2} = \frac{dv}{dx} + x.\frac{d2v}{dx^2} + \frac{dv}{dx}$$

$$\frac{d2y}{dx^2} = 2\frac{dv}{dx} + x.\frac{d2v}{dx^2} \text{ .... (3)}$$

Substitute (1), (2), and (3) in D.E.

$$2x.\frac{dv}{dx} + x^2.\frac{d2v}{dx^2} + v = v + x\,\frac{dv}{dx} \qquad \textbf{Euler equation}$$

$$2x.\frac{dv}{dx} + x^2.\frac{d2v}{dx^2} + v - v - x\ \frac{dv}{dx} = 0 \rightarrow\rightarrow\rightarrow\ x^2.\frac{d2v}{dx^2} + x.\frac{dv}{dx} = 0 \rightarrow\rightarrow\rightarrow$$

$$x(x.\frac{d2v}{dx^2} + \ \frac{dv}{dx}) = 0$$

either x = 0 or $x.\frac{d2v}{dx^2} + \ \frac{dv}{dx} = 0$

$$\boldsymbol{p = \frac{dv}{dx}\ \&\ \frac{dp}{dx} = \frac{d2v}{dx^2}}$$

$$x.\frac{dp}{dx} + p = 0 \rightarrow\rightarrow\rightarrow \frac{dp}{p} + \frac{dx}{x} = 0$$

$$\ln p + \ \ln x = \ \ln c_1 \rightarrow\rightarrow\rightarrow p = \frac{c_1}{x} = \frac{dv}{dx}$$

$$c_1.\frac{dx}{x} = \ dv$$

$$c_1 \ln x = v - c_1 \ln c_2 \rightarrow\rightarrow\rightarrow c_1 \ln x + c_1 \ln c_2 = v$$

$$c_1\ (\ln x + \ln c_2) = v = \frac{y}{x}$$

$$c_1\ (\ln x c_2) = \frac{y}{x} \quad \rightarrow\rightarrow\rightarrow \qquad y = c_1 x \ln x c_2$$

LECTURE (2)

SECOND ORDER DIFFERENTIAL EQUATIONS

PART TWO

❷ Linear Differential Equations:

1- Equations with constant coefficient.

2- Equations with constant coefficients as a function of the independent variable.

1- Equations with constant coefficient.

The general second order linear differential equation is:

$$\mathbf{A}\frac{d2y}{dx^2} + \mathbf{B}\frac{dy}{dx} + C_y = \mathrm{f(x)}$$

where A, B, & C are constants & $f(x)$ may a function of x or constant.

It is clear that equation has two solutions:

1) Complementary solution (y_c).

2) Particular solution (y_p).

The general solution is: $(y = y_c + y_p)$

1) **Complementary solution (y_c):**

We start with $\mathbf{A}\frac{d2y}{dx^2} + \mathbf{B}\frac{dy}{dx} + C_y = 0$

Putting this equation in the form of **D-Operator** $(\mathrm{A}\, D^2 + \mathrm{B}\, D + C)\mathrm{y} = 0$

Substitute for **D** by the constant **m**, we find that:

Either y= 0 **(not possible)**

Or A m^2 + B m + C = 0 **(auxiliary equation)**

1. **The roots are real and different (m_1& m_2)**

$$y_{c=} \text{ A.}e^{m_1x}\text{+ B. } e^{m_2x}$$

2. **The roots are equal ($m_1 = m_2 = \text{m}$)**

$$y_{c=} e^{mx}(\text{A+ B}x)$$

3. **The roots are complex ($m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$)**

$$y_{c=} e^{\alpha x}(\text{A.sin } \beta x + \text{B.cos } \beta x)$$

EX (1) Solve the equation: $\frac{d2y}{dx^2}$ - 2 $\frac{dy}{dx}$ - 3$y = 0$

Solution:

$$\text{A}\, m^2 + \text{B}\, m + C = 0$$

$$m^2 - 2\text{m} - 3 = 0$$

$$(\text{m-3})\ (\text{m+1}) = 0$$

$m_1 = 3$; m_2= -1 (**The roots are real and different (m_1& m_2)**)

$$y_{c=} \text{ A.}e^{m_1x}\text{+ B. } e^{m_2x}$$

$$= \text{A.}e^{3x}\text{+ B. } e^{-x}$$

EX (2) Solve the equation: $\frac{d2y}{dx^2}$ + 4 $\frac{dy}{dx}$ + 4$y = 0$

Solution: $\text{A}\, m^2 + \text{B}\, m + C = 0$

$$m^2 + 4\text{m} + 4 = 0$$

$$(\text{m+2})\ (\text{m+2}) = 0$$

$m_1 = -2$; m_2= -2 (**The roots are equal ($m_1 = m_2 = \text{m}$)**)

$$y_{c=}\ e^{mx}(\mathrm{A}+\mathrm{B}x)$$

$$= e^{-2x}(1+4x)$$

EX (3) Solve the equation: $\frac{d2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

Solution:

$$\mathrm{A}\,m^2 + \mathrm{B}\,m + C = 0$$

$$m^2 - 4\mathrm{m} + 5 = 0$$

$$\mathrm{m} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{-(4)\pm\sqrt{(4)^2-4(1)(5)}}{2\ (1)}$$

$$\mathrm{m} = \pm\, i \quad (\mathrm{m} = \alpha \pm \mathrm{i}\beta)$$

$$y_{c\,=}\ e^{\alpha x}(\mathrm{A.sin}\ \beta x + \mathrm{B.cos}\ \beta x)$$

$$= e^{2x}(\mathrm{A}.\,sin\ x + \mathrm{B}.cos\ x)$$

2) Particular solution (y_p):

There are many methods to find the particular solution, here, we consider two of the most common ones.

1. **Method of undetermined coefficient.**
2. **The inverse D-operator method.**

1.Method of undetermined coefficient:

In this method, we select the form of the particular solution then we calculate the coefficient in the function. If the particular solution is similar to a term in the complementary solution then we multiply the particular solution by the independent variable (x) and if this is, still, similar to another term we multiply by (x) again. This steps will be repeated until is no similarity.

EX (1) Solve the equation: $\frac{d2y}{dx^2} + 2\frac{dy}{dx} = 3e^x$

Solution:

$$m^2 + 2\text{m} = 0$$

$m_1 = 0;\ m_2$= -2 **(The roots are real and different)**

$$y_{c=}\ \text{A}.e^{m_1x} + \text{B}.\ e^{m_2x} = A + \text{B}.e^{-2x}$$

when the function is ($\propto e^x$) (where $\propto$ is a constant) then:

$$y = \text{A}.e^x;\ \frac{dy}{dx} = \text{A}.e^x;\ \frac{d2y}{dx^2} = \text{A}.e^x$$

Sub. in the main equation: $\text{A}.e^x + 2\ \text{A}.\ e^x = 3\ \text{A}.\ e^x$

$$\text{A} = 1;\ y_{p=}\ \text{A}.e^x = e^x$$

$$y = y_c + y_p = (A + \text{B}.e^{-2x}) + e^x$$

EX (2) Solve the equation: $\frac{d2y}{dx^2} + y = \sin x$

Solution:

$$m^2 + 1 = 0$$

$$m = \pm\sqrt{-1} \quad (\mathrm{m} = \alpha \pm \mathrm{i}\beta)$$

$$\alpha = 0; \beta = 1$$

$$y_c = e^{\alpha x}(\mathrm{A}.sin\ \beta x + \mathrm{B}.cos\ \beta x)$$

$$y_{c=}\ A \sin x + \ B \cos x$$

when $f(x)$is expressed by ($\propto \sin x$) (where $\propto$ is a constant) then:

$$y_{p\,=}\ A \sin x + B \cos x$$

$$(y_c \text{ is similar to } y_p)$$

thus we multiply the particular solution by the independent variable **(x):**

$$y_{p\,=} (A \sin x + B \cos x).x \quad = \quad A\,x\sin x + B\,x\cos x$$

$$\frac{dy}{dp} \ = A \sin x + A\,x\cos x + B \cos x \ - \ Bxsinx$$

$$\frac{d2y}{dp^2} = \mathrm{A}\cos x + A \cos x \ \ - A\,x\sin x - \ Bsinx - \ Bsinx - \ B\,x\cos x$$

$$\frac{d2y}{dp^2} = 2\,A \cos x \ \ - 2\,Bsinx - A\,x\sin x - B\,x\cos x$$

Sub. in the main equation: $\frac{d2y}{dx^2} + y = sin\ x$

$$(2\,A\,cos\,x \ \ - 2\,Bsinx - A\,xsin\,x - B\,xcos\,x\ \ ) + (A\,xsin\,x + B\,xcos\,x\,) = sinx$$

$$2\,A\,cos\,x \ \ - 2\,Bsinx - A\,xsin\,x - B\,xcos\,x + A\,xsin\,x + B\,xcos\,x \ = sinx$$

$$2\,A\cos x\ \ -2\,B\sin x\ = \sin x$$

$$\sin x\ \rightarrow\rightarrow\rightarrow\quad -2B = \ 1\ \ \rightarrow\rightarrow\rightarrow\ \ B = -\frac{1}{2}$$

$$\cos x\ \ \rightarrow\rightarrow\rightarrow\qquad 2A = 0\ \ \rightarrow\rightarrow\rightarrow\ \ A = 0$$

$$y_p = (A\,x\sin x + B\,x\cos x\,)\ \ = -\frac{1}{2}\,x\cos x$$

Now, (y_c is not similar to y_p) thus:

$$y = y_c + y_p$$

$$y = A\sin x\ + B\cos x - \frac{1}{2}\,x\cos x$$

The Inverse D-operator Method:

Definitions:

$D = \frac{d}{dx} \Rightarrow Dy = \frac{dy}{dx}$ $\quad * \; Dy \neq yD$

$D^2 = \frac{d^2}{dx^2} \Rightarrow D^2y = \frac{d^2y}{dx^2}$

$D^3 = \frac{d^3}{dx^3} \Rightarrow D^3y = \frac{d^3y}{dx^3}$

$D^n = \frac{d^n}{dx^n}$ means n number of differentiation.

$\frac{1}{D} = \int$ means integration.

$\frac{1}{D^n}$ means n number of integration.

Several cases can be used to find a particular solution:

First Rule:

$$\boxed{y_p = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}} \; , \; F(a) \neq 0$$

This rule is used when the right hand of D.E. are:
1. e^{ax} 2. Ae^{ax} 3. A

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{2x}$$

$m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

$m_1 = 3 \; \& \; m_2 = -2$

$y_c = Ae^{3x} + Be^{-2x}$

$(D^2 - D - 6)y = e^{2x}$

$y_p = \frac{1}{D^2 - D - 6} e^{2x} \Rightarrow y_p = \frac{1}{(2)^2 - 2 - 6} e^{2x}$

$y_p = \frac{-1}{4} e^{2x}$

$y = y_c + y_p \Rightarrow y = Ae^{3x} + Be^{-2x} - \frac{1}{4} e^{2x}$

Ex: Solve the equation :

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3e^{2x}$$

$m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0 \Rightarrow m_1 = 3$ & $m_2 = -2$

$y_c = Ae^{3x} + Be^{-2x}$

$(D^2 - D - 6)y = 3e^{2x}$

$y_p = \frac{1}{D^2 - D - 6} 3e^{2x} \Rightarrow y_p = \frac{1}{4 - 2 - 6} 3e^{2x}$

$y_p = -\frac{1}{4} 3e^{2x} \Rightarrow y_p = -\frac{3}{4} e^{2x}$

$y = y_c + y_p \Rightarrow y = Ae^{3x} + Be^{-2x} - \frac{3}{4} e^{2x}$

Ex: Solve the equation:

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=9$$

$$m^2+4m+3=0 \Rightarrow (m+3)(m+1)=0$$
$$m_1=-3 \quad \& \quad m_2=-1$$

$$y_c=Ae^{-3x}+Be^{-x}$$

$$(D^2+4D+3)y=9$$

$$y_p=\frac{1}{D^2+4D+3}\,9 \Rightarrow y_p=\frac{1}{(0)^2+4(0)+3}\,9$$

$$y_p=\frac{1}{3}\cdot 9 \Rightarrow y_p=3$$

$$y=y_c+y_p \Rightarrow y=Ae^{-3x}+Be^{-x}+3$$

Second Rule:

$$\boxed{y_p=\frac{1}{F(D)}e^{ax}F(x)=e^{ax}\frac{1}{F(D+a)}F(x)}$$

This rule is used when the right hand of D.E. is $(e^{ax}.F(x))$ and when the first rule is failer $(F(a)=0)$.

Ex: Solve the equation

$$\frac{d^2y}{dx^2}-y=e^{x}$$

$$m^2-1=0 \Rightarrow (m-1)(m+1)=0 \Rightarrow m_1=1 \;\&\; m_2=-1$$

$y_c = Ae^{x} + Be^{-x}$

$(D^2 - 1)y = e^{x}$

$y_p = \frac{1}{D^2 - 1} e^{x} \Rightarrow F(a) = 0$

$y_p = e^{x} \frac{1}{(D+1)^2 - 1} . 1 \Rightarrow y_p = e^{x} \frac{1}{D^2 + 2D + 1 - 1} . 1$

$y_p = e^{x} \frac{1}{D(D+2)} . 1$

$$\begin{array}{r|l} & \frac{1}{2} - \frac{1}{4}D + \frac{1}{8}D^2 \\ \hline D + 2 & 1 \\ & \mp 1 \mp \frac{1}{2}D \\ \hline & -\frac{1}{2}D \\ & \pm \frac{1}{2}D \pm \frac{1}{4}D^2 \\ \hline & +\frac{1}{4}D^2 \\ & \mp \frac{1}{4}D^2 \mp \frac{1}{8}D^3 \\ \hline \end{array}$$

$y_p = e^{x} \frac{1}{D} \left[\frac{1}{2} - \frac{1}{4}D + \frac{1}{8}D^2\right] . 1$

$y_p = e^{x} \left[\frac{x}{2} - \frac{1}{4} + \frac{1}{8}D\right]$

$y_p = e^{x} \left(\frac{x}{2} - \frac{1}{4}\right)$

$y_p = \frac{x}{2} e^{x} - \frac{1}{4} e^{x}$

$y = y_c + y_p \Rightarrow y = Ae^{x} + Be^{-x} + \frac{x}{2} e^{x} - \frac{1}{4} e^{x}$

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$$

$m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \Rightarrow m_1 = 3 \;\&\; m_2 = 2$

$y_c = Ae^{3x} + Be^{2x}$

$$(D^2-5D+6)y=e^{3x}$$

$$y_p=\frac{1}{D^2-5D+6}e^{3x} \quad , \quad F(a)=0$$

$$y_p=e^{3x}\frac{1}{(D+3)^2-5(D+3)+6}.1 \Rightarrow y_p=e^{3x}\frac{1}{D^2+D}.1$$

$$y_p=e^{3x}\frac{1}{D(D+1)}.1 \Rightarrow y_p=e^{3x}\frac{1}{D}(1+D)^{-1}.1$$

Note: $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots+\frac{1}{n!}x^n$

$$(1-D)^{-2}=1+(-2)(-D)+\frac{(-2)(-3)}{2!}(-D)^2+\frac{(-2)(-3)(-4)}{3!}(-D)^3+\dots$$

$$(1+D)^{-1}=1-D+\frac{(-1)(-2)}{2!}D^2+\frac{(-1)(-2)(-3)}{3!}D^3+\dots$$

$$\therefore y_p=e^{3x}\frac{1}{D}(1-D+D^2).1$$

$$y_p=e^{3x}\left(\frac{1}{D}-1+D\right).1 \Rightarrow y_p=e^{3x}(x-1+0)$$

$$y_p=xe^{3x}-e^{3x}$$

$$y=y_c+y_p \Rightarrow y=Ae^{3x}+Be^{2x}+xe^{3x}$$

Another solution:

$$y_p=e^{3x}\frac{1}{D(D+1)}.1$$

$$y_p=e^{3x}\frac{1}{D}(1-D+D^2).1$$

then continue ...

$$\begin{array}{r|l} & 1-D+D^2 \\ \hline D+1 & 1 \\ & \mp 1 \mp D \\ \hline & -D \\ & \pm D \pm D^2 \\ \hline & +D^2 \\ & \mp D^2 \mp D^3 \\ \hline \end{array}$$

Ex: Solve the equation:

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=xe^{2x}$$

$$m^2+4m+4=0 \Rightarrow (m+2)^2=0 \Rightarrow m_1=m_2=m=-2$$

$$y_c=e^{-2x}(A+Bx)$$

$$(D^2+4D+4)y=xe^{2x}$$

$$y_p=\frac{1}{D^2+4D+4}xe^{2x}$$

$$y_p=e^{2x}\frac{1}{(D+2)^2+4(D+2)+4}x$$

$$y_p=e^{2x}\frac{1}{D^2+8D+16}x$$

$$\begin{array}{r|l} & \frac{1}{16}-\frac{D}{32} \\ \hline D^2+8D+16 & 1 \\ & \mp 1\mp\frac{D}{2}\mp\frac{D^2}{16} \\ \hline & -\frac{D}{2}-\frac{D^2}{16} \\ & \pm\frac{D}{2}\pm\frac{D^2}{4}\pm\frac{D^3}{32} \\ \hline \end{array}$$

$$\therefore y_p=e^{2x}\left[\frac{1}{16}-\frac{D}{32}\right]x$$

$$y_p=e^{2x}\left[\frac{x}{16}-\frac{1}{32}\right]$$

$$y_p=\frac{1}{16}xe^{2x}-\frac{1}{32}e^{2x}$$

$$y=y_c+y_p \Rightarrow y=Ae^{-2x}+Bxe^{-2x}+\frac{1}{16}xe^{2x}-\frac{1}{32}e^{2x}$$

Third Rule :

$$\boxed{y_p = \frac{1}{F(D)} x^n = (a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n) x^n}$$

This rule is used when the right hand of D. E. are (x^n) or constant (A).

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2x^2$$

$$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m_1 = m_2 = m = -2$$

$$y_c = e^{-2x}(A + Bx)$$

$$(D^2 + 4D + 4)y = 2x^2$$

$$y_p = \frac{1}{D^2 + 4D + 4} 2x^2$$

$$\begin{array}{r|l} & \frac{1}{4} - \frac{D}{4} + \frac{3}{16}D^2 \\ \hline D^2 + 4D + 4 & 1 \\ & \mp 1 \mp D \mp \frac{1}{4}D^2 \\ \hline & -D - \frac{1}{4}D^2 \\ & \pm D \pm D^2 \pm \frac{D^3}{4} \\ \hline & \frac{3}{4}D^2 + \frac{D^3}{4} \end{array}$$

$$y_p = \left(\frac{1}{4} - \frac{D}{4} + \frac{3}{16}D^2\right) 2x^2$$

$$y_p = \frac{x^2}{2} - \frac{4x}{4} + \frac{3}{16} \cdot 4$$

$$y_p = \frac{x^2}{2} - x + \frac{3}{4}$$

$$y = y_c + y_p \Rightarrow y = Ae^{-2x} + Bxe^{-2x} + \frac{x^2}{2} - x + \frac{3}{4}$$

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x$$

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m_1 = m_2 = m = 1$$

$$y_c = e^x (A + Bx)$$

$$(D^2 - 2D + 1)\, y = x$$

$$y_p = \frac{1}{D^2 - 2D + 1}\, x \Rightarrow y_p = \frac{1}{(D-1)^2}\, x$$

$$\text{or } y_p = \frac{1}{(1-D)^2}\, x \Rightarrow y_p = (1-D)^{-2} \cdot x$$

$$(1-D)^{-2} = 1 + (-2)(-D) + \frac{(-2)(-3)}{2!}(-D)^2 + \dots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + \dots$$

$$\therefore y_p = (1 + 2D + 3D^2 + \dots)\, x$$

$$y_p = x + 2$$

$$y = y_c + y_p \Rightarrow y = Ae^x + Bxe^x + x + 2$$

<u>Another solution</u>:

$$y_p = \frac{1}{D^2 - 2D + 1}\, x$$

$$y_p = (1 + 2D + 3D^2)\, x$$

$$\begin{array}{r|l} & 1 + 2D + 3D^2 \\ \hline D^2 - 2D + 1 & 1 \\ & \mp 1 \pm 2D \mp D^2 \\ \hline & +2D - D^2 \\ & \mp 2D \pm 4D^2 \mp 2D^3 \\ \hline & +3D^2 - 2D^3 \end{array}$$

then continue ...

Fourth Rule :

$$y_p = \frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b) ,$$

$$F(-a^2) \neq 0$$

$$y_p = \frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-a^2)} \cos(ax+b) ,$$

$$F(-a^2) \neq 0$$

This rule is used when the right hand of D.E. are $\sin(ax+b)$ or $\cos(ax+b)$ and $F(-a^2) \neq 0$.

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} - 4y = \cos(2x+3)$$

$$m^2 - 4 = 0 \Rightarrow (m-2)(m+2) = 0 \Rightarrow m_1 = 2 \ \& \ m_2 = -2$$

$$y_c = Ae^{2x} + Be^{-2x}$$

$$(D^2-4)y = \cos(2x+3)$$

$$y_p = \frac{1}{D^2-4} \cos(2x+3)$$

$$y_p = \frac{1}{-(2)^2-4} \cos(2x+3)$$

$$y_p = \frac{1}{-8} \cos(2x+3)$$

$$y = y_c + y_p \Rightarrow y = Ae^{2x} + Be^{-2x} - \frac{1}{8} \cos(2x+3)$$

Ex: Solve the equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = \sin 2x$$

$$m^2+3m-4=0 \Rightarrow (m+4)(m-1)=0 \Rightarrow m_1=-4 \ \& \ m_2=1$$

$$y_c = Ae^{-4x} + Be^{x}$$

$$(D^2+3D-4)y = \sin 2x$$

$$y_p = \frac{1}{D^2+3D-4}\sin 2x$$

$$y_p = \frac{1}{-4+3D-4}\sin 2x \Rightarrow y_p = \frac{1}{3D-8}\sin 2x$$

$$y_p = \frac{(3D+8)}{(3D-8)(3D+8)}\sin 2x \Rightarrow y_p = \frac{3D+8}{9D^2-64}\sin 2x$$

$$y_p = \frac{3D+8}{9(-4)-64}\sin 2x \Rightarrow y_p = \frac{3D+8}{-100}\sin 2x$$

$$y_p = \frac{6\cos 2x + 8\sin 2x}{-100}$$

$$y_p = -\frac{6}{100}\cos 2x - \frac{8}{100}\sin 2x$$

$$y_p = -\frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

$$y = y_c + y_p \Rightarrow y = Ae^{-4x} + Be^{x} - \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

Fifth Rule :

$$y_p = \frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-(a+h)^2)} \sin((a+h)x+b),$$

$$F(-a^2) = 0$$

$$y_p = \frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-(a+h)^2)} \cos((a+h)x+b),$$

$$F(-a^2) = 0$$

This rule is used when the right hand of D. E. are $\sin(ax+b)$ or $\cos(ax+b)$ and $F(-a^2)=0$ (when the fourth rule is failer).

Ex: Solve the equation

$$\frac{d^2y}{dx^2} + 4y = \cos 2x$$

$$m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\mp\sqrt{-4} \Rightarrow m=\mp 2i$$

$$y_c = A\cos 2x + B\sin 2x \quad , \quad \alpha = 0$$

$$(D^2+4)y = \cos 2x$$

$$y_p = \frac{1}{D^2+4} \cos 2x \quad , \quad F(-a^2)=0$$

$$y_p = \frac{1}{-(2+h)^2+4} \cos(2+h)x$$

$$y_p = \frac{1}{-(4+4h+h^2)+4} \cos(2+h)x$$

$$y_p = \frac{1}{-h(4+h)} \cos(2+h)x$$

$$F(h) = F(h_0) + \frac{dF}{dh}\Big|_{h_0}(h-h_0) + \dots$$

<u>Note</u> : $F(x) = F(x_0) + \frac{dF}{dx}\Big|_{x_0}(x-x_0) + \frac{1}{2!}\frac{d^2F}{dx^2}\Big|_{x_0}(x-x_0)^2 + \dots$

$$\cos(2+h)x = \cos(2+\overset{0}{h_0})x + (-\sin(2+h_0)x \cdot x)(h-h_0)$$
$$= \cancel{\cos 2x} - hx\sin 2x \quad \text{(repeated in } y_c\text{)}$$
$$= -hx\sin 2x$$

$$\therefore y_p = \frac{1}{-h(4+h)}(-hx\sin 2x) \Rightarrow y_p = \frac{x\sin 2x}{4}$$

$$y = y_c + y_p \Rightarrow y = A\cos 2x + B\sin 2x + \frac{1}{4}x\sin 2x$$

Sixth Rule :

$$\boxed{y_p = \frac{1}{F(D)}xF(x) = x\frac{1}{F(D)}F(x) - \frac{F'(D)}{[F(D)]^2}F(x)}$$

This rule is used when the right hand of D.E. is $xF(x)$.

Ex: Solve the equation :

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x\sin 2x$$

$m^2+3m+2=0 \Rightarrow (m+2)(m+1)=0 \Rightarrow m_1=-2 \ \& \ m_2=-1$

$y_c = Ae^{-2x} + Be^{-x}$

$(D^2+3D+2)y = x\sin 2x$

$$y_p = \frac{1}{D^2+3D+2} x\sin 2x$$

$$y_p = x\frac{1}{D^2+3D+2}\sin 2x - \frac{2D+3}{(D^2+3D+2)^2}\sin 2x$$

$$y_p = x\frac{1}{-4+3D+2}\sin 2x - \frac{2D+3}{(-4+3D+2)^2}\sin 2x$$

$$y_p = x\frac{1}{3D-2}\sin 2x - \frac{2D+3}{(3D-2)^2}\sin 2x$$

$$y_p = x\frac{(3D+2)}{(3D-2)(3D+2)}\sin 2x - \frac{2D+3}{9D^2-12D+4}\sin 2x$$

$$y_p = x\frac{3D+2}{9D^2-4}\sin 2x - \frac{2D+3}{9(-4)-12D+4}\sin 2x$$

$$y_p = x\frac{3D+2}{9(-4)-4}\sin 2x - \frac{2D+3}{4(-8-3D)}\sin 2x$$

$$y_p = \frac{x}{-40}(3D+2)\sin 2x - \frac{2D+3}{-4(3D+8)}\sin 2x$$

$$y_p = \frac{x}{-40}(3D+2)\sin 2x + \frac{(2D+3)\ (3D-8)}{4(3D+8)\ (3D-8)}\sin 2x$$

$$y_p = \frac{x}{-40}(3D+2)\sin 2x + \frac{1}{4}\,\frac{(3D-8)(2D+3)}{9D^2-64}\sin 2x$$

$$y_p = \frac{x}{-40}\left(3(2)\cos 2x + 2\sin 2x\right) + \frac{1}{4}\,\frac{(6D^2-7D-24)}{9(-4)-64}\sin 2x$$

$$y_p = -\frac{x}{20}(3\cos 2x + \sin 2x) + \frac{1}{4}\,\frac{-24\sin 2x - 7(2)\cos 2x - 24\sin 2x}{-100}$$

$$y_p = -\frac{x}{20}(3\cos 2x + \sin 2x) + \frac{24\sin 2x + 7\cos 2x}{200}$$

$$y_p = -\frac{3}{20}x\cos 2x - \frac{1}{20}x\sin 2x + \frac{6}{50}\sin 2x + \frac{7}{200}\cos 2x$$

$$y = y_c + y_p$$

$$y = Ae^{-2x} + Be^{-x} - \frac{3}{20}x\cos 2x - \frac{1}{20}x\sin 2x + \frac{6}{50}\sin 2x + \frac{7}{200}\cos 2x$$

Simultaneous Differential Equation :

1. **Systematic Elimination** : in this method we eliminate the variables and their derivatives algebraically until we obtain an equation with only one dependent variable.

Ex: Solve the two simultaneous differential equations

$$\frac{dx}{dt} + 5x + \frac{dy}{dt} + 3y = e^{-t}$$

$$2\frac{dx}{dt} + x + \frac{dy}{dt} + y = 3$$

$$\frac{dx}{dt} + 5x + \frac{dy}{dt} + 3y = e^{-t}$$

$$\mp 10\frac{dx}{dt} \mp 5x \mp 5\frac{dy}{dt} \mp 5y = \mp 15$$

$$-9\frac{dx}{dt} - 4\frac{dy}{dt} - 2y = e^{-t} - 15$$

$$+2\frac{dx}{dt} + 10x + 2\frac{dy}{dt} + 6y = +2e^{-t}$$

$$\mp 2\frac{dx}{dt} \mp x \mp \frac{dy}{dt} \mp y = \mp 3$$

$$+9x + \frac{dy}{dt} + 5y = +2e^{-t} - 3$$

Differentiate with respect to t

$$9\frac{dx}{dt} + \frac{d^2y}{dt^2} + 5\frac{dy}{dt} = -2e^{-t}$$

$$-9\frac{dx}{dt} - 4\frac{dy}{dt} - 2y = +e^{-t} - 15$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = -e^{-t} - 15$$

$m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$m_1 = -2$ & $m_2 = 1$

$y_c = C_1 e^{-2t} + C_2 e^{t}$

$(D^2 + D - 2) y = -e^{-t} - 15$

$$y_p = \frac{1}{D^2 + D - 2} (-e^{-t}) - \frac{1}{D^2 + D - 2} (15)$$

2. D_Operator: Using the algebraic property of the D_operator.

Ex: Solve the two simultaneous differential equations

$$2\frac{dx}{dt} + 5x - 2\frac{dy}{dt} - 3y = t$$

$$\frac{dx}{dt} - 2x + \frac{dy}{dt} + 2y = 0$$

$$(2D+5)x - (2D+3)y = t \qquad * (D+2)$$
$$(D-2)x + (D+2)y = 0 \qquad * (2D+3)$$

$$(D+2)(2D+5)x - (D+2)(2D+3)y = (D+2).t$$
$$(D-2)(2D+3)x + (D+2)(2D+3)y = (2D+3)\cdot 0$$

$$(D+2)(2D+5)x + (D-2)(2D+3)x = (D+2).t$$
$$(2D^2+9D+10)x + (2D^2-D-6)x = (D+2).t$$
$$(4D^2+8D+4)x = (D+2).t$$
$$(4D^2+8D+4)x = 1+2t$$

$$4m^2+8m+4=0 \Rightarrow m^2+2m+1=0$$
$$(m+1)(m+1)=0 \Rightarrow m_1=m_2=m=-1$$

$$x_c = \bar{e}^{t}(At+B)$$

$$F(x) = 1+2t$$
$$x_p = C_1t + C_2 \quad , \quad x_p' = C_1 \quad , \quad x_p'' = 0$$

$$(4D^2+8D+4)x = 1+2t$$
$$(4(0)+8C_1+4)(C_1t+C_2) = 1+2t$$

$$4C_1t + 8C_1 + 4C_2 = 1 + 2t$$

$$8C_1 + 4C_2 = 1 \quad \& \quad 4C_1 = 2 \Rightarrow C_1 = \frac{1}{2}$$

$$\therefore C_2 = -\frac{3}{4}$$

$$x_p = C_1t + C_2 \Rightarrow x_p = \frac{1}{2}t - \frac{3}{4}$$

$$x = x_c + x_p \Rightarrow x = e^{-t}(At + B) + \frac{1}{2}t - \frac{3}{4}$$

Ex: Solve the two simultaneous differential equations

$$(D+2)x + (D+4)y = 1$$

$$(D+1)x + (D+5)y = 2$$

$$(D+1)(D+2)x + (D+1)(D+4)y = (D+1).1 = 0 + 1 = 1$$

$$\mp(D+2)(D+1)x \mp (D+2)(D+5)y = \mp(D+2).2 = 0 \mp 4 = \mp 4$$

$$(D+1)(D+4)y - (D+2)(D+5)y = -3$$

$$(D^2 + 5D + 4)y - (D^2 + 7D + 10)y = -3$$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y - \frac{d^2y}{dt^2} - 7\frac{dy}{dt} - 10y = -3$$

$$-2\frac{dy}{dt} - 6y = -3 \Rightarrow \frac{dy}{dt} = \frac{3}{2} - 3y \Rightarrow \frac{dy}{\frac{3}{2} - 3y} = dt$$

$$-\frac{1}{3}\ln\left(\frac{3}{2} - 3y\right) = t \Rightarrow \ln\left(\frac{3}{2} - 3y\right)^{-\frac{1}{3}} = t$$

$$\left(\frac{3}{2} - 3y\right)^{-\frac{1}{3}} = e^{t} \Rightarrow \frac{3}{2} - 3y = e^{-3t} \Rightarrow 3y = \frac{3}{2} - e^{-3t}$$

$$y = \frac{1}{2} - \frac{1}{3}e^{-3t}$$

Higher Order Differential Equations :

1. The roots are different $(m_1 \neq m_2 \neq m_3 \neq \dots)$

$$y_c = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + \dots$$

2. The roots are equal $(m = m_1 = m_2 = m_3 = \dots)$

$$y_c = e^{mx} (A + Bx + Cx^2 + \dots)$$

3. The roots are complex

$$y_c = e^{\alpha x} (A \cos\beta x + B \sin\beta x)$$

Ex: Solve the equation

$$\frac{d^3 y}{dx^3} + 5 \frac{d^2 y}{dx^2} + 9 \frac{dy}{dx} + 5y = 3e^{2x}$$

$m^3 + 5m^2 + 9m + 5 = 0$ (Auxiliary equation).
$m = -1$ (By inspection).
$(-1)^3 + 5(-1)^2 + 9(-1) + 5 = 0 \Rightarrow 0 = 0$

$(m+1)(m^2+4m+5) = 0$
$m+1 = 0 \Rightarrow m_1 = -1$
$m^2+4m+5 = 0$

$$\begin{array}{r|l} & m^2 + 4m + 5 \\ \hline m+1 & m^3 + 5m^2 + 9m + 5 \\ & \mp m^3 \mp m^2 \\ \hline & 4m^2 + 9m \\ & \mp 4m^2 \mp 4m \\ \hline & 5m + 5 \\ & \mp 5m \mp 5 \\ \hline & 0 \end{array}$$

$$m = \frac{-4 \mp \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$$

$$m = \frac{-4 \mp 2i}{2} \Rightarrow m = -2 \mp i$$

$m_2 = -2 + i$ & $m_3 = -2 - i$

$y_c = C_1 e^{m_1 x} + e^{\alpha x}(C_2 \cos\beta x + C_3 \sin\beta x)$

$y_c = C_1 e^{-x} + e^{-2x}(C_2 \cos x + C_3 \sin x)$

$y_p = A e^{2x}$, $\frac{dy}{dx} = 2Ae^{2x}$, $\frac{d^2y}{dx^2} = 4Ae^{2x}$,

$\frac{d^3y}{dx^3} = 8Ae^{2x}$.

$8Ae^{2x} + 5(4Ae^{2x}) + 9(2Ae^{2x}) + 5(Ae^{2x}) = 3e^{2x}$
$(8A + 20A + 18A + 5A)\, e^{2x} = 3e^{2x}$
$\therefore \quad A = 1/17$

$y_p = \frac{1}{17} e^{2x}$

$y = y_c + y_p \Rightarrow y = C_1 e^{-x} + e^{-2x}(C_2 \cos x + C_3 \sin x) + \frac{1}{17} e^{2x}$

Ex: Solve the equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$$

$m^3 - 2m^2 - 5m + 6 = 0$ (Auxiliary equation).
$m = 1$ (By inspection).
$(1)^3 - 2(1)^2 - 5(1) + 6 = 0 \Rightarrow 0 = 0$

$(m-1)(m^2 - m - 6) = 0$
$(m-1)(m-3)(m+2) = 0$
$m_1 = 1$
$m_2 = 3$
$m_3 = -2$

$$\begin{array}{r|l} & m^2 - m - 6 \\ \hline m-1 & m^3 - 2m^2 - 5m + 6 \\ & \mp m^3 \pm m^2 \\ \hline & -m^2 - 5m \\ & \pm m^2 \mp m \\ \hline & -6m + 6 \\ & \pm 6m \mp 6 \\ \hline & 0 \end{array}$$

$$y_c = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x}$$

$$y_c = Ae^{x} + Be^{3x} + Ce^{-2x}$$

$$(D^3 - 2D^2 - 5D + 6)y = e^{3x}$$

$$y_p = \frac{1}{D^3 - 2D^2 - 5D + 6} e^{3x} \quad , \quad F(a) = 0$$

$$y_p = e^{3x} \frac{1}{(D+3)^3 - 2(D+3)^2 - 5(D+3) + 6} \cdot 1$$

Note: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$\therefore y_p = e^{3x} \frac{1}{D^3 + 7D^2 + 10D} \cdot 1$$

$$y_p = e^{3x} \frac{1}{D(D^2 + 7D + 10)} \cdot 1$$

$$y_p = e^{3x} \frac{1}{D} \cdot \frac{1}{10}$$

$$\begin{array}{r|l} & \frac{1}{10} \\ \hline D^2 + 7D + 10 & 1 \\ & \mp 1 \mp \frac{7}{10}D \mp \frac{1}{10} \\ \hline \end{array}$$

$$y_p = e^{3x} \frac{x}{10}$$

$$y = y_c + y_p$$

$$y = Ae^{x} + Be^{3x} + Ce^{-2x} + \frac{1}{10} x e^{3x}$$

Another Solution:

$$y_p = \frac{1}{D^3 - 2D^2 - 5D + 6} e^{3x}$$

$$y_p = \frac{1}{(D-1)(D-3)(D+2)} e^{3x}$$

$$y_p = \frac{1}{(D-3)(3-1)(3+2)} e^{3x} \Rightarrow y_p = \frac{1}{(D-3)(10)} e^{3x}$$

$$y_p = e^{3x} \frac{1}{10(D+3-3)} \cdot 1 \Rightarrow y_p = e^{3x} \frac{1}{D} \cdot \frac{1}{10}$$

$$y_p = \frac{1}{10} x e^{3x}$$

$$y = y_c + y_p \Rightarrow y = Ae^{x} + Be^{3x} + Ce^{-2x} + \frac{1}{10} x e^{3x}$$

Ex: Solve the equation :

$$\frac{d^4 y}{dx^4} + 2\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} = x^2 + 3e^{2x}$$

$m^4 + 2m^3 - 3m^2 = 0$
$m^2(m^2 + 2m - 3) = 0$
$m^2(m+3)(m-1) = 0$
$m_1 = 0 \ , \ m_2 = 0 \ , \ m_3 = -3 \ , \ m_4 = 1$

$$y_c = e^{mx}(A + Bx) + Ce^{mx} + Ee^{mx}$$

$$y_c = A + Bx + Ce^{-3x} + Ee^{x}$$

$$(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x}$$

$$y_p = \frac{1}{D^4 + 2D^3 - 3D^2} x^2 + \frac{1}{D^4 + 2D^3 - 3D^2} 3e^{2x}$$

Factorial Function

The classical case of the integer form of the factorial function, $n!$, consists of the product of n and all integers less than n, down to 1, as follows

$$n! = \begin{cases} n(n-1)(n-2)\ldots 3\cdot 2\cdot 1 & n = 1, 2, 3, \ldots \\ 1 & n = 0 \end{cases} \tag{1.1}$$

where by definition, $0! = 1$.

Gamma Function

The factorial function can be extended to include non-integer arguments through the use of Euler's second integral given as

$$z! = \int_0^\infty e^{-t}\, t^z \, dt \tag{1.7}$$

Equation 1.7 is often referred to as the *generalized factorial function.*

Through a simple translation of the $z-$ variable we can obtain the familiar *gamma function* as follows

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt = (z-1)! \tag{1.8}$$

The gamma function is one of the most widely used special functions encountered in advanced mathematics because it appears in almost every integral or series representation of other advanced mathematical functions.

Let's first establish a direct relationship between the gamma function given in Eq. 1.8 and the integer form of the factorial function given in Eq. 1.1. Given the gamma function $\Gamma(z+1) = z!$ use integration by parts as follows:

$$\int u\, dv = uv - \int v\, du$$

where from Eq. 1.7 we see

$$u = t^z \quad \Rightarrow \quad du = zt^{z-1}\, dt$$

$$dv = e^{-t}\, dt \quad \Rightarrow \quad v = -e^{-t}$$

which leads to

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z\, dt = \Big[- e^{-t}\, t^z \Big]_0^\infty + z \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

Given the restriction of $z > 0$ for the integer form of the factorial function, it can be seen that the first term in the above expression goes to zero since, when

$$t = 0 \quad \Rightarrow \quad t^n \rightarrow 0$$

$$t = \infty \quad \Rightarrow \quad e^{-t} \rightarrow 0$$

Therefore

$$\Gamma(z+1) = z \underbrace{\int_0^\infty e^{-t}\, t^{z-1}\, dt}_{\Gamma(z)} = z\, \Gamma(z), \qquad z > 0 \tag{1.9}$$

When $z = 1 \quad \Rightarrow \quad t^{z-1} = t^0 = 1$, and

$$\Gamma(1) = 0! = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1$$

and in turn

$$\begin{aligned} \Gamma(2) &= 1\, \Gamma(1) = 1 \cdot 1 = 1! \\ \Gamma(3) &= 2\, \Gamma(2) = 2 \cdot 1 = 2! \\ \Gamma(4) &= 3\, \Gamma(3) = 3 \cdot 2 = 3! \end{aligned}$$

In general we can write

$$\Gamma(n+1) = n! \qquad\qquad n = 1, 2, 3, \ldots$$

The gamma function constitutes an essential extension of the idea of a factorial, since the argument z is not restricted to positive integer values, but can vary continuously.

From Eq. 1.9, the gamma function can be written as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

From the above expression it is easy to see that when $z = 0$, the gamma function approaches ∞ or in other words $\Gamma(0)$ is undefined.

Given the recursive nature of the gamma function, it is readily apparent that the gamma function approaches a singularity at each negative integer.

However, for all other values of z, $\Gamma(z)$ is defined and the use of the recurrence relationship for factorials, i.e.

$$\Gamma(z+1) = z\,\Gamma(z)$$

effectively removes the restriction that x be positive, which the integral definition of the factorial requires. Therefore,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad z \neq 0, -1, -2, -3, \ldots \tag{1.11}$$

A plot of $\Gamma(z)$ is shown in Figure 1.1.

Several other definitions of the Γ-function are available that can be attributed to the pioneering mathematicians in this area

Other forms of the gamma function are obtained through a simple change of variables, as follows

$$\Gamma(z) = 2\int_0^\infty y^{2z-1}\, e^{-y^2}\, dy \qquad \text{by letting } t = y^2 \tag{1.15}$$

$$\Gamma(z) = \int_0^1 \left(\ln \frac{1}{y}\right)^{z-1} dy \qquad \text{by letting } e^{-t} = y \tag{1.16}$$

Relations Satisfied by the Γ-Function

Recurrence Formula

$$\Gamma(z+1) = z\,\Gamma(z) \tag{1.17}$$

Duplication Formula

$$2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z) \tag{1.18}$$

Reflection Formula

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{1.19}$$

Some Special Values of the Gamma Function

Using Eq. 1.15 or Eq. 1.19 we have

$$\Gamma(1/2) = (-1/2)! = 2\,\underbrace{\int_0^\infty e^{-y^2}\,dy}_{I} = \sqrt{\pi} \tag{1.20}$$

where the solution to I is obtained from Schaum's Handbook of Mathematical Functions (Eq. 18.72).

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$$

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\mathrm{d}t$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

$$\Gamma(1) = 1$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$\Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})} = \tfrac{4}{3}\sqrt{\pi}$$

(a) For n a positive integer

$$\Gamma(n+1) = n\Gamma(n) = n!$$

$$\Gamma(1) = 1; \quad \Gamma(0) = \infty; \quad \Gamma(-n) = \pm\infty$$

(b) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi};$ $\qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}; \qquad \Gamma(-\tfrac{3}{2}) = \frac{4}{3}\sqrt{\pi}$$

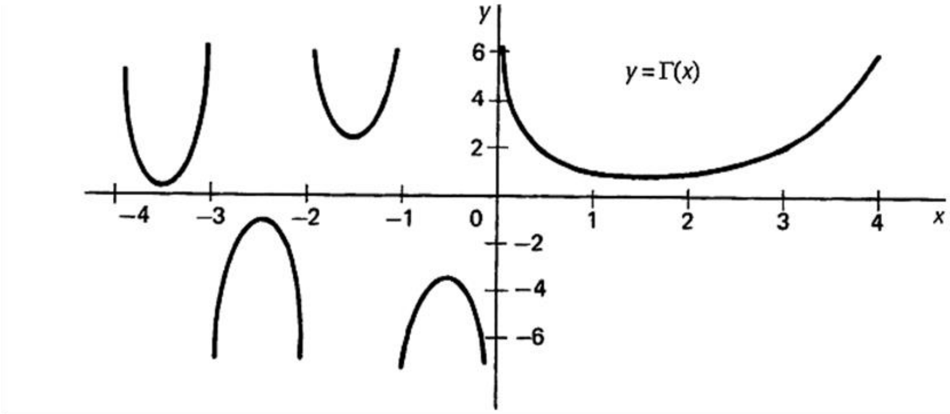
$$\Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}; \qquad \Gamma(-\tfrac{5}{2}) = -\frac{8}{15}\sqrt{\pi}$$

$$\Gamma(\tfrac{7}{2}) = \frac{15\sqrt{\pi}}{8}; \qquad \Gamma(-\tfrac{7}{2}) = \frac{16}{105}\sqrt{\pi}$$

$y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



$$\int_0^\infty x^7 e^{-x} \mathrm{d}x. \qquad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t \qquad \Gamma(v) = \int_0^\infty x^{v-1} e^{-x} \mathrm{d}x \qquad v = 8$$

$$I = \Gamma(v) = \Gamma(8) \qquad \Gamma(8) = 7! = 5040$$

Evaluate $\int_0^\infty x^3 \, e^{-4x} \, \mathrm{d}x.$

$$y = 4x \quad \mathrm{d}y = 4\,\mathrm{d}x \qquad I = \frac{1}{4^4}\int_0^\infty y^3 \, e^{-y} \, \mathrm{d}y = \frac{1}{4^4}\Gamma(v) \qquad v = 4$$

$$I = \frac{1}{4^4}\,\Gamma(4 \qquad I = \frac{3}{128}$$

Evaluate $\int_0^\infty x^{1/2}\, e^{-x^2}\, \mathrm{d}x$.

$$y = x^2 \qquad \mathrm{d}y = 2x\, \mathrm{d}x$$

$$I = \int_0^\infty y^{1/4}\, e^{-y}\, \mathrm{d}y/2x = \int_0^\infty \frac{y^{1/4}\, e^{-y}\, \mathrm{d}y}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty y^{-1/4}\, e^{-y}\, \mathrm{d}y$$

$$= \frac{1}{2}\int_0^\infty y^{v-1}\, e^{-y}\mathrm{d}y \quad \text{where } v = \frac{3}{4} \quad \therefore\ I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

From tables, $\Gamma(0{\cdot}75) = 1{\cdot}2254$ $\qquad I = 0{\cdot}613$

The beta function

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$B(m, n) = B(n, m)$$

Alternative form

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta$$

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

$$B(4, 3) = \frac{(3!)(2!)}{(6!)}$$

$$B(5, 3) = \frac{(4!)(2!)}{(7!)}$$

$$B(k, 1) = B(1, k) = \frac{1}{k}$$

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$$

Relation between the gamma and beta functions

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$B(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\sqrt{\pi}/2 \times \sqrt{\pi}}{1} = \frac{\pi}{2}$$

Evaluate $I = \int_0^1 x^5(1-x)^4\, dx$

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$m-1=5 \qquad n-1=4$$

$$m=6 \qquad n=5$$

$$I = B(6, 5) = \frac{5!\,4!}{10!} = \frac{1}{1260}$$

$$I = \int_0^1 x^4 \sqrt{1 - x^2}\,dx$$

$$x^2 = y \qquad x = y^{\frac{1}{2}} \quad dx = \tfrac{1}{2} y^{-\frac{1}{2}}\,dy$$

$$I = \int_0^1 y^2 (1 - y)^{\frac{1}{2}} \frac{1}{2} y^{-\frac{1}{2}}\,dy = \frac{1}{2} \int_0^1 y^{\frac{3}{2}} (1 - y)^{\frac{1}{2}}\,dy$$

$$m - 1 = \tfrac{3}{2} \qquad n - 1 = \tfrac{1}{2}$$

$$I = \tfrac{1}{2} \mathrm{B}\left(\tfrac{5}{2}, \tfrac{3}{2}\right) \qquad I = \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \qquad \Gamma\left(\tfrac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} \qquad \Gamma(4) = 3! \qquad I = \frac{\pi}{32}$$

$$I = \int_0^3 \frac{x^3\,dx}{\sqrt{3 - x}}.$$

$$I = \int_0^3 \frac{x^3\,dx}{\sqrt{3 - x}} = \int_0^3 x^3 (3 - x)^{-\frac{1}{2}}\,dx = 3^{-\frac{1}{2}} \int_0^3 x^3 \left(1 - \frac{x}{3}\right)^{-\frac{1}{2}}\,dx$$

$$\frac{x}{3} = y \qquad x = 3y \qquad dx = 3\,dy$$

Limits: $x = 0,\ y = 0$; $x = 3,\ y = 1$

$$I = 27\sqrt{3} \int_0^1 y^3 (1 - y)^{-\frac{1}{2}}\,dy \qquad \begin{array}{ll} m - 1 = 3 & m = 4 \\ n - 1 = -\frac{1}{2} & n = \frac{1}{2} \end{array}$$

$$I = 27\sqrt{3}\ \mathrm{B}\left(4, \tfrac{1}{2}\right) = 27\sqrt{3}\ \frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\ (9/2)}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi};\ \ \Gamma(9/2) = \frac{105\sqrt{\pi}}{16};\ \ \Gamma(4) = 3!$$

$$I = 27\sqrt{3} \times 6 \times \sqrt{\pi} \times \frac{16}{105\sqrt{\pi}} = \frac{864\sqrt{3}}{35} = 42{\cdot}76$$

Evaluate $I = \int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta.$

$$\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$2m - 1 = 5 \quad \therefore \; m = 3; \quad 2n - 1 = 4 \quad \therefore \; n = 5/2$$

$$I = \frac{1}{2}\,\mathrm{B}(3, 5/2) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma(5/2)}{\Gamma(11/2)}$$

$$= \frac{1}{2}\cdot\frac{2!(3\sqrt{\pi})/4}{(945\sqrt{\pi})/32} = \frac{3\sqrt{\pi}}{4}\cdot\frac{32}{945\sqrt{\pi}} = \frac{8}{315}$$

Evaluate $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta.$

$$\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta \, d\theta$$

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$\therefore \; 2m - 1 = \frac{1}{2} \quad \therefore \; m = \frac{3}{4}; \quad 2n - 1 = -\frac{1}{2} \quad \therefore \; n = \frac{1}{4}$$

$$\therefore \; I = \frac{1}{2}\,\mathrm{B}\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}$$

$$\Gamma(0{\cdot}25) = 3{\cdot}6256 \quad \text{and} \quad \Gamma(0{\cdot}75) = 1{\cdot}2254$$

$$I = \frac{1}{2}\cdot\frac{(1{\cdot}2254)(3{\cdot}6256)}{1{\cdot}0000} = 2{\cdot}2214$$

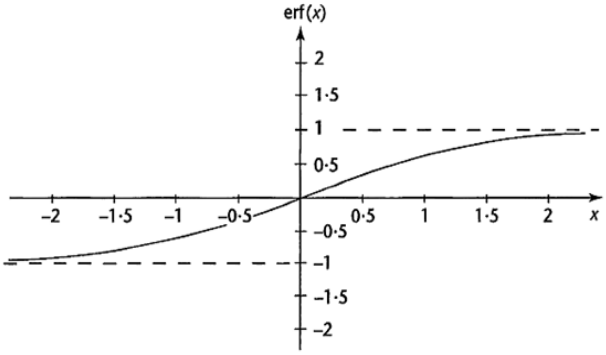
The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = 1$$

$$\lim_{x\to\infty}(\operatorname{erf}(x)) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = 1$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\\ &= \frac{2}{\sqrt{\pi}}\int_0^x\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}\right)dt\\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\int_0^x\frac{(-1)^n t^{2n}}{n!}\,dt\right)\\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\end{aligned}$$



$$\operatorname{erf}(-x) = -\operatorname{erf}(x)$$

erf (x) is an odd function.

The complementary error function erfc (x)

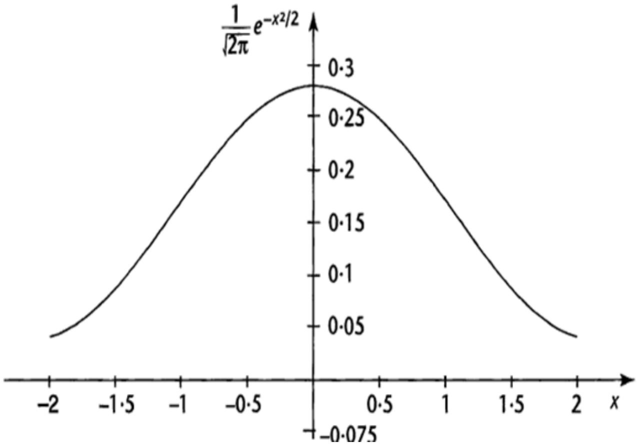
$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

In statistics the integral

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

is the area beneath the Gaussian or normal probability distribution



$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\left(2\int_{0}^{\infty} e^{-t^2/2}\,\mathrm{d}t\right)$$

the integrand is even

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$\begin{aligned}
\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-t^2/2}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-t^2/2}\,\mathrm{d}t \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \times \sqrt{2}\int_{0}^{x/\sqrt{2}} e^{-u^2}\,\mathrm{d}u \quad \text{where } u = t/\sqrt{2} \\
&= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)
\end{aligned}$$

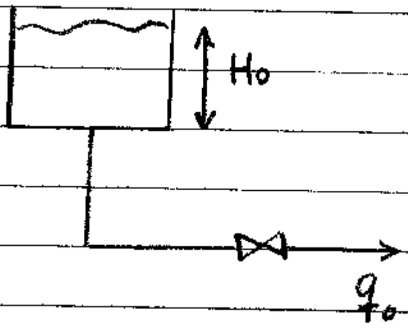
Formulation of Chemical Engineering Problems :

The mathematical model is an expression that represent a phenomenon or an operation. When deriving the model we make use of the basic theoretical principles and the validity of the model is, then, tested experimentally.

The main problems to be solved are :

1. Storage tanks.
2. Mixing tanks.
3. Chemical reaction vessels.
4. Heat transfer proplems.
5. Mass transfer problems.
6. Momentum transfer problems.
7. Process control systems.
8. Another problems.

Example : A vertical tank is filled with liquid to a height (H_o). The base of the tank is connected to a valve, if the valve is opend. Derive the equation which relate the variation of height with time, given that the flow through the valve is laminar.



Material balance on the tank

In = Out + Accumulation

$$0 = \rho q_o + \rho \frac{dV}{dt}$$

$$0 = \rho q_o + \rho A \frac{dH}{dt}$$

Laminar flow $\Rightarrow q_o \propto H \Rightarrow q_o = KH \quad , K = \frac{m^3/hr}{m} = \frac{m^2}{hr}$

$$0 + KH + A\frac{dH}{dt} \Rightarrow \frac{A}{K}\frac{dH}{dt} + H = 0$$

$\tau\frac{dH}{dt} + H = 0$ Taking Laplace Transform

$$\tau[s\bar{H}(s) - H(0)] + \bar{H}(s) = 0$$

at $t=0 \quad H = H_o$

$$\tau[s\bar{H}(s) - H_o] + \bar{H}(s) = 0$$

$$(\tau s + 1)\bar{H}(s) = \tau H_o \Rightarrow \bar{H}(s) = \frac{\tau H_o}{\tau s + 1} \Rightarrow \bar{H}(s) = \frac{\tau H_o}{\tau(s + \frac{1}{\tau})}$$

$\bar{H}(s) = \frac{H_o}{s + \frac{1}{\tau}}$ Taking Inverse Laplace Transform

$$H(t) = H_o e^{-\frac{1}{\tau}t} \Rightarrow H(t) = H_o e^{-\frac{t}{\tau}}$$

Another solution,

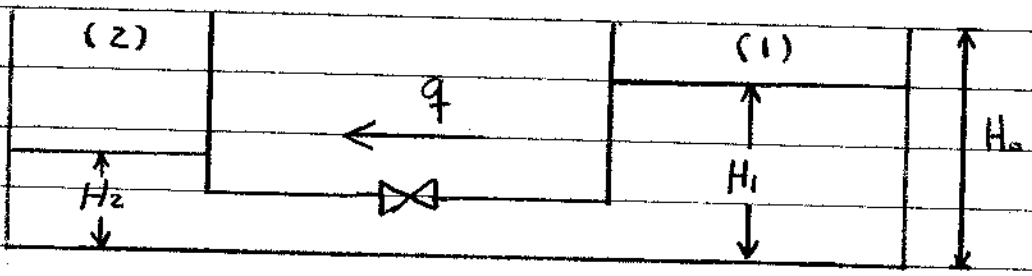
$$\tau\frac{dH}{dt} + H = 0 \Rightarrow \int\frac{dH}{H} = -\int\frac{dt}{\tau}$$

$$\ln H = -\frac{1}{\tau}t + \ln C \Rightarrow H = Ce^{-t/\tau}$$

at $t=0 \quad H = H_o \Rightarrow C = H_o$

$$H = H_o e^{-t/\tau} \quad \text{or} \quad H(t) = H_o e^{-t/\tau}$$

Example: Two tanks are connected as shown below. Tank 1 contain a liquid to height H_o and tank 2 is empty. The valve between the two tanks is opened. Find the relation which relate the height in tank 2 with time. Assuming that all resistance to flow was due to the valve and the flow is laminar.



Material balance on tank (1)

In = Out + Accumulation

$$0 = \rho q + \rho A_1 \frac{dH_1}{dt} \Rightarrow 0 = q + A_1 \frac{dH_1}{dt}$$

Material balance on tank (2)

In = Out + Accumulation

$$\rho q = 0 + \rho A_2 \frac{dH_2}{dt} \Rightarrow q = 0 + A_2 \frac{dH_2}{dt}$$

The flow is laminar $\Rightarrow q \propto H \Rightarrow q = KH$

or $q = K(H_1 - H_2)$

$$0 = K(H_1 - H_2) + A_1 \frac{dH_1}{dt}$$

$$K(H_1 - H_2) = 0 + A_2 \frac{dH_2}{dt}$$

By taking Laplace Transform

$$0 = K(\bar{H}_1(s) - \bar{H}_2(s)) + A_1(S\bar{H}_1(s) - H(0))$$
$$K(\bar{H}_1(s) - \bar{H}_2(s)) = 0 + A_2(S\bar{H}_2(s) - H(0))$$

At $t=0$ $H=H_o$ in tank (1)
At $t=0$ $H=0$ in tank (2)

$$0 = K(\bar{H}_1(s) - \bar{H}_2(s)) + A_1(S\bar{H}_1(s) - H_o)$$
$$K(\bar{H}_1(s) - \bar{H}_2(s)) = 0 + A_2(S\bar{H}_2(s) - 0)$$
$$K\bar{H}_1(s) - K\bar{H}_2(s) = A_2 S\bar{H}_2(s)$$

$$\bar{H}_1(s) = \bar{H}_2(s) + \frac{A_2}{K} s\bar{H}_2(s) \Rightarrow \bar{H}_1(s) = \left(\frac{A_2}{K}s + 1\right)\bar{H}_2(s)$$

$$\bar{H}_2(s) = \bar{H}_1(s) + \frac{A_1}{K}(S\bar{H}_1(s) - H_o)$$

$$\bar{H}_2(s) = \left(\frac{A_1}{K}s + 1\right)\bar{H}_1(s) - \frac{A_1}{K}H_o$$

$$\bar{H}_2(s) = \left(\frac{A_1}{K}s + 1\right)\left(\frac{A_2}{K}s + 1\right)\bar{H}_2(s) - \frac{A_1}{K}H_o$$

let $\tau_1 = A_1/K$ & $\tau_2 = A_2/K$

$$\bar{H}_2(s) = (\tau_1 s + 1)(\tau_2 s + 1)\bar{H}_2(s) - \tau_1 H_o$$
$$(\tau_1 s + 1)(\tau_2 s + 1)\bar{H}_2(s) - \bar{H}_2(s) = \tau_1 H_o$$
$$[(\tau_1 s + 1)(\tau_2 s + 1) - 1]\bar{H}_2(s) = \tau_1 H_o$$

$$\bar{H}_2(s) = \frac{\tau_1 H_o}{(\tau_1 s + 1)(\tau_2 s + 1) - 1} \Rightarrow \bar{H}_2(s) = \frac{\tau_1 H_o}{\tau_1\tau_2 s^2 + \tau_1 s + \tau_2 s + 1 - 1}$$

$$\bar{H}_2(s) = \frac{\tau_1 H_o}{s(\tau_1\tau_2 s + \tau_1 + \tau_2)} \Rightarrow \bar{H}_2(s) = \frac{\tau_1 H_o}{s\,\tau_1\tau_2\left(s + \frac{\tau_1 + \tau_2}{\tau_1\tau_2}\right)}$$

let $K_1 = \frac{\tau_1 + \tau_2}{\tau_1 \tau_2}$

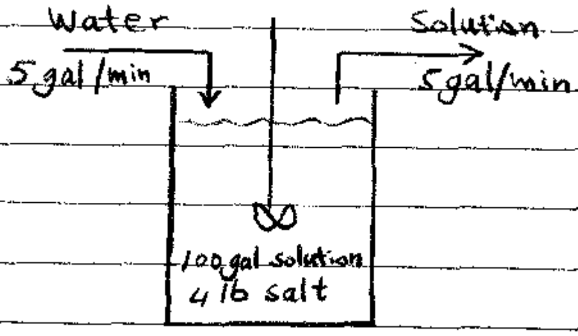
$$\bar{H}_2(s) = \frac{H_0}{\tau_2 s\,(s + K_1)} \Rightarrow$$

Taking Inverse Laplace Transform

$$H_2(t) = \frac{H_0}{\tau_2 K_1}\left(1 - e^{-K_1 t}\right)$$

$$H_2(t) = \frac{\tau_1 H_0}{\tau_1 + \tau_2}\left(1 - e^{-\left(\frac{\tau_1 + \tau_2}{\tau_1 \tau_2}\right)t}\right)$$

Example: A tank holds 100 gal of water salt solution in which 4 lb of salt is dissolved. Water runs into the tank at the rate of 5 gal/min and salt solution overflows at the same rate. If the mixing in the tank is adequate to keep the concentration of salt in the tank uniform at all times, how much salt is in the tank at the end of 50 min?



Salt material balance

In = Out + Accumulation

$$0 = \frac{5x}{100} + \frac{dx}{dt}$$

Note: $\text{Out} = 5\,\frac{\text{gal}}{\text{min}} \cdot x\ \text{lb} \cdot \frac{1}{100\ \text{gal}} = \frac{5x}{100}\ \frac{\text{lb}}{\text{min}}$

where x is lb of salt in solution

$$\frac{dx}{dt} = -0.05x \Rightarrow \int \frac{dx}{x} = -0.05 \int dt$$

At $t = 0$ $x = 4$

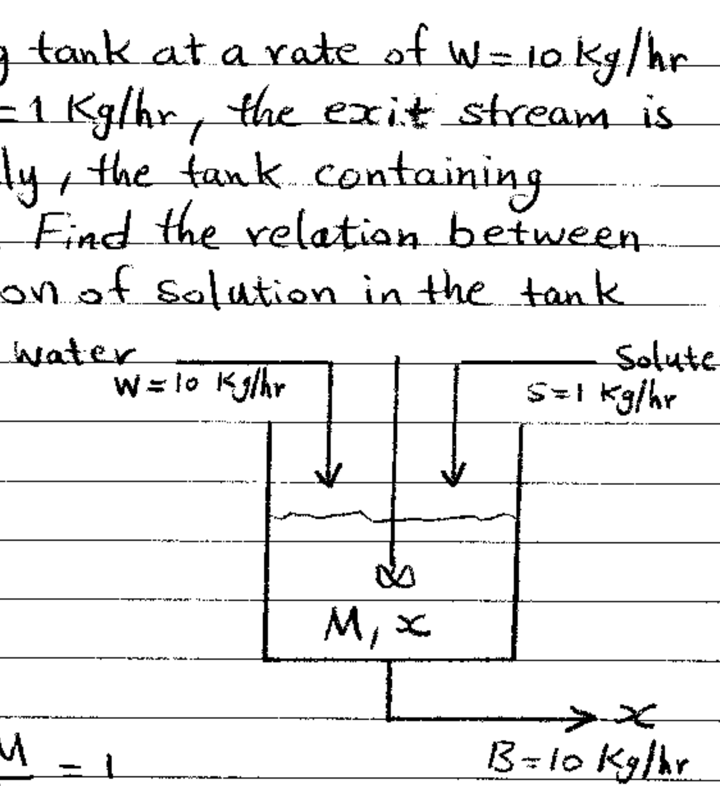
& $t = 50$ $x = x$

$$\int_4^x \frac{dx}{x} = -0.05 \int_0^{50} dt \Rightarrow \ln x \Big]_4^x = -0.05\, t \Big]_0^{50}$$

$$\ln x - \ln 4 = -0.05\,(50 - 0) \Rightarrow \ln \frac{x}{4} = -0.05(50)$$

$\ln \frac{x}{4} = -2.5 \Rightarrow x = 0.328$ lb salt.

Example: Water enter a mixing tank at a rate of $W = 10$ Kg/hr and solute added $S = 1$ Kg/hr, the exit stream is $B = 10$ kg/hr. Initially, the tank containing $M_o = 100$ Kg water. Find the relation between change in concentration of solution in the tank with time?



Overall material balance

In = Out + Accumulation

$$W + S = B + \frac{dM}{dt}$$

$$10 + 1 = 10 + \frac{dM}{dt} \Rightarrow \frac{dM}{dt} = 1$$

$$\int dM = \int dt$$

At $t = 0$ $M = M_o = 100$

At $t = t$ $M = M$

$$\int_{100}^{M} dM = \int_{0}^{t} dt \Rightarrow M\Big]_{100}^{M} = t\Big]_{0}^{t} \Rightarrow M - 100 = t - 0$$

$$M = 100 + t$$

Solute material balance

In = Out + Accumulation

$$W(0) + S(1) = B(x) + \frac{d(Mx)}{dt}$$

$$1 = 10x + x\frac{dM}{dt} + M\frac{dx}{dt}$$

$$1 = 10x + x(1) + (100+t)\frac{dx}{dt}$$

$$1 = 11x + (100+t)\frac{dx}{dt} \Rightarrow (1-11x) = (100+t)\frac{dx}{dt}$$

$$\int \frac{dx}{(1-11x)} = \int \frac{dt}{(100+t)}$$

At $t = 0$ $x = 0$

At $t = t$ $x = x$

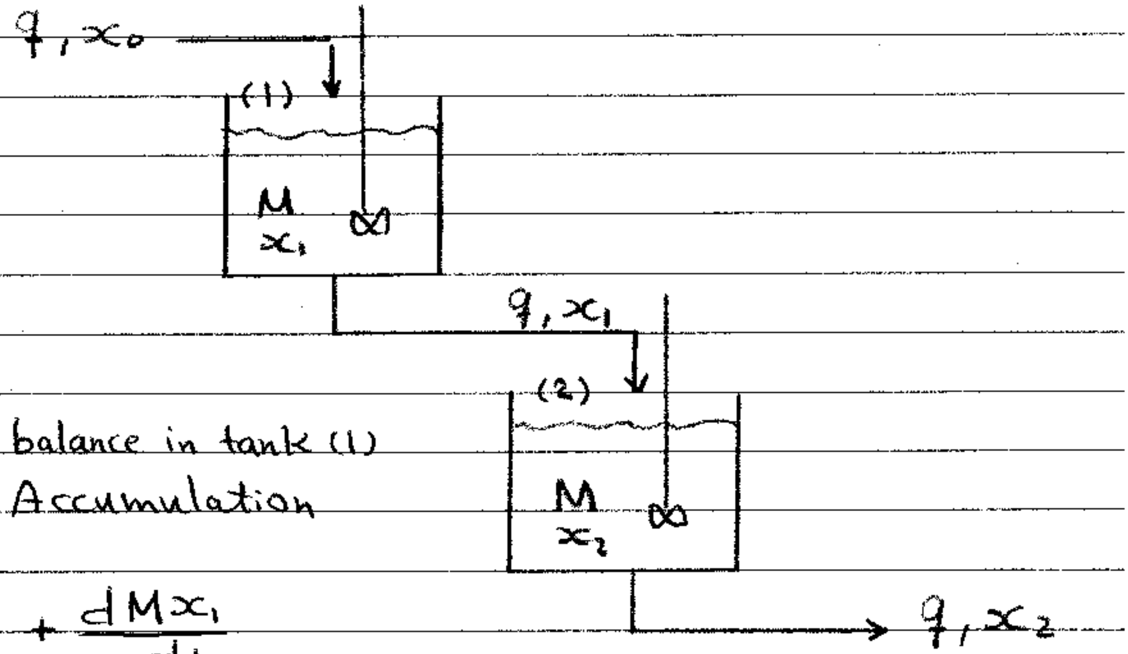
$$\int_{0}^{x} \frac{dx}{(1-11x)} = \int_{0}^{t} \frac{dt}{(100+t)}$$

$$-\frac{1}{11}\ln(1-11x)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\frac{1}{11}\ln\left(\frac{1}{1-11x}\right)\Big]_{0}^{x} = \ln(100+t)\Big]_{0}^{t}$$

$$\ln\left(\frac{1}{1-11x}\right)^{1/11} = \ln\left(\frac{100+t}{100}\right) \Rightarrow \left(\frac{1}{1-11x}\right)^{1/11} = \frac{100+t}{100}$$

Example: Two mixer are connected in series, each of them contain M kg of water. Intitially, q (kg/hr) of water flows to the first mixer containing solute with x_o. Find the concentration in the second mixer when a step change Δx_o is take place in the inlet stream to mixer (1).



Solute material balance in tank (1)

In = Out + Accumulation

$$q x_o = q x_1 + \frac{d M x_1}{dt}$$

$$q x_o = q x_1 + M \frac{d x_1}{dt}$$

$$\frac{M}{q} \frac{d x_1}{dt} + x_1 = x_o \Rightarrow \tau \frac{d x_1}{dt} + x_1 = x_o$$

$$\tau \frac{d x_1'}{dt} + x_1' = \Delta x_o$$ Taking Laplace Transform

$$\tau \left(s \overline{x_1'}(s) - x_1'(0) \right) + \overline{x_1'}(s) = \frac{\Delta x_o}{s}$$

At $t=0$ $x_1 = 0$ or $x_1' = 0$

$$\tau \left(s \overline{x_1'}(s) - 0 \right) + \overline{x_1'}(s) = \frac{\Delta x_o}{s}$$

$$(\tau s + 1) \overline{x_1'}(s) = \frac{\Delta x_o}{s} \Rightarrow \overline{x_1'}(s) = \frac{\Delta x_o}{s(\tau s + 1)} \quad \ldots (1)$$

Solute material balance in tank (2)

In = Out + Accumulation

$$q x_1 = q x_2 + \frac{d M x_2}{dt} \Rightarrow q x_1 = q x_2 + M \frac{d x_2}{dt}$$

$$\frac{M}{q} \frac{d x_2}{dt} + x_2 = x_1 \Rightarrow \tau \frac{d x_2}{dt} + x_2 = x_1$$

$\tau \frac{d x'_2}{dt} + x'_2 = x'_1$ Taking Laplace Transform

$$\tau \left(s \bar{x}'_2(s) - x'_2(0) \right) + \bar{x}'_2(s) = \bar{x}'_1(s)$$

At $t=0$ $x_2 = 0$ or $x'_2 = 0$

$$(\tau s + 1) \bar{x}'_2(s) = \bar{x}'_1(s) \Rightarrow \bar{x}'_2(s) = \frac{\bar{x}'_1(s)}{\tau s + 1} \quad \ldots (2)$$

Substitute equation (1) in uation (2)

$$\bar{x}'_2(s) = \frac{\Delta x_0}{s(\tau s + 1)^2}$$

$$\frac{1}{s(\tau s+1)^2} = \frac{A}{s} + \frac{B}{\tau s+1} + \frac{C}{(\tau s+1)^2}$$

$$1 = A(\tau s+1)^2 + B s(\tau s+1) + C s$$
$$1 = A\tau^2 s^2 + 2A\tau s + A + B\tau s^2 + B s + C s$$
$$1 = (A\tau^2 + B\tau)s^2 + (2A\tau + B + C)s + A$$
$$A = 1 \quad , \quad A\tau^2 + B\tau = 0 \quad , \quad 2A\tau + B + C = 0$$
$$\therefore B = -\tau \quad \& \quad C = -\tau$$

$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{\tau}{\tau s+1} - \frac{\tau}{(\tau s+1)^2}$$

$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{\tau}{\tau(s+\frac{1}{\tau})} - \frac{\tau}{\tau^2(s+\frac{1}{\tau})^2}$$

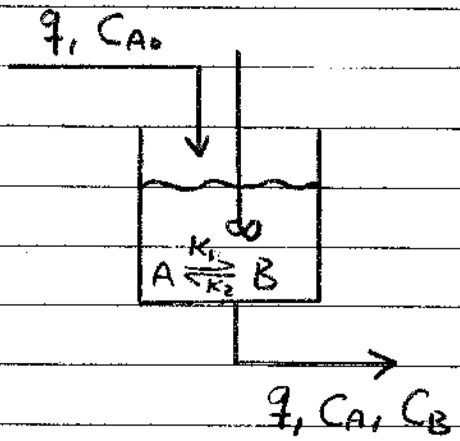
$$\frac{1}{s(\tau s+1)^2} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} - \frac{1}{\tau(s+\frac{1}{\tau})^2}$$

Taking Inverse Laplace Transform

$$x_2'(t) = \Delta x_0 \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}} \right)$$

$$x_2(t) - x_2(0) = \Delta x_0 \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}} \right)$$

Example: The first order reversible reaction $A \underset{K_2}{\overset{K_1}{\rightleftharpoons}} B$ occur in continuous stirred tank reactor. Find the differential equation which relate C_A with time?



Material Balance on A

In + Generation = Out + Consumption + Accⁿ:

$$qC_{Ao} + K_2 C_B V = qC_A + K_1 C_A V + \frac{d\,C_A V}{dt}$$

$$V\frac{dC_A}{dt} + (q + K_1 V)\,C_A = qC_{Ao} + K_2 C_B V$$

$$\frac{V}{q+K_1V}\;\frac{dC_A}{dt} + C_A = \frac{q}{q+K_1V}\,C_{Ao} + \frac{K_2V}{q+K_1V}\,C_B$$

$$\tau_1\,\frac{dC_A}{dt} + C_A = C_1 C_{Ao} + C_2 C_B \quad \ldots\ldots (1)$$

Material Balance on B

In + Generation = Out + Consumption + Accumulation

$$0 + K_1 C_A V = qC_B + K_2 C_B V + \frac{d\,C_B V}{dt}$$

$$V\frac{dC_B}{dt} + (q + K_2V)\,C_B = K_1 C_A V$$

$$\frac{V}{q+K_2V}\;\frac{dC_B}{dt} + C_B = \frac{K_1V}{q+K_2V}\,C_A$$

$$\tau_2\,\frac{dC_B}{dt} + C_B = C_3 C_A \quad \ldots\ldots (2)$$

Divided equation (1) by C_2

$$\frac{\tau_1}{C_2}\,\frac{dC_A}{dt} + \frac{1}{C_2}\,C_A - \frac{C_1}{C_2}\,C_{Ao} = C_B \quad \ldots\ldots (3)$$

Differentiate with respect to t

$$\frac{\tau_1}{c_2}\frac{d^2C_A}{dt^2}+\frac{1}{c_2}\frac{dC_A}{dt}-\frac{C_1}{C_2}(0)=\frac{dC_B}{dt} \quad \ldots\ldots(4)$$

Substitute equations (3) & (4) in equation (2)

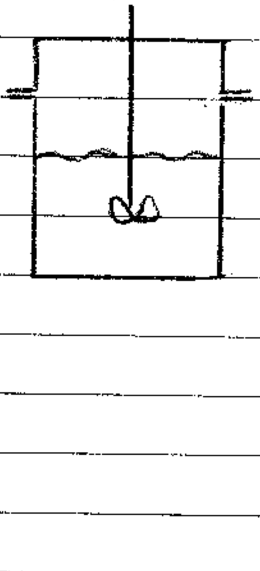
$$\tau_2\left[\frac{\tau_1}{C_2}\frac{d^2C_A}{dt^2}+\frac{1}{C_2}\frac{dC_A}{dt}\right]+\frac{\tau_1}{C_2}\frac{dC_A}{dt}+\frac{1}{C_2}C_A-\frac{C_1}{C_2}C_{A_0}=C_3C_A$$

$$\frac{\tau_1\tau_2}{C_2}\frac{d^2C_A}{dt^2}+\frac{\tau_2}{C_2}\frac{dC_A}{dt}+\frac{\tau_1}{C_2}\frac{dC_A}{dt}+\frac{1}{C_2}C_A-\frac{C_1}{C_2}C_{A_0}=C_3C_A$$

$$\frac{\tau_1\tau_2}{C_2}\frac{d^2C_A}{dt^2}+\left(\frac{\tau_1+\tau_2}{C_2}\right)\frac{dC_A}{dt}+\left(\frac{1}{C_2}-C_3\right)C_A=\frac{C_1}{C_2}C_{A_0}$$

$$\frac{d^2C_A}{dt^2}+\left(\frac{\tau_1+\tau_2}{\tau_1\tau_2}\right)\frac{dC_A}{dt}+\left(\frac{1-C_2C_3}{\tau_1\tau_2}\right)C_A=\frac{C_1}{\tau_1\tau_2}C_{A_0}$$

Example: The first order reversible reaction $A \underset{K_2}{\overset{K_1}{\rightleftharpoons}} B$ occur in batch reactor. Find the differential equation which relate CA with time?



Material Balance on A

In + Generation = Out + Consumption + Accumulation

$$0 + K_2 C_B V = 0 + K_1 C_A V + V \frac{dC_A}{dt}$$

$$K_2 \cancel{V} C_B = K_1 \cancel{V} C_A + \cancel{V} \frac{dC_A}{dt}$$

$$K_2 C_B = K_1 C_A + \frac{dC_A}{dt} \quad \ldots \quad (1)$$

Material Balance on B

In + Generation = Out + Consumption + Accumulation

$$0 + K_1 V C_A = 0 + K_2 V C_B + V \frac{dC_B}{dt}$$

$$K_1 \cancel{V} C_A = K_2 \cancel{V} C_B + \cancel{V} \frac{dC_B}{dt}$$

$$K_1 C_A = K_2 C_B + \frac{dC_B}{dt} \quad \ldots\ldots \quad (2)$$

From equation (1)

$$\frac{dC_A}{dt} + K_1 C_A = K_2 C_B \Rightarrow (D + K_1) C_A = K_2 C_B$$

$$C_B = \frac{(D + K_1)}{K_2} C_A$$

From equation (2)

$$\frac{dC_B}{dt} + K_2 C_B = K_1 C_A \Rightarrow (D + K_2) C_B = K_1 C_A$$

$$(D + K_2) \frac{(D + K_1)}{K_2} C_A = K_1 C_A$$

$$(D + K_2)(D + K_1) C_A = K_1 K_2 C_A$$

$$(D^2 + K_2 D + K_1 D + K_1 K_2) C_A = K_1 K_2 C_A$$

$$(D^2 + (K_1 + K_2) D + K_1 K_2) C_A = K_1 K_2 C_A$$

$$\frac{d^2 C_A}{dt^2} + (K_1 + K_2) \frac{dC_A}{dt} + K_1 K_2 C_A = K_1 K_2 C_A$$

$$\frac{d^2 C_A}{dt^2} + (K_1 + K_2) \frac{dC_A}{dt} = 0 \quad , \quad \lambda = K_1 + K_2$$

$$D^2 + \lambda D = 0$$

$$D(D + \lambda) = 0 \Rightarrow D_1 = 0 \; , \; D_2 = -\lambda$$

$$C_A = C_1 e^{0t} + C_2 e^{-\lambda t} \Rightarrow C_A = C_1 + C_2 e^{-\lambda t}$$

At $t = 0$ $\quad C_A = C_{A_o}$

$t = \infty$ $\quad C_A = C_{Ae}$

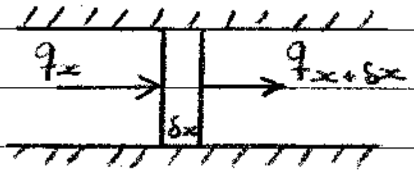
B.C (1) $\Rightarrow C_{A_o} = C_1 + C_2$

B.C (2) $\Rightarrow C_{Ae} = C_1 + 0 \Rightarrow C_{Ae} = C_1$

$\therefore C_2 = C_{A_o} - C_{Ae}$

$$C_A = C_1 + C_2 e^{-\lambda t} \Rightarrow C_A = C_{Ae} + (C_{A_o} - C_{Ae}) e^{-(K_1+K_2)t}$$

Example: A hot liquid flow through a pipe (insulated) with constant velocity u. Find the differential equation which describe the variation of liquid temperature in axial distance with time?



Heat Balance

In = Out + Accumulation

$$q_x = q_{x+\delta x} + M C_p \frac{\partial T}{\partial t}$$

$$\cancel{q_x} = \cancel{q_x} + \frac{\partial q_x}{\partial x}\delta x + M C_p \frac{\partial T}{\partial t}$$

$$0 = \frac{\partial q_x}{\partial x}\delta x + M C_p \frac{\partial T}{\partial t}$$

$$0 = \dot{m} C_p \frac{\partial T}{\partial x}\delta x + \rho V C_p \frac{\partial T}{\partial t}$$

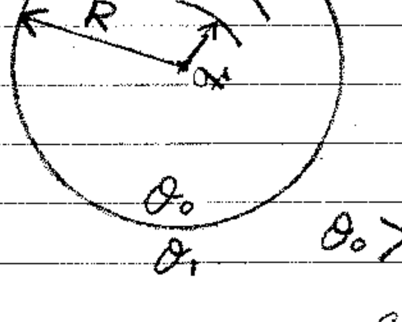
$$0 = \rho u A C_p \frac{\partial T}{\partial x}\delta x + \rho A \delta x C_p \frac{\partial T}{\partial t}$$

$$0 = u\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} \Rightarrow \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0$$

Example : Derive heat transfer equation through a spherical body ?

In = Out + Accumulation

$$q_r(4\pi r^2) = q_{r+\delta r}\,(4\pi(r+\delta r)^2) + M\,C_p\,\frac{\partial\theta}{\partial t}$$



$\theta_o > \theta_1$

$$q_r(4\pi r^2) = (q_r + \frac{\partial q_r}{\partial r}\delta r)(4\pi r^2 + 8\pi r\delta r + 4\pi\delta r^2) + 4\pi r^2\delta r$$

$$\rho C_p \frac{\partial\theta}{\partial t}$$

$$q_r(4\pi r^2) = q_r(4\pi r^2) + q_r(8\pi r\delta r) + q_r(4\pi\delta r^2) + \frac{\partial q_r}{\partial r}\delta r$$

$$(4\pi r^2) + \frac{\partial q_r}{\partial r}\delta r^2(8\pi r) + \frac{\partial q_r}{\partial r}\delta r^3(4\pi) + 4\pi r^2\delta r\rho C_p\frac{\partial\theta}{\partial t}$$

δr is small, δr^2 & δr^3 are very small $\Rightarrow$ neglected

$$0 = 2q_r + r\frac{\partial q_r}{\partial r} + r\rho C_p\frac{\partial\theta}{\partial t} \qquad \div r$$

$$0 = \frac{2}{r}q_r + \frac{\partial q_r}{\partial r} + \rho C_p\frac{\partial\theta}{\partial t}$$

$$q_r = -K\frac{\partial\theta}{\partial r}$$

$$0 = \frac{2}{r}\left(-K\frac{\partial\theta}{\partial r}\right) - K\frac{\partial^2\theta}{\partial r^2} + \rho C_p\frac{\partial\theta}{\partial t}$$

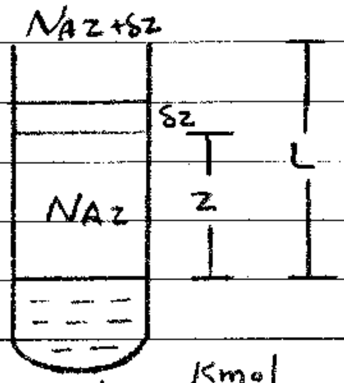
$$\frac{\partial\theta}{\partial t} = \frac{K}{\rho C_p}\left[\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r}\right]$$

$t=0 \qquad \theta = \theta_0$

$r=0 \qquad \frac{\partial\theta}{\partial r} = 0$

$r=R \qquad \theta = \theta_1$

Example: A glass tube of cross sectional (S) is filled with a volatile liquid to a certain level. The level is kept constant. It's open end is subjected to a stream air. Find the equation describing the rate of diffusion of the vapor of volatile liquid.



$$N_A = \frac{Kmol}{m^2 . s}$$

Mass Balance

In = Out + Accumulation

$$N_A \, . \, S = \left[N_A + \frac{\partial N_A}{\partial z} \delta z \right] S + \overset{V}{\underbrace{S \, \delta z}} \frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_A}{\partial z} \delta z . S + \delta z S \frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_A}{\partial z} + \frac{\partial C_A}{\partial t} \Rightarrow \frac{\partial C_A}{\partial t} = - \frac{\partial N_A}{\partial z}$$

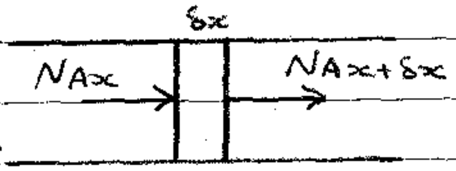
$$N_{A_z} = - D_A \frac{\partial C_A}{\partial z} + U_z C_A$$ Fick's law

$U_z C_A$ *is neglected* $\Rightarrow N_A = - D_A \dfrac{\partial C_A}{\partial z}$

$$\frac{\partial C_A}{\partial t} = D_A \frac{\partial^2 C_A}{\partial z^2}$$

$$\begin{array}{llll}
t = 0 & C_A = 0 & t = 0 & C = 0 \\
z = 0 & C_A = C_A^* & x = 0 & C = C_i \\
z = L & C_A = 0 & x = \infty & C = 0
\end{array}$$

Example : The liquid phase reaction $A \xrightarrow{K} B$ is carried out in tubular packed bed reactor. The liquid enters at constant velocity u and the concentration of A is C_{A0}. The reactor initially has only inert material ($C_{A0}=0$). Find the differential equation which describe this system?



Material Balance on A

In + Generation = Out + Consumption + Acc$^{\underline{n}}$.

$$N_{Ax}(\pi R^2) + 0 = N_{Ax+\delta x}(\pi R^2) + K C_A V + V \frac{\partial C_A}{\partial t}$$

$$N_{Ax}(\pi R^2) = \left(N_{Ax} + \frac{\partial N_{Ax}}{\partial x}\delta x\right)(\pi R^2) + K C_A V + V \frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_{Ax}}{\partial x}\delta x(\pi R^2) + K C_A(\pi R^2 \delta x) + (\pi R^2 \delta x)\frac{\partial C_A}{\partial t}$$

$$0 = \frac{\partial N_{Ax}}{\partial x} + K C_A + \frac{\partial C_A}{\partial t} \Rightarrow \frac{\partial C_A}{\partial t} = -\frac{\partial N_{Ax}}{\partial x} - K C_A$$

$$N_{Ax} = -D\frac{\partial C_A}{\partial x} + u_x C_A$$

$$\frac{\partial N_{Ax}}{\partial x} = -D\frac{\partial^2 C_A}{\partial x^2} + u_x \frac{\partial C_A}{\partial x}$$

$$\frac{\partial C_A}{\partial t} = D\frac{\partial^2 C_A}{\partial x^2} + u_x \frac{\partial C_A}{\partial x} - K C_A$$

$D\frac{\partial^2 C_A}{\partial x^2}$ is neglected

$$\therefore \frac{\partial C_A}{\partial t} = u_x \frac{\partial C_A}{\partial x} - K C_A$$