Numerical Anylsis

Solution of non-linear equations

1 simple Iterative method

As mentioned above, open methods employ a formula to predict the root. such a formula can be developed for simple Iterative by rearanging the function $f(x)=0$ so that x is one the left-hand side of the equation:

$$x = g(x) \quad \text{---} \; (1)$$

this transformation can be accomplished either by algebraic manipulation or by simply adding x to both sides of the original equation.

for example, $x^2 - 2x + 3 = 0$

can be simply maimipulated to yield $x = \frac{x^2+3}{2}$

whereas $\sin x = 0$ would be put into the form of eq. (1) by adding x to both sides to yield $x = \sin x + x$

the utility of eq. (1) is that it provides a formula to predict a value of x as a function of x, Thus, given an initial guess at the root x_i, eq. (1) can be used to compute a new estimate x_{i+1}, as expressed by the Iterative formula : $X_{i+1} = g(X_i)$

And the approximate error for this equation can be determined using the error in the following equation

$$|\epsilon| = \left| \frac{X_{i+1} - X_i}{X_{i+1}} \right| 100\%$$

ex.1 use simple Iterative method to locate the root of $f(x) = e^{-x} - x = 0$

sol. For $x_{i+1} = g(x_i) \Rightarrow X_{i+1} = e^{-X_i}$

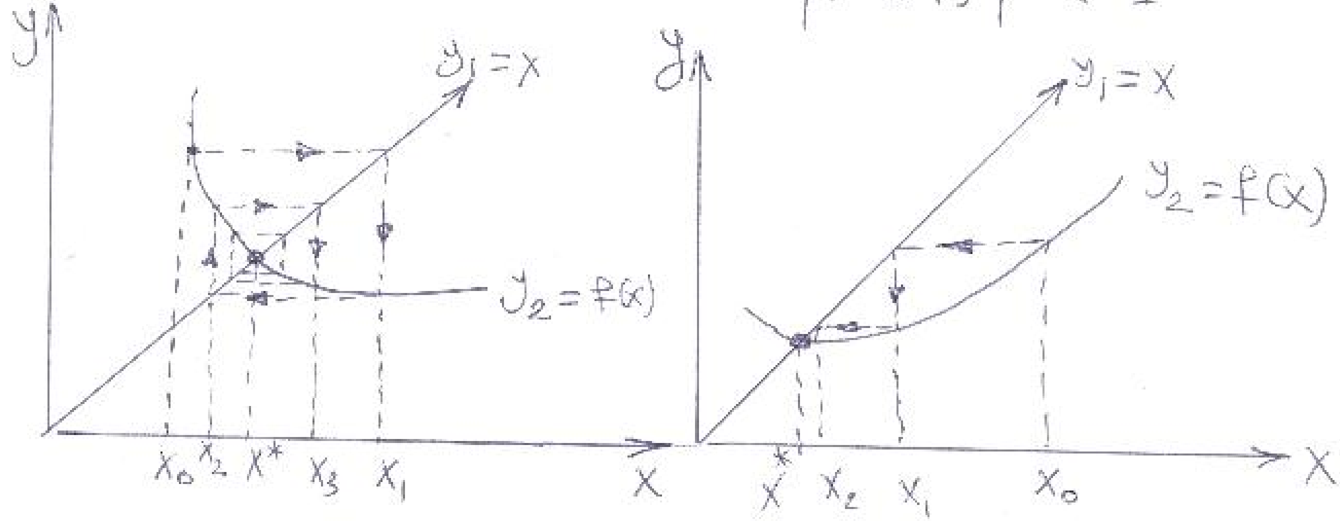
and starting with initial value of $x_0 = 0$

i	x_i	$\lvert\epsilon\rvert$
0	0	
1	1.000000	100
2	0.367879	171.8
3	0.692201	46.9
4	0.500473	38.3
5	0.606244	17.4
6	0.545396	11.2
7	0.579612	5.9
8	0.560115	3.48
9	0.571143	1.93
10	0.564879	1.11
⋮	⋮	⋮

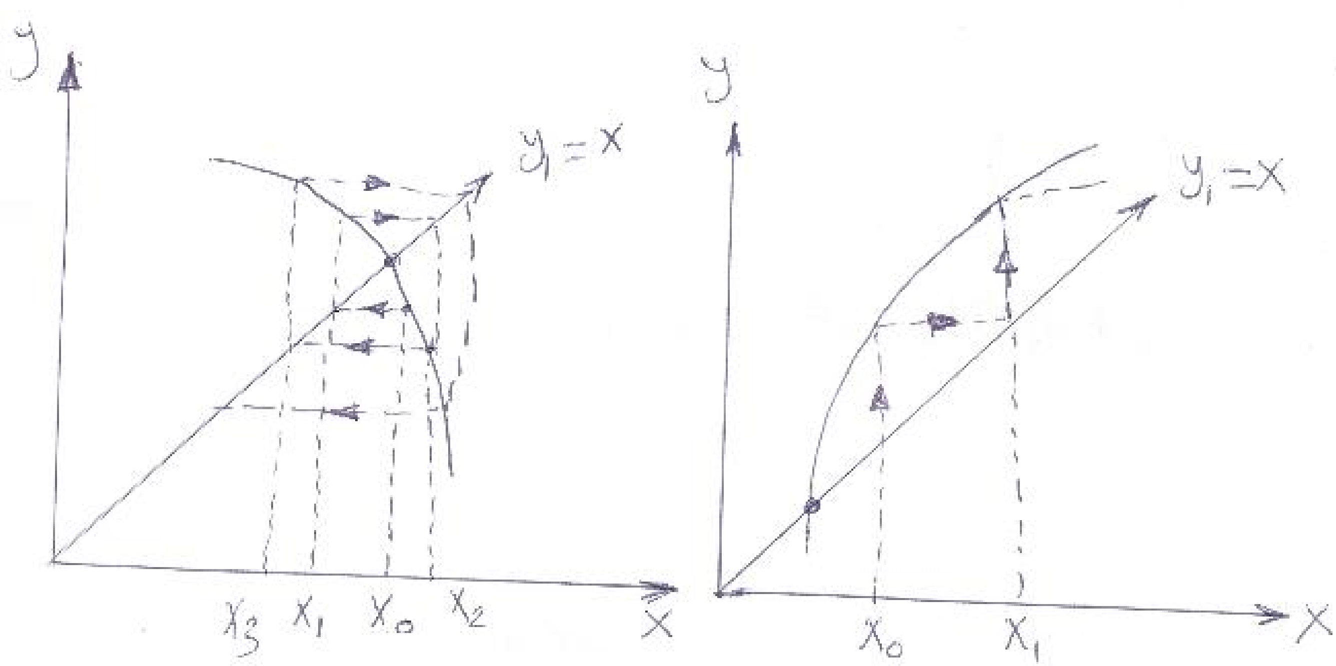
the true value $X^* = 0.567143$

Condition of convergence and divergence for simple Iteration method

the function $x_{i+1} = g(x_i)$ to converge must be satisfied $|f'(x_i)| < 1$



Convergence cases

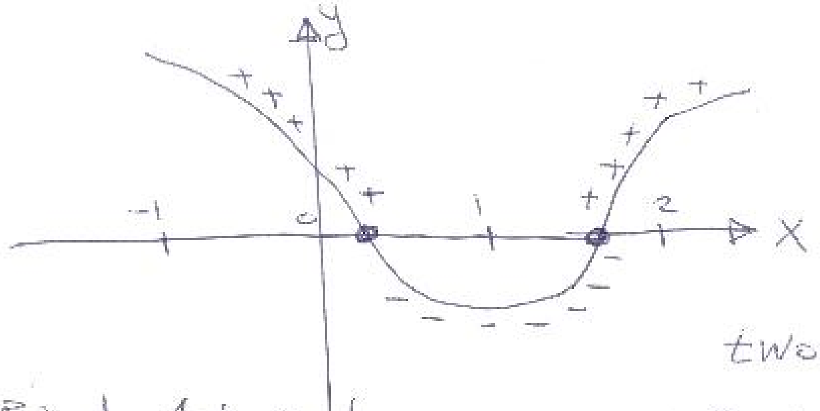


divergence cases

ex. 2 Find the root of the equation $2x^3 - 7x + 2 = 0$ using the simple iterative method.

sol. to find the initial value of iteration x_0 we must graph the function.

then $y = f(x) = 2x^3 - 7x + 2$



x	y
-1	+
0	+
1	−
2	+

two roots

$0 \leqslant x_1^* \leqslant 1$

$1 \leqslant x_2^* \leqslant 2$

* to find 1st root

$0 \leqslant x_1^* \leqslant 1 \Rightarrow x_{i+1} = \frac{2}{7}(x_i^3 + 1)$

and $f'(x) = \frac{6}{7}x^2$

$|f'(0)| = 0 < 1$

$|f'(1)| = \frac{6}{7} < 1$

$\Rightarrow$ the solution is Convergence

i	x_i	x_{i+1}
0	1.000	0.5714
1	0.5714	0.3390
2	0.3390	0.2968
3	0.2968	0.2932
4	0.2932	0.2929
5	0.2929	0.2929

$\Rightarrow x_1^* = 0.2929$

* to find 2nd root

$1 \leqslant x_2^* \leqslant 2$

$|f'(1)| = \frac{6}{7} < 1$ the solution

$|f'(2)| = \frac{24}{7} > 1$ $\Rightarrow$ is divergence

therefor, the equation must be change

$$x_{i+1} = \sqrt[3]{\frac{7}{2}x_i - 1}$$

and $f'(x) = \frac{1}{3}\left(\frac{7}{2}x - 1\right)^{-2/3} * \frac{7}{2}$

$|f'(1)| = \quad < 1$ $\Rightarrow$ the solution

$|f'(2)| = 0.3533 < 1$ is convergence

i	x_i	x_{i+1}
0	1.000	1.3572
1	1.3572	1.5536
2	1.5536	1.6433
3	1.6433	1.6812
4	1.6812	1.6967
5	1.6967	1.7029
6	1.7029	1.7054
7	1.7054	1.7064
8	1.7064	1.7068
10	1.7068	1.7070
11	1.7070	1.7071
12	1.7071	1.7071

$\Rightarrow x^* = 1.7071$

ex find the root of the equation $x = \cos x$ using the simple Iteration method

Sol. $f(x) = x - \cos x$ to graph

x	$f(x)$
-1	-
0	-
1	+
2	+

then $0 \leq x^* \leq 1$, $x_{i+1} = \cos x_i$

$\Rightarrow f'(x) = -\sin x$

$|f'(0)| = 0 < 1$

$|f'(1)| = 0.841 < 1$

$\Rightarrow$ this form of the equation $x = \cos x$ will be converge

i	x_i	x_{i+1}
0	0.00	1.00
1	1.00	0.54
2	0.54	0.86
3	0.86	0.65
4	0.65	0.79
5	0.79	0.70
6	0.70	0.76
7	0.76	0.72
8	0.72	0.75
9	0.75	0.73
10	0.73	0.74
11	0.74	0.74

$\Rightarrow x^* = 0.74$

ex. find the root of the following equation $x^2 - 4 = \ln x$ use $x_0 = 1.000$

sol. Let $x_{i+1} = \sqrt{\ln x_i + 4} = f(x) = (\ln x + 4)^{1/2}$

$\Rightarrow f'(x) = \frac{1}{2}(\ln x + 4)^{-1/2} * \frac{1}{x}$

$\Rightarrow f'(1) = 0.25 < 1$

i	x_i	x_{i+1}
0	1.000	2.000
1	2.000	2.166
2	2.166	2.185
3	2.185	2.187
4	2.187	2.187

$\Rightarrow x^* = 2.187$

ex. Find one root of the equation $2x^5 - 2x - 1 = 0$ start with $x_0 = 0.000$

Sol. Let $x_{i+1} = \sqrt[5]{\frac{2x_i + 1}{2}} \Rightarrow f'(x) = \frac{1}{5}\left(\frac{2x+1}{2}\right)^{-4/5} * 1$

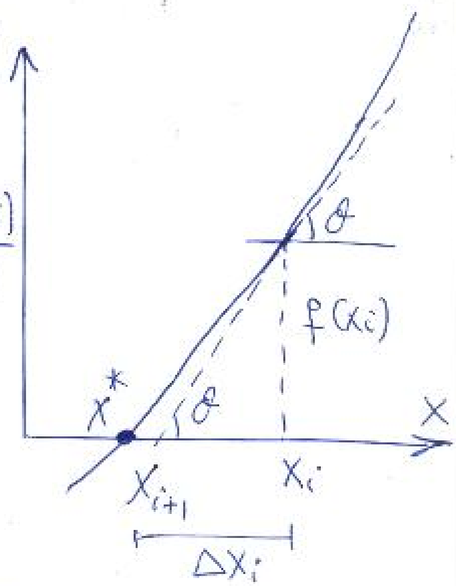
$\Rightarrow f'(0) = 0.348 < 1$

i	x_i	x_{i+1}
0	0.000	0.871
1	0.871	1.065
2	1.065	1.094
3	1.094	1.098
4	1.098	1.098

$\Rightarrow x^* = 1.098$

② Newton-Raphson method

Let $f(x) = 0$



using Taylor series method

$$f(x) = 0 = f(x_i) + \Delta x_i \cdot \frac{f'(x_i)}{1!} + \Delta x_i^2 \cdot \frac{f''(x_i)}{2!} + \Delta x_i^3 \cdot \frac{f'''(x_i)}{3!} + \cdots$$

for more accuracy $\Delta x_i \ll 0$

then $\Rightarrow$ $f(x_i) + \Delta x_i \cdot \frac{f'(x_i)}{1!} = 0$

$$\Rightarrow \Delta x_i = \frac{-f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i + \Delta x_i$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

ex. solve the following equation using Newton Raphson method:

sol. $\Rightarrow$ $f(x) = \frac{1}{x} + 1$, $f'(x) = -1/x^2$

$$\Rightarrow x_{i+1} = x_i - \frac{(1/x + 1)}{(-1/x^2)}$$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	-0.500	-1.000	-4.000	-0.750
1	-0.750	-0.333	-1.770	-0.937
2	-0.937	-0.067	-1.137	-0.997
3	-0.997	-0.003	-1.006	-1.000

Application of special cases for Newton-Raphson method

(A) Square roots

Let $n > 0 \Rightarrow$ any number, and $x = \sqrt{n}$

$$\Rightarrow x^2 - n = 0 = f(x) \quad , \quad f'(x) = 2x$$

then by N-R-M

$$x_{i+1} = x_i - \frac{x_i^2 - n}{2x_i} = \frac{1}{2}\left[x_i + \frac{n}{x_i}\right]$$

ex. Find the square root of 10 using Newton Raphson method, starting with $x_0 = 3.0000$

sol. $n = 10 \Rightarrow x_{i+1} = x_i - \dfrac{x_i^2 - 10}{2x_i}$

i	x_i	x_{i+1}
1	3.0000	3.1667
2	3.1667	3.1623
3	3.1623	3.1623

$$\Rightarrow x^* = 3.1623$$

(B) Roots of any arbitrary order

Let $x = \sqrt[k]{n}$ $\Rightarrow$ $x^k - n = f(x)$

$\Rightarrow$ $f'(x) = k\,x^{k-1}$ where n = any number
k = Integar number

then by N-R-M

$$x_{i+1} = x_i - \frac{x_i^k - n}{k\,x_i^{k-1}}$$

ex. Compute $\sqrt[3]{7}$, using Newton-Raphson method starting from $x_0 = 1.5$, take an accuracy 5D places.

Sol. $n = 7$, $K = 3$ $\Rightarrow$ $x_{i+1} = x_i - \frac{x_i^3 - 7}{3x_i^2}$

i	x_i	x_{i+1}
1	1.50000	2.03704
2	2.03704	1.92034
3	1.92034	1.91296
4	1.91296	1.91293
5	1.91293	1.91293

$\Rightarrow$ $x^* = 1.91293$

(c) The Reciprocal of any number

Let $x = \frac{1}{n} \Rightarrow n = \frac{1}{x} \Rightarrow f(x) = \frac{1}{x} - n = 0$

$\Rightarrow f'(x) = -\frac{1}{x^2}$

by N-R.M $\Rightarrow X_{i+1} = X_i - \frac{\left(\frac{1}{X_i} - n\right)}{\left(-1/X_i^2\right)}$

ex. Find the reciprocal of 2, using Newton Raphson method, starting with $X_0 = 0.1$ work to 4D ?

Sol. $n = 2 \Rightarrow X_{i+1} = X_i - \frac{\left(\frac{1}{X_i} - 2\right)}{\left(-1/X_i^2\right)}$

i	X_i	X_{i+1}
0	0.1000	0.1800
1	0.1800	0.2952
2	0.2952	0.4161
3	0.4161	0.4852
4	0.4852	0.4995
5	0.4995	0.4999
6	0.4999	0.4999

The following Linear System of equations

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n = b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n = b_2$$
$$\vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + \cdots + a_{mn}X_n = b_m$$

where $\Rightarrow a_{ij}$ $\begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{matrix}$ are the coefficients of n.

and X_1, X_2, X_n are variables;

b_1, b_2, b_m are constants

The above system can be written in the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A \cdot X = B$$

To solve this ~~system~~ system we have two types of ~~m~~methods :-

① $\boxed{\text{The direct methods.}}$

② $\boxed{\text{The indirect methods.}}$

A - The Matrix Inversion Method

when $A \cdot X = B$ then $X = A^{-1} \cdot B$

where A^{-1} is the inverse of A

Determinant of $A = |A| \neq 0$

ex.-① Use the matrix inversion method to find the values of (X_1, X_2, X_3) for the following set of linear equations:

$$3X_1 - 6X_2 + 7X_3 = 3$$
$$9X_1 \qquad - 5X_3 = 3$$
$$5X_1 - 8X_2 + 6X_3 = -4$$

Sol.-
$$\begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & 8 & 6 \end{bmatrix}$$

$\because |A| = 462 \neq 0 \Rightarrow$ there is a solution of this set of eq.

محددة المصفوفة

$$\begin{bmatrix} 3 & -6 & 7 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

مصفوفة الوحدة

* First Step :-

$\boxed{New\ R_1} = R_1 / a_{11}$

$$\begin{bmatrix} +1 & -2 & 2.33 \\ 9 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix} \begin{bmatrix} 0.33 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{New\ R_2} = R_2 - a_{21} \cdot R_1$

$\underline{New\ R_3} = R_3 - a_{31} \cdot R_1$

$$\begin{bmatrix} +1 & -2 & 2.33 \\ 0 & 18 & -26 \\ 0 & 2 & -5.65 \end{bmatrix} \begin{bmatrix} 0.33 & 0 & 0 \\ -3 & 1 & 0 \\ -1.65 & 0 & 1 \end{bmatrix}$$

* Second Step :-

$\boxed{New\ R_2} = R_2 / a_{22}$

$$\begin{bmatrix} 1 & -2 & 2.33 \\ 0 & 1 & -1.44 \\ 0 & 2 & -5.65 \end{bmatrix} \begin{bmatrix} 0.33 & 0 & 0 \\ -0.17 & 0.06 & 0 \\ -1.65 & 0 & 1 \end{bmatrix}$$

$\underline{New\ R_1} = R_1 - a_{12} \cdot R_2$

$\underline{New\ R_3} = R_3 - a_{32} \cdot R_2$

$$\begin{bmatrix} 1 & 0 & -0.55 \\ 0 & 1 & -1.44 \\ 0 & 0 & -2.77 \end{bmatrix} \begin{bmatrix} -0.01 & 0.12 & 0 \\ -0.17 & .006 & 0 \\ -1.31 & -0.12 & 1 \end{bmatrix}$$

* Third Step :-

$\boxed{\text{New } R_3} = R_3 / a_{33}$

$$\begin{bmatrix} 1 & 0 & -0.55 \\ 0 & 1 & 1.44 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.01 & 0.12 & 0 \\ -0.17 & 0.06 & 0 \\ 0.48 & 0.04 & -0.36 \end{bmatrix}$$

$\underline{\text{New } R_1} = R_1 - a_{13} \cdot R_3$

$\underline{\text{New } R_2} = R_2 \cdot a_{23} \cdot R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.04 & -0.36 \end{bmatrix}$$

$$\boxed{\therefore A^{-1}} = \begin{bmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.4 & -0.36 \end{bmatrix}$$

$$\boxed{\therefore X = A^{-1} \cdot B}$$

$$X = \begin{bmatrix} 0.26 & 0.14 & -0.2 \\ 0.52 & 0.12 & -0.52 \\ 0.48 & 0.4 & -0.36 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

$\therefore X_1 = (0.26 * 3) + (0.14 * 3) + (-0.2 * -4)$

$= 2$

$X_2 = 4$

$X_3 = 3$

(B) Gauss Elimination Method :-

- Form the matrix $[a_{ij} | b_i]$:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Downarrow$$

Forward Elimination

الحذف الأمامي

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{0} & a'_{22} & a'_{23} \\ \boxed{0} & \boxed{0} & a''_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

$$\Downarrow$$

Back Substitution

التعويض التراجعي

$$X_3 = b''_3 / a''_{33}$$

$$X_2 = (b'_2 - a'_{23} X_3) / a'_{22}$$

$$X_1 = (b'_1 - a_{12} X_2 - a_{13} X_3) / a_{11}$$

ex. ① find the solution of the following set of equations using the Gauss Elimination method. work 4D.

$$2.37X_1 + 3.06X_2 - 4.28X_3 = 1.76$$
$$1.46X_1 - 0.78X_2 + 3.75X_3 = 4.69$$
$$-3.6X_1 + 5.13X_2 - 1.06X_3 = 5.74$$

Sol.:-

$$\begin{bmatrix} 2.37 & 3.06 & -4.28 \\ 1.46 & -0.78 & 3.75 \\ -3.6 & 5.13 & 1.06 \end{bmatrix} \begin{bmatrix} 1.76 \\ 4.69 \\ 5.74 \end{bmatrix}$$

$$\Rightarrow \text{New } R_2 = R_2 - R_1 \frac{a_{21}}{a_{11}}$$
$$\text{New } R_3 = R_3 - R_1 \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} 2.37 & 3.06 & -4.28 \\ 0 & -2.6650 & 6.3865 \\ 0 & 9.8944 & -5.6040 \end{bmatrix} \begin{bmatrix} 1.7600 \\ 3.6058 \\ 8.4803 \end{bmatrix}$$

$$\Rightarrow NR_3 = R_3 - R_2 \frac{a_{32}}{a_{22}}$$

$$\begin{bmatrix} 2.37 & 3.06 & -4.28 \\ 0 & -2.6650 & 6.3865 \\ 0 & 0 & 18.1072 \end{bmatrix} \begin{bmatrix} 1.7600 \\ 3.6058 \\ 21.8676 \end{bmatrix}$$

$$\therefore X_3 = \frac{21.8676}{18.1072} = 1.2077$$

$$\therefore X_2 = 1.5412$$

$$\therefore X_1 = 0.9337$$

(c) Gauss-Jordan Elimination Method :-

- Form the matrix [A|B], and follow the same elimination steps to solve the eqn.

ex. (1) Solve :-

$$2X_1+3X_2-X_3=11$$
$$4X_1+4X_2-3X_3=17$$
$$-2X_1+3X_2-X_3=-1$$

Sol.

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 11 \\ 17 \\ -1 \end{bmatrix} \Rightarrow NR_1=\frac{R_1}{a_{11}}=\frac{R_1}{2}$$

$$\begin{bmatrix} 1 & 1.5 & -0.5 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 5.5 \\ 17 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} NR_2=R_2-a_{21}\cdot R_1 \\ NR_3=R_3-a_{31}\cdot R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1.5 & -0.5 \\ 0 & -2 & -1 \\ 0 & 6 & -2 \end{bmatrix}\begin{bmatrix} 5.5 \\ -5 \\ 10 \end{bmatrix} \Rightarrow NR_2=\frac{R_2}{a_{22}}=\frac{R_2}{-2}$$

$$\begin{bmatrix} 1 & 1.5 & -0.5 \\ 0 & 1 & 0.5 \\ 0 & 6 & -2 \end{bmatrix}\begin{bmatrix} 5.5 \\ 2.5 \\ 10 \end{bmatrix} \Rightarrow \begin{matrix} NR_1=R_1-a_{12}\cdot R_2 \\ NR_3=R_3-a_{32}\cdot R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & -5.0 \end{bmatrix}\begin{bmatrix} 1.75 \\ 2.5 \\ -5.0 \end{bmatrix} \Rightarrow NR_3=\frac{R_3}{a_{33}}=\frac{R_3}{-5}$$

$$\begin{bmatrix} 1 & 0 & -1.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1.75 \\ 2.5 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} NR_1=R_1-a_{13}\cdot R_3 \\ NR_2=R_2-a_{23}\cdot R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore X_1=3 \;;\; X_2=2 \;;\; X_3=1$

2 The Indirect Methods

In this type of methods we have a sufficient condition for a solution to be found:-

$$\left| a_{ij} \right| > \sum_{\substack{i=1 \\ j \neq i}} \left| a_{ij} \right| \; ; \; i = 1, 2, \dots, n$$

A- Jacob's Method

The following set of equations:

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 = b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = b_3$$

could be written as:

$$X_1^{K+1} = \left(b_1 - a_{12} X_2^{K} - a_{13} X_3^{K} \right) / a_{11}$$
$$X_2^{K+1} = \left(b_2 - a_{21} X_1^{K} - a_{23} X_3^{K} \right) / a_{22}$$
$$X_3^{K+1} = \left(b_3 - a_{31} X_1^{K} - a_{32} X_2^{K} \right) / a_{33}$$

where $(K = 0, 1, 2, 3 \dots)$

$K+1$ = new value ; K = old value.

The solution starts with $X_1^{0}, X_2^{0}, X_3^{0}$ sub.

then find $X_1^{(1)}, X_2^{(1)}, X_3^{(1)}$,

$X_1^{(2)}, X_2^{(2)}, X_3^{(2)}$ sub.

and so on.

Ex. 1 Solve the following set of equations using Jacob's method.

$$8X_1 - X_2 - X_3 = 8$$
$$X_1 - 7X_2 + 2X_3 = -4$$
$$2X_1 - X_2 + 9X_3 = 12$$

Sol.

$$X_1^{K+1} = 1 + 0.125\,X_2^{K} + 0.125\,X_3^{K}$$

$$X_2^{K+1} = 0.571 + 0.143\,X_1^{K} + 0.286\,X_3^{K}$$

$$X_3^{K+1} = 1.333 - 0.222\,X_1^{K} - 0.111\,X_2^{K}$$

assume $X_1^{0} = X_2^{0} = X_3^{0} = 0$

<table>
<tr><th>K</th><th>X1</th><th>X2</th><th>X3</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td><td>0.571</td><td>1.333</td></tr>
<tr><td>2</td><td>[illegible]</td><td>[illegible]</td><td>[illegible]</td></tr>
<tr><td>3</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>4</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>5</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>6</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>7</td><td>1.000</td><td>1.000</td><td>1.000</td></tr>
<tr><td>8</td><td>1.000</td><td>1.000</td><td>1.000</td></tr>
</table>

∴ actual values

$$X_1 = 1$$
$$X_2 = 1$$
$$X_3 = 1$$

B- Gauss - Seidel Method

The following set of equations:

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2$$
$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3$$

can be written as:

$$X_1^{K+1} = (b_1 - a_{12}X_2^{K} - a_{13}X_3^{K})/a_{11} \text{ ——— } (1)$$

$$X_2^{K+1} = (b_2 - a_{12}X_1^{K+1} - a_{23}X_3^{K})/a_{22} \text{ ——— } (2)$$

$$X_3^{K+1} = (b_3 - a_{31}X_1^{K+1} - a_{32}X_2^{K+1})/a_{33} \text{ ——— } (3)$$

Ex. (1) solve the following set of equations:

$$5X_1 - 2X_2 + X_3 = 4$$
$$X_1 + 4X_2 - 2X_3 = 3$$
$$X_1 + 2X_2 + 4X_3 = 17$$

sol.

Applying eq. (1), (2), and (3)

$$X_1 = 0.8 + 0.4X_2 - 0.2X_3 \text{ ——— } (1)$$
$$X_2 = 0.75 - 0.25X_1 + 0.5X_3 \text{ ——— } (2)$$
$$X_3 = 4.25 - 0.25X_1 - 0.5X_2 \text{ ——— } (3)$$

assume $X_2^{0} = X_3^{0} = 0 \Longrightarrow$ sub. in (1)

$\therefore X_1 = 0.8 \Longrightarrow X_2 = 0.55 \Longrightarrow X_3 = 3.75$

K	1	2	3	4	5	6	7
X_1	0.8	0.265	~~	~~	~~	1.001	0.999
X_2	0.55	2.571	~~	~~	~~	1.999	2.000
X_3	3.75	2.898	~~	~~	~~	3.000	3.000

$\therefore$ actual values $\Rightarrow X_1 = 0.999,\ X_2 = 2.000,\ X_3 = 3.000$

Ex. 2 Solve the following set of equations using a) Jacob's method.
b) Gauss-Seidel method.

$$5x_1 + 2x_2 + x_3 = 12$$
$$x_1 + 4x_2 + 2x_3 = 15$$
$$x_1 + 2x_2 + 5x_3 = 20$$

Sol.

assume $x_1^0 = x_2^0 = x_3^0 = 0$

Jacob's method	Gauss-Seidel method
$x_1^{k+1} = (12 - 2x_2^k - x_3^k)/5$	$x_1^{k+1} = (12 - 2x_2^k - x_3^k)/5$
$x_2^{k+1} = (15 - x_1^k - 2x_3^k)/4$	$x_2^{k+1} = (15 - x_1^{k+1} - 2x_3^k)/4$
$x_3^{k+1} = (20 - x_1^k - 2x_2^k)/5$	$x_3^{k+1} = (20 - x_1^{k+1} - 2x_2^{k+1})/5$

K	x_1	x_2	x_3	x_1	x_2	x_3
1	2.40	3.75	4.00	2.4	3.1	2.3
2	0.10	1.15	2.02	0.7	2.4	2.9
3	1.54	2.72	3.52	0.86	2.08	3.00
4	0.71	1.17	2.60	0.97	2.10	3.00
5	1.41	2.29	3.41	0.996	2.00	3.00
6	0.81	1.69	3.20			
7	1.08	1.95	3.16			
8	0.988	1.90	3.00			

$\therefore$ actual values are:

$x_1 = 0.996$, $x_2 = 2.00$, $x_3 = 3.00$

So;

we noted that Gauss-Seidel method is shorter and has more accuracy.

C- Relaxation Method

It is a modification of Gauss-Seidel method where the new value of (X) has to be corrected using the following formula:

$$\overset{new}{X_i^{*}} = \lambda X_i^{new} + (1-\lambda) X_i^{old}$$

Where $\Rightarrow$ λ is the correction term with value of $0 \leq \lambda \leq 2$

$X_i^{*} \Rightarrow$ is the corrected value.

The following set of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written as:

$$X_1^{k+1} = (b_1 - a_{12}x_1^{k} - a_{13}x_3^{k}) / a_{11}$$

$$\boxed{X_1^{k+1*} = X_1^{k} + \lambda (X_1^{k+1} - X_1^{k})}$$

$$X_2^{k+1} = (b_2 + a_{21}x_1^{k+1*} - a_{23}x_3^{k}) / a_{22}$$

$$\boxed{X_2^{k+1*} = X_2^{k} + \lambda (X_2^{k+1} - X_2^{k})}$$

$$X_3^{k+1} = (b_3 - a_{31}x_1^{k+1} - a_{32}X_2^{k+1*}) / a_{33}$$

$$\boxed{X_3^{k+1*} = X_3^{k} + \lambda (X_3^{k+1} - X_3^{k})}$$

Then subtitute to find the new values of (X) and so on.

Ex. 1 Solve the following equations using relaxation method with ($\lambda = 1.1$).

$$10X_1 + X_2 + X_3 = 12$$
$$X_1 + 10X_2 + X_3 = 12$$
$$X_1 + X_2 + 10X_3 = 12$$

Sol.

$$X_1 = 1.2 - 0.1X_2 - 0.1X_3 \quad \text{——— (1)}$$
$$X_2 = 1.2 - 0.1X_1 - 0.1X_3 \quad \text{——— (2)}$$
$$X_3 = 1.2 - 0.1X_1 - 0.1X_3 \quad \text{——— (3)}$$

assume: $X_2 = X_3 = 0$; Sub. in (1)

$$\therefore X_1 = 1.2$$
$$X_1^{K+1*} = X_1^{K} + \lambda(X_1^{K+1} - X_1^{K})$$
$$= 0 + 1.1(1.2 - 0) = \boxed{1.3200}$$

sub. in (2) $\Longrightarrow$ $X_2 = 1.068$

$$\therefore X_2^{K+1*} = \boxed{1.1748}$$

sub. in (3) $\Longrightarrow$ $X_3 = 0.9505$

$$\therefore X_3^{K+1*} = \boxed{1.0456}$$

K	X_1^*	X_2^*	X_3^*
1	1.3200	1.1748	1.0456
2	0.9550	0.9931	1.0010
3	1.0051	1.0000	0.9993
4	0.9996	1.0001	1.0001
5	1.0000	1.0000	1.0000

The actual values are:

$$X_1^* = 1.0000 \;,\; X_2^* = 1.0000 \;,\; X_3^* = 1.0000$$

Chapter Four

Interpolation

It is one of the mathematical methods to find approximate values of the function $f(x)$ for new values of x that lie between the given values (points) $(x_0, x_1, x_2, \dots, x_n)$.
These given values can be written in the form:- $f_0 = f(x_0)$, $f_1 = f(x_1)$, $f_n = f(x_n)$
or as orderd pairs :- (x_0, f_0), (x_1, f_1), (x_n, f_n)

* Differences Tables

a- Forward Differences (Δ)

$$\Delta f_0 = f_1 - f_0 \quad , \quad \Delta f_1 = f_2 - f_1 \quad , \quad \Delta f_2 = f_3 - f_2$$

$$\Rightarrow \quad f_n = f_{n+1} - f_n$$

for higher orders :-

$$\Delta^n f_n = \Delta^{n-1}(f_{n+1} - f_n)$$

b- Backward Differences (∇

$$\nabla f_n = f_n - f_{n-1}$$

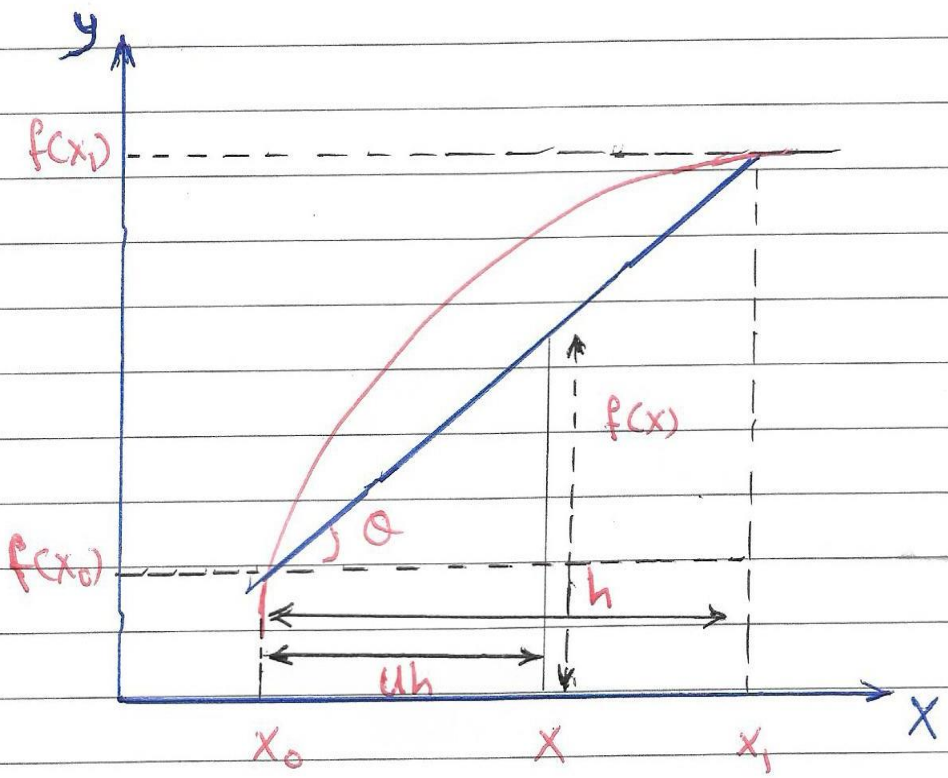
$$\nabla^n f_n = \nabla^{n-1}(f_n - f_{n-1})$$

Ex. $\Delta^2 f_1 = \Delta(\Delta f_1) = \Delta(f_2 - f_1)$

$$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta(f_1 - f_0)$$

Types of Interpolation :-

(1) Linear Interpolation



$$\tan\theta = \text{slope} = \frac{f(x_1) - f(x_0)}{h} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$x = x_0 - uh$$

$$\therefore \frac{f(x_1) - f(x_0)}{h} = \frac{f(x) - f(x_0)}{uh}$$

$$\therefore f(x) = f(x_0) + u\left[f(x_1) - f(x_0)\right]$$

$$f(x) = f(x_0) + u\Delta f(x_0)$$

$$\boxed{P(\cancel{x}) = f_{(x)} = f_0 + u\Delta f_0} \qquad 0 \leq u \leq 1$$

Ex. (1) Find the value of ($\ln 9.2$) for the following table using linear interpolation.

between 9.0 and 9.5

X	9.0	9.5	10.0
$\ln X$	2.1970	2.2510	2.3026

Sol.

$$f(x) = P(x) = f_0 + u \cdot \Delta f_0$$

$$u = \frac{X - X_0}{X_1 - X_0} = \frac{9.2 - 9.0}{9.5 - 9.0} = 0.4$$

$$\therefore f(x) = \ln x = \ln 9.2$$
$$= \ln 9 + 0.4 [\ln 9.5 - \ln 9.0]$$
$$= 2.219$$

Ex. (2) Find the value of ($\sin 22$) for the following table using linear interpolation.

X	0	10	20	30	40
$\sin x$	0	0.17365	0.34202	0.50000	0.69279

$$f(x) = P(x) = f_0 + u \Delta f_0$$

$$u = \frac{X - X_0}{X_1 - X_0} = \frac{22 - 20}{30 - 20} = 0.2$$

$$\therefore f(x) = \sin 22 = \sin 20 + 0.2 [\sin 30 - \sin 20]$$
$$= 0.34202 + 0.2 [0.5 - 0.34202]$$
$$= 0.34202 + 0.031596 = 0.37361$$

② Quadratic Interpolation

$$f(x) = f_0 + \frac{u \Delta f_0}{1!} + \frac{u(u-1)}{2!} \Delta^2 f_0$$

$$0 \leqslant u \leqslant 2$$

Ex. ①

find the value of $\ln(9.2)$ for the following table using Quadratic Interpolation.

x	9.0	9.5	10.0
$\ln x$	2.1970	2.2510	2.3026

Sol.

$$f(x) = \frac{u \Delta f_0}{1!} + \frac{u(u-1)}{2!} \cdot \Delta^2 f_0$$

$$= 2.1972 + 0.4 \frac{0.0541}{1 * 1} + \frac{0.4(0.6)}{2 * 1} \cdot (0.0028)$$

$$= 2.219$$

③ Newton Interpolation

a- Newton-Gregory Forward Formula :-

x_1	x_2	x_3	x_{n+1}
$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_{n+1})$

$$f(x_n) = P(x_n) = f_0 + u\frac{\Delta f_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 f_0$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0 + \cdots +$$

$$\frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^n f_0$$

where; $u = \frac{x - x_0}{h} \Rightarrow h = \text{constant}$

$0 \leq u \leq h$

b- Newton-Gregory Backward Formula

$$f(x) = P(x) = f_n + u\nabla f_1 + \frac{u(u+1)}{2!}\nabla^2 f_n$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n + \cdots +$$

$$\frac{u(u+1)(u+2)\cdots(u+n-1)}{n!}\nabla^n f_n$$

$u = \frac{x - x_n}{h} \Rightarrow h = \text{Constant}$

Ex. (1) Q.

Construct difference table and find the Polynomial of minimum degree which fits the following data and compute

1- $f(10.6)$.

2- $f'(7.3)$.

X	3	5	7	9	11
$f(x)$	6	24	58	108	174

Sol. for five points $\Rightarrow$ degree of polynomial $= 4$
$= n-1$

X	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
$3 = x_0$	6				
		18			
$5 = x_1$	24		16		
		34		0	
$7 = x_2$	58		16		0
		50		0	
$9 = x_3$	108		16		
		66			
$11 = x_4$	174				

$$P(x) = f(x) = f_0 + \frac{u \Delta f_0}{1!} + \frac{u(u-1)}{2!} \cdot \Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 f_0 + \cdots$$

$$u = \frac{x - x_0}{h} = \frac{x-3}{2} = 0.5(x-3)$$

$x - x_0$

$$\therefore f(x) = 6 + \frac{u * 18}{1!} + \frac{u(u-1)}{2!} * 16 + 0$$

$$f(x) = 2x^2 - 7x + 9$$

$$\therefore f(10.6) = 2(10.6)^2 - 7(10.6) + 9$$
$$= 224.72 - 74.2 + 9 = 307.92$$

$$f'(x) = 4x - 7$$

$$\therefore f'(7.3) = 4(7.3) - 7$$
$$= 22.2$$

Interpolation

Ex. Construct Newton interpolation polynomial on the interval $(3.5 \rightarrow 3.7)$ for the function $(y = e^x)$ using $(h = 0.05)$.

sol.

x	3.5	3.55	3.60	3.65	3.70
y	33.115	34.813	36.598	38.457	40.447

$+0.05$

$$u = \frac{x - x_0}{h} \Rightarrow h = 0.05 \Rightarrow u = \frac{1}{0.05}(x - 3.5)$$

$$\therefore u = 20(x - 3.5)$$

x غير محددة في السؤال

X	$y = e^x$	Δf	$\Delta^2 f$	$\Delta^3 f$	
3.5	33.115				
		1.698			
3.55	34.813		0.087		
		1.785		0.005	
3.60	36.598		0.092		
		1.877		0.003	
3.65	38.457		0.095		
		1.972			
3.70	40.447				

$$f(x) = f_0 + \frac{u}{1!}\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0$$

$$f(x) = 33.115 + \frac{20(x-3.5)}{1*1}(1.698) + \cdots\cdots$$

~~f(x) = [33.115~~

$$f(x) = 33.115 + 1.698 * 20(x-3.5) + 0.087 * 20(x-3.5) * \frac{20(x-3.5)-1}{2} + \frac{0.005}{6}[20(x-3.5)] * [20(x-3.5)-1] * [20(x-3.5)-2]$$

Interpolation

Ex. Find the velocity of a rocket by using Newton interpolating polynomial at $(t=150)$ sec.

t(sec)	0	60	120	180	240	300
V (mile/sec)	0	0.0824	0.2747	0.6502	1.3851	3.2224

t	V	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0	0					
		0.0824				
60	0.0824		0.1099			
		0.1932		0.0733		
120	0.2747		0.1831		0.1029	
		0.3755		0.1762		0.4644
180	0.6502		0.3594		0.5673	
		0.7344		0.7435		
240	1.3851		1.1029			
		1.8878				
300	3.2224					

sol. $h = 60$, $u = \frac{150-0}{60} = 2.5$

$$\therefore f(x) = f_0 + \frac{u}{1!}\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!} \cdot \Delta^5 f_0$$

$\therefore f(x) = 0.4365$ mile/s

Ex. Given the following function $y = \log_{10} X$
find $y = \log_{10} 1044$ for $(X = 1000(10)1050)$ by
using interpolation.

Sol. assume $x_0 = 1050$; $X = 1044$

$$u = \frac{x - x_n}{h} = \frac{1044 - 1050}{10} = -0.6$$

~~answer at the end~~

X	$y = \log_{10} X$	∇	∇^2	∇^3
1000	3.00000			
		0.00432		
1010	3.00432		-0.000426	
		0.004278		0.0008
1020	3.00860		-0.000418	
		0.00423		0.00009
1030	3.01283		-0.000409	
		0.00419		0.00009
1040	3.017033		-0.000401	
		0.00415		
1050	3.021189			

$x = 1044$ and it is located at the end of the table.

$$f(x) = \log_{10} 1044 = f_0 + \frac{u \nabla f_0}{1!} + \frac{u(u+1)}{2!} \nabla^2 f_0 + \frac{u(u+1)(u+2)}{3!} \nabla^3 f_0$$

$$\therefore f(x) = \log_{10} 1044 = 3.01887005$$

(24) Lagrange Interpolation Formula

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k = L_0 f(x_0) + L_1 f(x_1) + \cdots + L_n f(x_n)$$

where: $L_k = \prod_{\substack{i=0 \\ i \neq k}} \left(\frac{x - x_i}{x_k - x_i} \right)$ $i = 0, 1, 2, 3, \ldots, n$; $k = 0, 1, 2, 3, \ldots, n$

$\prod$ – is the product of .....
L_k – polynomial coefficients

for example $n = 1$ = first order

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

for Second order

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_2 - x_1)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$\Rightarrow L_0 = \frac{(x - x_1)}{(x_0 - x_1)}$$

$$L_1 = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

$$L_2 = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

$$\Rightarrow P_2(x) = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2)$$

Ex. Use $x_0 = 2$, $x_1 = 2.5$, $x_2 = 4$ find the second order polynomial of the function $f(x) = \frac{1}{x}$.

sol. $n = 2$

$$P_1(x) = \sum_{k=0}^{n=2} f(x_k) L_k$$

L_k = polynomial coefficients

$$L_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i} \right)$$

$$\therefore P_2(x) = f(x_0) L_0 + f(x_1) L_1 + f(x_2) L_2$$

$f(x) = \frac{1}{x}$, $x_0 = 2$, $x_1 = 2.5$, $x_2 = 4$

$$\therefore f(x_0) = \frac{1}{2} = 0.5$$

$$f(x_1) = \frac{1}{2.5} = 0.4$$

$$f(x_2) = \frac{1}{4} = 0.25$$

$$L_0 = \left(\frac{x - x_1}{x_0 - x_1} \right)\left(\frac{x - x_2}{x_0 - x_2} \right) = \left(\frac{x - 2.5}{2 - 2.5} \right)\left(\frac{x - 4}{2 - 4} \right) = x^2 - 6.5x + 10$$

$$L_1 = \frac{x^2 - 6x + 8}{0.0.75} \Rightarrow \left(\frac{x - x_0}{x_1 - x_0} \right)\left(\frac{x - x_2}{x_1 - x_2} \right) = \left(\frac{x - 2}{2.5 - 2} \right)\left(\frac{x - 4}{2.5 - 4} \right)$$

$$L_2 = \frac{x^2 - 4.5x + 5}{3} \Rightarrow \left(\frac{x - x_0}{x_2 - x_0} \right)\left(\frac{x - x_1}{x_2 - x_1} \right) = \left(\frac{x - 2}{4 - 2} \right)\left(\frac{x - 2.5}{4 - 2.5} \right)$$

$$P_2(x) = 0.5(x^2 - 6.5x + 10) + 0.4\left(\frac{x^2 - 6x + 8}{0.75} \right)$$

$$+ 0.25\left(\frac{x^2 - 4.5x + 5}{3} \right)$$

$$\Rightarrow P_2(x) = \boxed{0.05x^2 - 0.425x + 1.15}$$

Ex. Find the second degree interpolation polynomial.

Sol. $(n=2)$

$i = 1+2 = 3 \;;\; i = 0,1,2$
$K = 1+2 = 3 \;;\; K = 0,1,2$

$$P_n(x) = \sum_{K=0}^{n} f(x_K)\, L_K$$

$$L_K = \prod_{\substack{i=0 \\ i\neq K}} \frac{(x-x_i)}{(x_K-x_i)} \;;\; K=0,\; \underset{i\neq K}{i=1,2}$$

$$\therefore L_0 = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)$$

$K=1,\; i=0,2$

$$\therefore L_1 = \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)$$

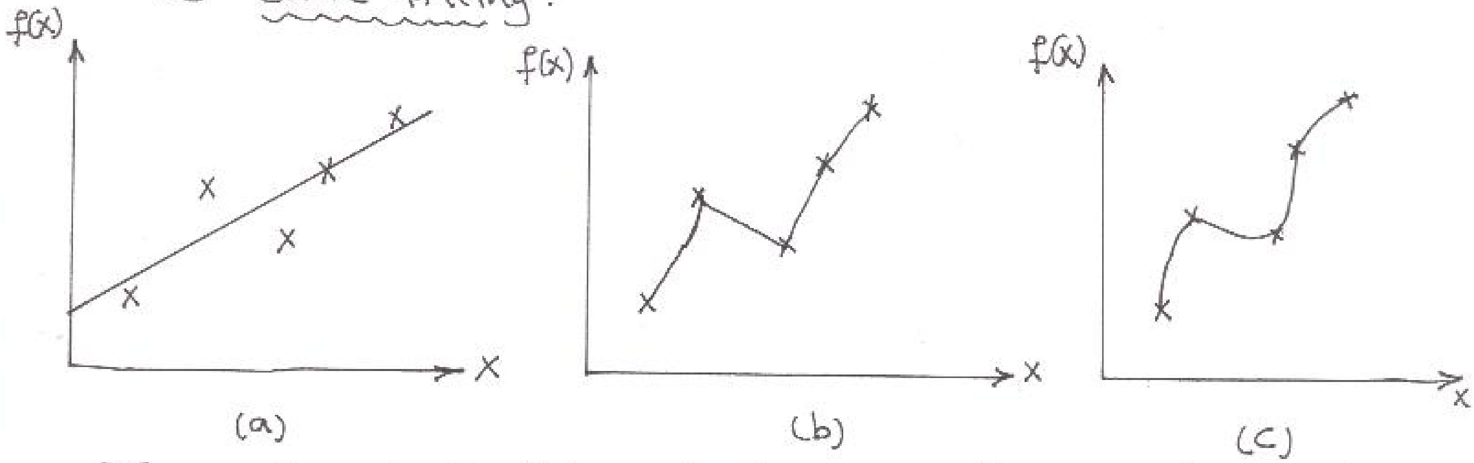
$K=2,\; i=0,1$

$$\therefore L_2 = \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)$$

$$\therefore P_2(x) = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2)$$

Curves Fitting

One way to do Fit the data is to compute values of the function at a number of descrete values along the range of interest. Then, a simpler function may be derived to fit these values. Both of these applications are known as curve fitting.



Three attempts to fit a best curve through five data points
(a) Least-squares regression (b) linear interpolation (c) curvilinear interpolation

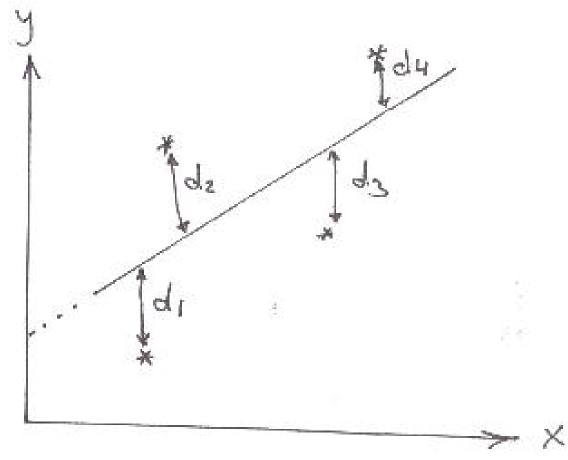
Least-Squares Regression

linear Regression

The simplest example of a least-squares approximation is fitting a straight line to a set of paired observations: (x_1, y_1), (x_2, y_2) ---- (x_n, y_n) the mathmatical expression for the straight line is

$$\boxed{\bar{y} = a_0 + a_1 x}$$

Deviation $= d = y - \acute{y}$



where

$$d_1 = y_1 - \bar{y}_1 = y_1 - f(x_1)$$
$$d_2 = y_2 - \bar{y}_2 = y_2 - f(x_2)$$
$$d_3 = y_3 - \bar{y}_3 = y_3 - f(x_3)$$
$$\vdots \qquad \vdots$$
$$d_m = y_m - \bar{y}_m = y_m - f(x_m)$$

$m =$ No. of points

Now we applied to minimize the sum of the squares of the residuals between the measured y and the y calculated with the linear model

$$S = \sum_{i=1}^{m} d_i^2 = \sum_{i=1}^{m} (y_i - \bar{y}_i)^2 = \sum_{i=1}^{m} (y_i - a_0 - a_1 x_i)^2$$

to find a_0, a_1 must $\frac{\partial S}{\partial a_0} = \frac{\partial S}{\partial a_1} = 0$ minimum values

Then,

$$\frac{\partial S}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i) = 0$$

$$\frac{\partial S}{\partial a_1} = -2 \sum \left[(y_i - a_0 - a_1 x_i) x_i \right] = 0$$

Now, realizing that $\sum_{i=1}^{m} a_0 = m a_0$, we can express the equations as a set of two simultaneous linear equations with two unknowns (a_0 and a_1)

$$n a_0 + a_1 \cdot \left(\sum x_i\right) = \left(\sum y_i\right)$$
$$\left(\sum x_i\right) a_0 + a_1 \cdot \left(\sum x_i^2\right) = \left(\sum x_i y_i\right)$$

there are called the normal equations they can be solved as

$$a_0 = \frac{\sum y_i \cdot \sum x_i^2 - \sum x_i \cdot \sum x_i \cdot y_i}{m \sum x_i^2 - \left(\sum x_i\right)^2} \qquad a_1 = \frac{m \sum x_i y_i - \sum x_i \cdot \sum y_i}{m \sum x_i^2 - \left(\sum x_i\right)^2}$$

ex. Use linear regression to fit the following experimental data :

x :	1	3	4	6	8	9	11	14
y :	1	2	4	4	5	7	8	9

sol.

Let $\bar{y} = a_0 + a_1 x$ Then $m = 8$

i	x_i	y_i	x_i^2	$x_i \cdot y_i$
1	1	1	1	1
2	3	2	9	6
3	4	4	16	16
4	6	4	36	24
5	8	5	64	40
6	9	7	81	63
7	11	8	121	88
8	14	9	196	126
Σ	56	40	564	364

$$a_0 = \frac{\Sigma y_i \cdot \Sigma x_i^2 - \Sigma x_i \cdot \Sigma x_i y_i}{m \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$\Rightarrow a_0 = \frac{40 * 524 - 56 * 364}{8 * 524 - 56^2}$$

$$\Rightarrow a_0 = 6/11$$

also

$$a_1 = \frac{m \Sigma x_i y_i - \Sigma x_i \Sigma y_i}{m \Sigma x_i^2 - (\Sigma x_i)^2}$$

$$\Rightarrow a_1 = \frac{8 * 364 - 56 * 40}{8 * 524 - 56^2}$$

$\Rightarrow a_1 = 7/11 \quad \Rightarrow \bar{y} = \frac{6}{11} + \frac{7}{11} x$ or $11\bar{y} - 7x = 6$

ex: Fit a straight line to the following data

x :	1	2	3	4	5	6	7
y :	0.5	2.5	2.0	4.0	3.5	6.0	5.5

sol: the following quantities can be computed

$m = 7$, $\Sigma x_i y_i = 119.5$, $\Sigma x_i^2 = 140$, $\Sigma x_i = 28$

$\Sigma y_i = 24$

Then $a_0 = 0.07142857$

$a_1 = 0.8392857$

Linearization of nonlinear relationships

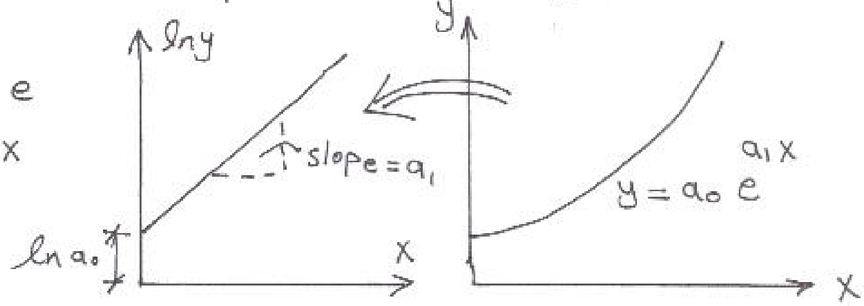
Linear regression provides a powerful technique for fitting a best line to data.

Case ① $\Rightarrow$ Exponential equation $\qquad y = a_0 e^{a_1 x}$

then,

$$\ln y = \ln a_0 + a_1 x \ln e$$

$$\Rightarrow \ln y = \ln a_0 + a_1 x$$

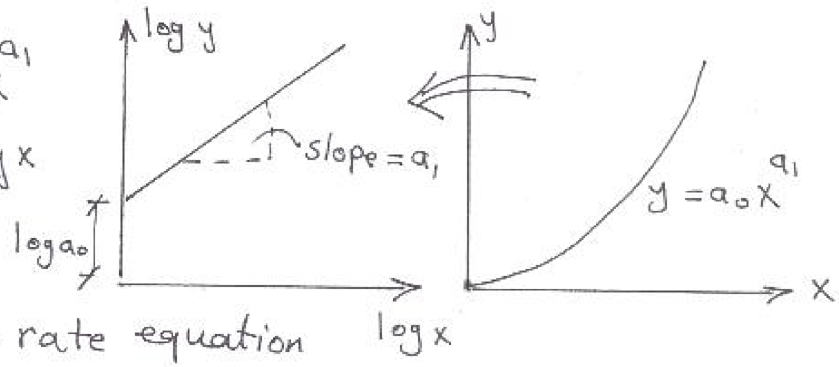


Case ② $\Rightarrow$ power equation $\qquad y = a_0 x^{a_1}$

then

$$\log y = \log a_0 + \log x^{a_1}$$

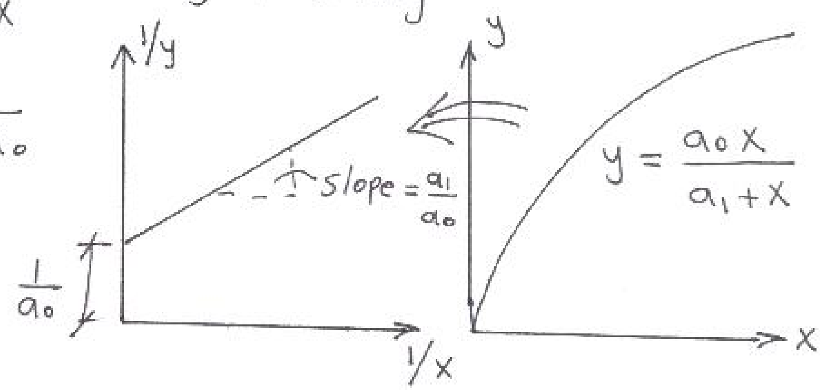
$$\log y = \log a_0 + a_1 \log x$$



Case ③ $\Rightarrow$ Growth-rate equation

$$y = a_0 \cdot \frac{x}{a_1 + x} \qquad \text{by inverting}$$

$$\Rightarrow \frac{1}{y} = \frac{a_1}{a_0} \qquad + \frac{1}{a_0}$$



ex: Fit the data in the following table using a logarithmic transformation of the data

x :	1	2	3	4	5
y :	0.5	1.7	3.4	5.7	8.4

sol: logarithmic transformation $\Rightarrow$ applied for power eq.

$$y = a_0 x^{a_1} \xrightarrow[\text{to}]{\text{Linearization}} \log y = \log a_0 + a_1 \log x$$

then

$$Y = a_0 + a_1 X$$

x	y	log x	log y	$(\log x)^2$	$\log x \cdot \log y$
1	0.5	0.000	-0.301	0.00	0.00
2	1.7	0.301	0.226	0.090	0.080
3	3.4	0.477	0.534	0.227	0.254
4	5.7	0.602	0.753	0.362	0.453
5	8.4	0.699	0.922	0.488	0.644
Σ		2.079	2.134	1.167	1.431

then

$$a_0 = \frac{\Sigma \log y \cdot \Sigma \log x^2 - \Sigma \log x \cdot \Sigma \log x \cdot \log y}{m \cdot \Sigma \log x^2 - (\Sigma \log x)^2} = \frac{2.134 \times 1.167 - 2.079 \times 1.431}{5 \times 1.167 - 2.079^2}$$

$$\Rightarrow a_0 = -0.320 = \log a_0 \Rightarrow a_0 = 0.478$$

also

$$a_1 = \frac{m \cdot \Sigma \log x \cdot \log y - \Sigma \log x \cdot \Sigma \log y}{m \ \Sigma \log x^2 - (\Sigma \log x)^2} = \frac{5 \times 1.431 - 2.079 \times 2.134}{5 \times 1.167 - 2.079^2}$$

$$\Rightarrow a_1 = 1.796 = a_1$$

then $y = 0.478\, x^{1.796}$ or $\log y = -0.32 + 1.796 \log x$

Polynomial Regression

Let $\bar{y} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$$\Rightarrow S^t = \sum_{i=1}^{m} [y_i - \bar{y}] = \sum_{i=1}^{m} [y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n]$$

$$\Rightarrow \frac{\partial S}{\partial a_0} = -2 \sum [y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n] = 0$$

$$\Rightarrow \frac{\partial S}{\partial a_1} = -2 \sum [x_i (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

$$\Rightarrow \frac{\partial S}{\partial a_2} = -2 \sum [x_i^2 (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

$$\vdots \qquad \vdots$$

$$\frac{\partial S}{\partial a_n} = -2 \sum [x_i^n (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n)] = 0$$

then

$$a_0 m + a_1 \sum x_i + a_2 \sum x_i^2 + \cdots + a_n \sum x_i^n = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 + \cdots + a_n \sum x_i^{n+1} = \sum x_i y_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 + \cdots + a_n \sum x_i^{n+2} = \sum x_i^2 \cdot y_i$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\Rightarrow a_0 \sum x_i^n + a_1 \sum x_i^{n+1} + a_2 \sum x_i^{n+2} + \cdots + a_n \sum x_i^{2n} = \sum x_i^n \cdot y_i$$

ex: fit a second-order polynomial to the data in the following table:

x :	0	1	2	3	4	5
y :	2.1	7.7	13.6	27.2	40.9	61.1

sol $\bar{y} = a_0 + a_1 x + a_2 x^2$

then $n = 2$, $m = 6$

$\sum x_i = 15$, $\sum y_i = 152.6$, $\sum x_i^2 = 55$, $\sum x_i^3 = 225$

$\sum x_i^4 = 979$, $\sum x_i y_i = 585.6$, $\sum x_i^2 \cdot y_i = 2488.8$

$\Rightarrow$

$$\Rightarrow \quad 6a_0 + 15a_1 + 55a_2 = 152.6$$
$$15a_0 + 55a_1 + 225a_2 = 585.6$$
$$55a_0 + 225a_1 + 979a_2 = 2488.8$$

by Guass method

$a_0 = 2.47857$

$a_1 = 2.35929$ $\quad \Rightarrow \quad y = 2.47857 + 2.35929x + 1.8607x^2$

$a_2 = 1.86071$

(1) Newton - Forward Formula

$$f(x) = y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$u = \frac{x - x_0}{h} \Rightarrow \Delta u = \frac{\Delta x}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{h} \text{ then}$$

$$\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{du} \quad \text{——— (1)}$$

$$y = y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0 + \cdots$$

$$\therefore \frac{dy}{du} = 0 + \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots \quad (2)$$

sub. (2) in (1)

$$\therefore y'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 \right. \quad \text{—— (3)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right) \cdot \frac{du}{dx}$$

$$\therefore y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \cdots\right] \quad \text{——— (4)}$$

* Special cases:-

When $x = x_0 \Rightarrow u = \frac{x_0 - x_0}{h} = 0$

$$\therefore y'(x_0) = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right.$$

$$y''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\cdots\right.$$

Ex. ① Find the value of the derivative for the following function of $x = 2.3$; $X = X_0$.
$y = f(x) = x^4 - \ln x$ when $x = 1(0.5)3.5$

Sol.

X	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1.0	1.0			
1.5	4.657	3.657	6.993	
2·3 ← 2.0	15.307	10.650	12.189	5.196
2.5	38.146	22.839	18.916	6.727
3.0	79.901	41.755	27.154	8.238
3.5	148.810	68.909		

$$u = \frac{x - x_0}{h} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y'(2.3) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(2.3) = \boxed{48.255} \Longrightarrow \text{approximate value}$$

$*$ at $x = X_0 \Rightarrow u = \frac{X_0 - X_0}{h} = 0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(X_0) = \frac{1}{0.5}\left[3.657 - \frac{1}{2}(6.993) + \frac{2}{6}(5.196)\right]$$

$$= 3.785$$

$*$ $y = x^4 - \ln x \Longrightarrow y' = 4x^3 - \frac{1}{x}$

$$\therefore y'(2.3) = 4(2.3)^3 - \frac{1}{2.3}$$

$$= \boxed{48.233} \Longrightarrow \text{exact value}$$

(2) Newton - Backward Formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

and $u = \frac{x-x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du} \qquad \text{——— (1)}$$

$$y = y_n + u\nabla y_n + \frac{u^2+u}{2!}\nabla^2 y_n + \frac{u^3+3u^2+2u}{3!}\nabla^3 y_n$$

$$\therefore \frac{dy}{du} = 0 + \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \qquad \text{——— (2)}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n\right] \qquad \text{——— (3)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx}$$

(4)

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \cdots\right]$$

* special cases

when $x = x_n$ then $u = \frac{x_n - x_n}{h} = 0$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$y''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

Ex. (4) Find the exact and approximate value of the derivative for the following function at $X = 2.3$ and $X = X_0$; $f(x) = X^4 \ln x$;

$X = 1(0.5)3.5$

Sol.

X	1.0	1.5	2.0	2.5	3.0	3.5
y	1.0	4.657	15.307	38.146	79.901	148.810

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0				
		3.657			
1.5	4.657		6.993		
		10.650		5.196	
2.0	15.307		12.189		1.531
		22.839		6.727	
2.5	38.146		18.916		1.511
		41.755		8.238	
3.0	79.901		27.15		
		68.909			
3.5	148.81				

$$u = \frac{X - X_0}{h} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(2.3) = \frac{1}{0.5}\left\{3.657 + \frac{2(2.6)-1}{2}(6.993) + \frac{3(2.6)^2 - 6(2.6) + 2}{6}(5.196)\right]$$

$y'(2.3) = 48.255 \Longrightarrow$ approximate value

$$y' = 4X^3 - \frac{1}{X}$$

$$= 4(2.3)^3 - \frac{1}{2.3} = 48.233 \Longrightarrow \text{exact value}$$

* at $X = X_0$

$$y'(X_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right]$$

$$= \frac{1}{0.5}\left[3.657 - \frac{6.993}{2} + \frac{5.196}{3}\right]$$

$$= 3.785$$

Ex. ② By using Newton Backward formula, find the value of the derivative for the following data at $x = 2.5$.

x	0	1	2	3	4
y	-8	-7	0	19	56

Sol.

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0	-8				
1	-7	1	6		
2	0	7	12	6	0
3	19	19	18	6	
4	56	37			

$\nabla^4 y_n = 0$

$$u = \frac{x - x_n}{h} = \frac{2.5 - 4}{1} = (-1.5) \quad ; \quad h = x - x_0 = 1$$

$$\therefore y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3(u)^2 + 6u + 2}{6}\nabla^3 y_n\right]$$

$$= \frac{1}{1}\left[37 + \frac{2(-1.5)+1}{\cancel{2}}\,\cancel{(18)}^{\,9} + \frac{3(-1.5)^2 + 6(-1.5) + 2}{\cancel{6}}\,\cancel{(6)}\right]$$

$$\therefore y'(x) = 18.75$$

$\nabla^4 y_n = 0$

$$y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n\right]$$

$$\therefore y''(x) = \frac{1}{1}\left[18 + (-1.5+1)(6)\right] = 18 - 3$$
$$= 16$$

3) Lagrange Formula

$$y_{(x)} = L_0 y_0 + L_1 y_1 + L_2 y_2 + \dots + L_n y_n$$

$$\therefore \frac{dy}{dx} = \frac{dL_0}{dx} y_0 + \frac{dL_1}{dx} y_1 + \frac{dL_2}{dx} y_2 + \dots + \frac{dL_n}{dx} y_n$$

$$L_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{x^2 - x_1 x - x_2 x - x_1 x_2}{(x_0-x_1)(x_0-x_2)}$$

$$\therefore \frac{dL_0}{dx} = \frac{2x - x_1 - x_2}{(x_0-x_1)(x_0-x_2)}$$

$$L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{x^2 - x_2 x - x_0 x - x_0 x_2}{(x_1-x_0)(x_1-x_2)}$$

$$\therefore \frac{dL_1}{dx} = \frac{2x - x_2 - x_0}{(x_1-x_0)(x_1-x_2)}$$

$$\vdots \qquad \vdots$$

$$\frac{dL_n}{dx} = \,?$$

Ex. (3) Find the first derivative of the function tabulated bellow at the point (1·3).

X	1·2	1·5	1·7
y	0·1823	0·4055	0·5306

Sol.

$$\frac{dy}{dx} = \frac{dL_0}{dx} y_0 + \frac{dL_1}{dx} y_1 + \frac{dL_2}{dx} y_2$$

$$\frac{dL_0}{dx} = \frac{2X - X_1 - X_2}{(X_0 - X_1)(X_0 - X_2}$$

$$= \frac{2(1 \cdot 3) - 1 \cdot 5 - 1 \cdot 7}{(-0 \cdot 3)(-0 \cdot 5)}$$

$$\frac{dL_1}{dx} = \frac{2X - X_2 - X_0}{(X_1 - X_0)(X_1 - X_2)}$$

$$= \frac{2(1 \cdot 3) - 1 \cdot 7 - 1 \cdot 2}{(0 \cdot 3)(-0 \cdot 2)}$$

$$\frac{dL_2}{dx} = \frac{2X - X_0 - X_1}{(X_2 - X_0)(X_2 - X_1)}$$

$$= \frac{2(1 \cdot 3) - 1 \cdot 2 - 1 \cdot 5}{(0 \cdot 5)(0 \cdot 2)}$$

$$\therefore \frac{dy}{dx} =$$

Repeat this example (at the point 1·5).

(1) Newton-Forward Formula

$$f(x) = y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$u = \frac{x - x_0}{h} \Rightarrow \Delta u = \frac{\Delta x}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{h} \text{ then}$$

$$\frac{dy}{dx} = \frac{1}{h}\cdot\frac{dy}{du} \quad \text{——— (1)}$$

$$y = y_0 + u\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{(u^3 - 3u^2 + 2u)}{3!}\Delta^3 y_0 + \cdots$$

$$\therefore \frac{dy}{du} = 0 + \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots \quad (2)$$

sub. (2) in (1)

$$\therefore y'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 \right. \quad \text{——— (3)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx}$$

$$\therefore y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \cdots\right] \quad \text{——— (4)}$$

* Special cases:-

When $x = x_0 \Rightarrow u = \dfrac{x_0 - x_0}{h} = 0$

$$\therefore y'(x_0) = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right.$$

$$y''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\cdots\right.$$

Ex. ① Find the value of the derivative for the following function of $x = 2.3$; $X = X_0$.
$y = f(x) = x^4 - \ln x$ when $x = 1(0.5)3.5$

Sol.

X	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1.0	1.0			
		3.657		
1.5	4.657		6.993	
		10.650		5.196
2.0 ← 2.3	15.307		12.189	
		22.839		6.727
2.5	38.146		18.916	
		41.755		8.238
3.0	79.901		27.154	
		68.909		
3.5	148.810			

$$u = \frac{x - x_0}{h} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y'(2.3) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(2.3) = \boxed{48.255} \Longrightarrow \text{approximate value}$$

$*$ at $x = X_0 \Rightarrow u = \frac{X_0 - X_0}{h} = 0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2!}\Delta^2 y_0 + \frac{2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(X_0) = \frac{1}{0.5}\left[3.657 - \frac{1}{2}(6.993) + \frac{2}{6}(5.196)\right]$$

$$= 3.785$$

$*$ $y = x^4 - \ln x \Longrightarrow y' = 4x^3 - \frac{1}{x}$

$$\therefore y'(2.3) = 4(2.3)^3 - \frac{1}{2.3}$$

$$= \boxed{48.233} \Longrightarrow \text{exact value}$$

(2) Newton - Backward Formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

and $u = \frac{x - x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du} \qquad (1)$$

$$y = y_n + u\nabla y_n + \frac{u^2+u}{2!}\nabla^2 y_n + \frac{u^3+3u^2+2u}{3!}\nabla^3 y_n$$

$$\therefore \frac{dy}{du} = 0 + \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \qquad (2)$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n\right] \qquad (3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \cdots\right] \qquad (4)$$

* special cases

when $x = x_n$ then $u = \frac{x_n - x_n}{h} = 0$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$y''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

Ex. (4) Find the exact and approximate value of the derivative for the following function at $X = 2.3$ and $X = X_0$; $f(x) = x^4 \ln x$;
$x = 1(0.5)3.5$

Sol.

x	1.0	1.5	2.0	2.5	3.0	3.5
y	1.0	4.657	15.307	38.146	79.901	148.810

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0				
		3.657			
1.5	4.657		6.993		
		10.650		5.196	
2.0	15.307		12.189		1.531
		22.839		6.727	
2.5	38.146		18.916		1.511
		41.755		8.238	
3.0	79.901		27.15		
		68.909			
3.5	148.81				

$$u = \frac{x - x_0}{h} = \frac{2.3 - 1}{0.5} = 2.6$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

$$\therefore y'(2.3) = \frac{1}{0.5}\left[3.657 + \frac{2(2.6)-1}{2}(6.993) + \frac{3(2.6)^2 - 6(2.6) + 2}{6} (5.196)\right]$$

$y'(2.3) = 48.255 \Rightarrow$ approximate value

$$y' = 4x^3 - \frac{1}{x}$$

$$= 4(2.3)^3 - \frac{1}{2.3} = 48.233 \Rightarrow \text{exact value}$$

* at $X = X_0$

$$y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right]$$

$$= \frac{1}{0.5}\left[3.657 - \frac{6.993}{2} + \frac{5.196}{3}\right]$$

$$= 3.785$$

Ex. ② By using Newton Backward formula, find the value of the derivative for the following data at $x = 2.5$.

x	0	1	2	3	4
y	-8	-7	0	19	56

Sol.

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0	-8				
		1			
1	-7		6		
		7		6	
2	0		12		0
		19		6	
3	19		18		
		37			
4	56				

$$u = \frac{x - x_n}{h} = \frac{2.5 - 4}{1} = -1.5 \quad ; \quad h = x - x_0 = 1$$

$\nabla^4 y_n = 0$

$$\therefore y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3(u)^2 + 6u + 2}{6}\nabla^3 y_n\right]$$

$$= \frac{1}{1}\left[37 + \frac{2(-1.5)+1}{\cancel{2}}(\cancel{18})^{9} + \frac{3(-1.5)^2 + 6(-1.5) + 2}{\cancel{6}}(\cancel{6})\right]$$

$$\therefore y'(x) = 18.75$$

$$y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n\right]$$

$\nabla^4 y_n = 0$

$$\therefore y''(x) = \frac{1}{1}\left[18 + (-1.5+1)(6)\right] = 18 - 3$$

$$= 15$$

(3) Lagrange Formula

$$y_{(x)} = L_0 y_0 + L_1 y_1 + L_2 y_2 + \cdots + L_n y_n$$

$$\therefore \frac{dy}{dx} = \frac{dL_0}{dx} y_0 + \frac{dL_1}{dx} y_1 + \frac{dL_2}{dx} y_2 + \cdots + \frac{dL_n}{dx} y_n$$

$$L_0 = \frac{(X - X_1)(X - X_2)}{(X_0 - X_1)(X_0 - X_2)}$$

$$= \frac{X^2 - X_1 X - X_2 X - X_1 X_2}{(X_0 - X_1)(X_0 - X_2)}$$

$$\therefore \frac{dL_0}{dx} = \frac{2X - X_1 - X_2}{(X_0 - X_1)(X_0 - X_2)}$$

$$L_1 = \frac{(X - X_0)(X - X_2)}{(X_1 - X_0)(X_1 - X_2)}$$

$$= \frac{X^2 - X_2 X - X_0 X - X_0 X_2}{(X_1 - X_0)(X_1 - X_2)}$$

$$\therefore \frac{dL_1}{dx} = \frac{2X - X_2 - X_0}{(X_1 - X_0)(X_1 - X_2)}$$

$$\vdots \qquad \vdots$$

$$\frac{dL_n}{dx} = ?$$

Ex. ③ Find the first derivative of the function tabulated bellow at the point (1.3).

X	1.2	1.5	1.7
y	0.1823	0.4055	0.5306

Sol.

$$\frac{dy}{dx} = \frac{dL_0}{dx} y_0 + \frac{dL_1}{dx} y_1 + \frac{dL_2}{dx} y_2$$

$$\frac{dL_0}{dx} = \frac{2X - X_1 - X_2}{(X_0 - X_1)(X_0 - X_2}$$

$$= \frac{2(1.3) - 1.5 - 1.7}{(-0.3)(-0.5)}$$

$$\frac{dL_1}{dx} = \frac{2X - X_2 - X_0}{(X_1 - X_0)(X_1 - X_2)}$$

$$= \frac{2(1.3) - 1.7 - 1.2}{(0.3)(-0.2)}$$

$$\frac{dL_2}{dx} = \frac{2X - X_0 - X_1}{(X_2 - X_0)(X_2 - X_1)}$$

$$= \frac{2(1.3) - 1.2 - 1.5}{(0.5)(0.2)}$$

$$\therefore \frac{dy}{dx} =$$

Repeat this example (at the point 1.5).

A- Trapezoidal Rule

$$\text{Integral} = I = \int_a^b f(x)\,dx \; ; \; \underbrace{h}_{\text{step size}} = \frac{b-a}{N}$$

$b-a \Rightarrow$ interval

$N \Rightarrow$ number of areas

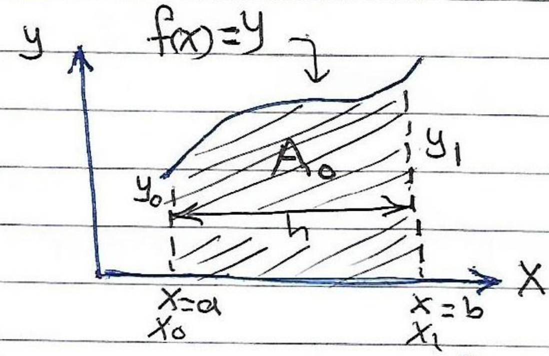


fig. (1)

$$A_0 = \frac{1}{2}h(y_0 + y_1) \Longrightarrow \text{for fig. (1)}$$

$$\int_a^b f(x)\,dx = \text{Total area} = \underbrace{A_0 + A_1 + A_2 + A_3 + \cdots + A_n}_{\text{for fig. (2)}}$$

$$= \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \cdots + \frac{1}{2}h(y_{n-1} + y_n)$$

$$\therefore \int_a^b f(x)\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_n) + y_n \right]$$

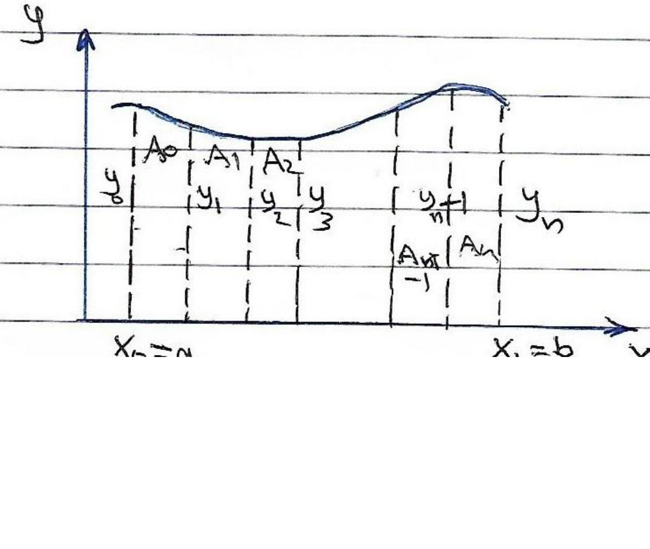


fig. (2)

Ex. ① Evaluate $\int_0^1 \frac{dx}{1+x^2}$ to (4D) by trapezoidal rule

where the interval $(0 \rightarrow 1)$ is sub-divided into (6) equal parts.

Sol.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_{n-1} + y_n\right]$$

$\because N = 6$

$\therefore h = \frac{b-a}{N} = \frac{1-0}{6} = \frac{1}{6} \Longrightarrow$ Step size

X	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1.0
$f(x) = \frac{1}{1+x^2}$	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	---	---		

$$\therefore \int_0^1 \frac{dx}{1+x^2} = 0.7842$$

Ex. ② Use the trapezoidal rule with 4 intervals to evaluate the integral $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places.

Sol.

$$h = \frac{b-a}{N} = \frac{3-1}{4} = 0.5$$

X	1.0	1.5	2.0	2.5	3.0
$y = \frac{2}{\sqrt{x}}$	2.000	1.633	1.414	1.2649	1.154

$$\therefore \int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_{n-1} + y_n\right]$$

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx = 2.945$$

B- Simpson's $\frac{1}{3}$ Rule :-

$$f(x) = f(x_0) + u\cdot\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \cdots$$

or: $$f(x) = y_0 + u\cdot\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

for second order $$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\left[y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0\right]dx$$

$$\therefore \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\Rightarrow \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + y_{n-1} + y_n)$$

Ex: (1) Use Simpson's $\frac{1}{3}$ Rule for evaluation of $\int_0^1 \frac{dx}{1-x^2}$, considering (6) strips.

Sol. $h = \frac{b-a}{N} = \frac{1-0}{6} = \frac{1}{6}$

X	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1.0
$f(x)=\frac{1}{1-x^2}$	1	0.97297	0.9	0.8	0.6923	0.59016	0.5

$$I = \int_0^1 \frac{dx}{1-x^2} = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \frac{h}{3}(y_4 + 4y_5 + y_6)$$

$$= 0.78539$$

C- Simpson's $\frac{3}{8}$ Rule :-

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_3} \left\{ y_0 + u \cdot \Delta y_0 + \frac{u(u-1)}{2!} \cdot \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 \right\} dx$$

$\Delta y_0 = y_1 - y_0$; $\Delta^2 y_0 = y_2 - 2y_1 + y_0$; $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

$$u = \frac{x - x_0}{h}$$

$$\therefore I = \int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8} h (y_0 + 3y_1 + 3y_2 + y_3)$$

Ex. ① Use Simpson's $\frac{3}{8}$ rule to evaluate $\int_0^1 x^4\,dx$ considering (6) strips.

Sol. $h = \frac{b-a}{N} = \frac{1-0}{6} = \frac{1}{6}$

X	0	1/6	2/6	3/6	4/6	5/6	1.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
$y = x^4$	0	0.00077	0.01234	0.06251	0.1975	0.482253	1.0

$$I = \frac{3}{8} h \left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) \right]$$

$$= 0.2002243$$

D- Multiple Integrals :-

$$\iint_A f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$h_x = \dfrac{b-a}{N}$ step size for x ; $x \longrightarrow (a-b)$

$h_y = \dfrac{d-c}{N}$ step size for y ; $y \longrightarrow (c-d)$

Ex.① Evaluate the double integral $\int_{0.2}^{0.6} \int_{1.5}^{3.0} f(x,y)\,dx\,dy$ using trapezoidal rule in x-direction and Simpson rule in y-direction, and f(x,y) as a given in the following table

x \ y	0.1	0.2	0.3	0.4	0.5	0.6
0.5	0.165	0.428	0.687	0.942	1.190	1.431
1.0	0.271	0.640	1.003	1.359	1.703	2.035
1.5	0.447	0.990	1.524	2.045	2.549	3.031
2.0	0.738	1.568	2.384	3.177	3.943	4.672
2.5	1.216	2.520	3.800	5.044	6.241	7.379
3.0	2.005	4.090	6.139	8.122	10.030	11.841
3.5	3.306	6.679	9.986	13.196	16.277	19.198

Sol. ⇒ Interval $x = (1.5 \longrightarrow 3.0)$ and $y = (0.2 \longrightarrow 0.6)$

if $y = 0.2$ = constant then $h_x = 2 - 1.5 = 0.5$

$$I_x = \int_{1.5}^{3.0} f(x,y)\,dx = \int_{1.5}^{3.0} f(x,0.2)\,dx = \frac{h_x}{2}\left[y_1 + 2(y_1 + y_3) + y_4\right]$$

$$I_{x_0}(y=0.2) = \frac{0.5}{2}\left[0.99 + 2(1.568 + 2.520) + 4.090\right] = 3.3140$$

$$I_{x_1}(y=0.3) = \frac{0.5}{2}\left[1.524 + 2(2.384 + 3800) + 6.139\right] = 5.007$$

$I_{x_2} = 6.6522$, $I_{x_3} = 8.2368$, $I_{x_4} = 9.7435$

$h_y = 0.3 - 0.2 = 0.1$

$$\therefore \int_{0.2}^{0.6} f(x,y)\,dy = \frac{h_y}{3}\left[y_0 + 4(y_1 + y_3) + 2(y_2) + y_4\right]$$

$$= \frac{0.1}{3}\left[3.314 + 4(5.007 + 8.2368) + 2(6.6522) + 9.7435\right]$$

$$= 2.6446$$