SHEAR FORCE AND BENDING MOMENT

The load applied on a beam gets transferred to supports. To see how this transfer takes place, consider a simply supported beam subject to the loads as shown in Fig. 9.26.

$$\sum M_B = 0, \text{ gives}$$

$$R_A \times 7 = 20 \times 5 + 40 \times 3 + 60 \times 1$$

$$R_A = 40 \text{ kN}$$

$$R_B = (20 + 40 + 60) - 40$$

$$= 80 \text{ kN}$$

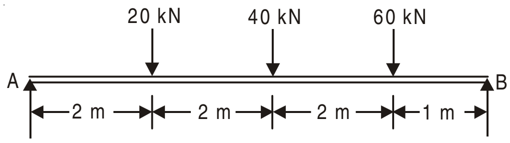

Fig. 9.26

Now to find what is happening at a section, consider the section at C which is at a distance of 3 m from A. Imagining a cut at this section, left hand side portion and right hand side portions are drawn separately in Fig. 9.27.

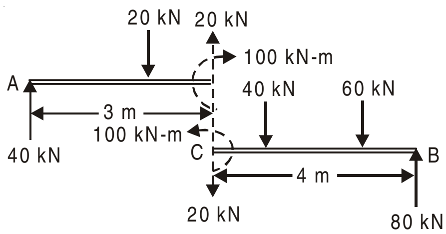

Fig. 9.27

Considering the algebraic sum of vertical forces acting on left hand side portion, it is found that a net vertical force of 40 – 20 = 20 kN is experienced at the section. This effect is shown in Fig. 9.27 by dotted line. Again considering the portion on left hand side, the moment experienced at *C* is given by

$$M_C = 40 \times 3 - 20 \times 1$$
$$= 100 \text{ kN-m clockwise.}$$

This moment is also shown on left hand side portion of the beam at C by dotted line.

Now, considering the right hand side portion:

Force experienced at *C*

$$= 80 - 60 - 40 = -20 \text{ kN}$$
$$= 20 \text{ kN. downward}$$

and the moment experienced is $M = 80 \times 4 - 60 \times 3 - 40 \times 1$

$$= 100 \text{ kN-m. anticlockwise}$$

These forces and moments are also shown in Fig. 9.27 on right hand side portion of the beam at *C*.

Thus the section *C* is subjected to a force of 20 kN, which is trying to shear off the beam as shown in Fig. 9.28(*a*), and is also subjected to a moment of 100 kN-m which is trying to bend the beam as shown in Fig. 9.28(*b*). Since this force of 20 kN is trying to shear off the section, it is called as shear force at section *C*. The moment is trying to bend the beam at *C* and hence it is called as bending moment at that section. The shear force and bending moment at a section in a beam may be defined as follows:

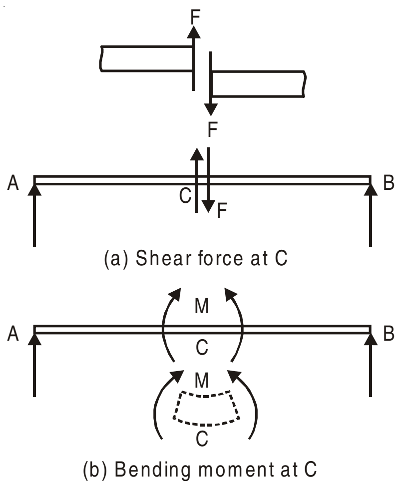

Fig. 9.28

"*Shear Force* at a section in a beam (or any structural member) is the force that is trying to shear off the section and is obtained as algebraic sum of all the forces acting normal to the axis of beam either to the left or to the right of the section".

"*Bending Moment* at a section in a beam is the moment that is trying to bend the beam and is obtained as algebraic sum of moment of all the forces about the section, acting either to the left or to the right of the section".

Hence to find shear force or bending moment at a section, a cut at the section is to be imagined and any one portion with all the forces acting on that portion, is to be considered. It may be noted that for finding bending moment at a section, the moment of the forces are to be found about the section considered.

SIGN CONVENTION

Although different sign conventions may be used, most of the engineers use the following sign conventions for shear forces and bending moment.

(*a*) The shear force that tends to move left portion upward relative to the right portion shall be called as positive shear force (Fig. 9.29).

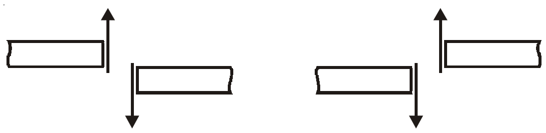

Fig. 9.29

(*b*) The bending moment that is trying to sag the beam shall be taken as positive bending moment. If left portion is considered positive bending moment comes out to be clockwise moment (Fig. 9.30).

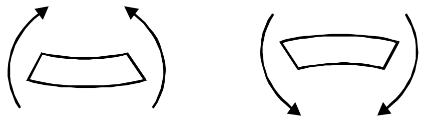

Fig. 9.30

To decide the sign of moment due to a force about a section, assume the beam is held tightly at that section and observe the deflected shape. Then looking at the shape sign can be assigned. Thus in the problem shown in Fig. 9.26 and 9.27, 40 kN reaction at *A* produces positive moment at *C* and 20 kN load produces negative moment.

RELATIONSHIP BETWEEN LOAD INTENSITY, SHEAR FORCE AND BENDING MOMENT

Consider the beam *AB* subject to a general loading as shown in Fig. 9.31(*a*). The free body diagram of a segment of beam at a distance x from *A* and of length δx is shown in Fig. 9.31(*b*). The intensity of loading on this elemental length may be taken as constant. Let it be w/unit length.

Let shear force and bending moment acting on the section at a distance x be F and M respectively. At section at a distance $x + \delta x$, these values be $F + \delta F$ and $M + \delta M$ respectively. Now from the equilibrium of the element.

$$\Sigma V = 0 \text{ leads to}$$

$$-F + F + \delta F - w\delta x = 0$$

or
$$\frac{\delta F}{\delta x} = w$$

In the limiting case as $\delta x \to 0$, ...(9.1)

$$\frac{dF}{dx} = w$$

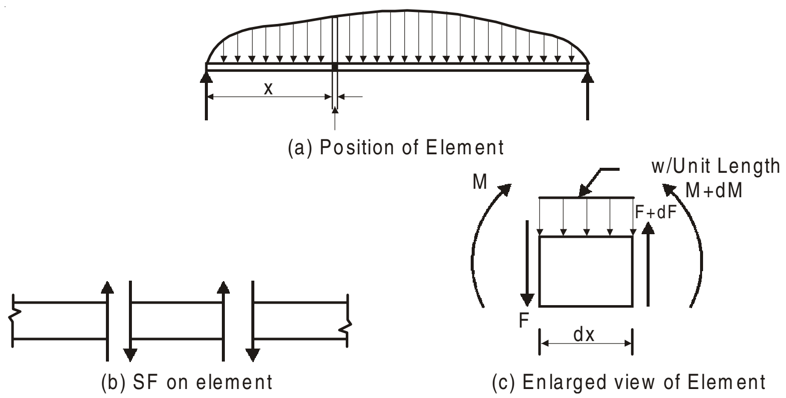

Fig. 9.31

The moment equilibrium condition at section $x + dx$ leads to

$$M - F\delta x - w\delta x \times \frac{\delta x}{2} - (M + \delta M) = 0$$

Neglecting the small quantity of higher order, we get

$$F\delta x + \delta M = 0$$

or
$$\frac{\delta M}{\delta x} = -F$$

In the limiting case as $x \to 0$, we get ...(9.2)

$$\frac{dM}{dx} = -F$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS

Shear force and bending moment in a beam vary from section to section. The graphical representation of shear force in which ordinate represents shear force and the abscissa represents the position of the section is called *Shear Force Diagram* (*SFD*). The diagram in which the ordinate represent bending moment the abscissa represent the position of the section is called *Bending Moment Diagram (BMD).* In drawing *SFD* and *BMD*, the sign conventions explained earlier are used. These diagrams are usually located below the load diagram.

From equations 1 and 2, it may be concluded that the rate of change of shear force (slope of shear force diagram curve) at any section is equal to the intensity of loading at that section and the rate of change of bending moment (*i.e.,* shape of bending moment diagram curve) is equal to the shear force at that section. From equation 2, it can also be concluded that the bending moment will be maximum/minimum where shear force (dM/dx) is zero.

At any section, if moment changes its sign the point representing that section is called the point of contraflexure. Obviously, the moment at that section is zero.

SFD AND BMD FOR A FEW STANDARD CASES

The methods of drawing shear force and bending moment diagrams have been explained here in case of the following beams subjected to standard loading conditions.

(*a*) Cantilever beams

(*b*) Simply supported beams and

(*c*) Overhanging beams

Cantilever Subject to a Concentrated Load at Free End

Consider the section *X–X* at a distance *x* from free end in a cantilever beam shown in Fig. 9.32(*a*).

From left hand side segment of beam,

$$F = -W$$

Thus shear force is constant i.e., it will not vary with *x*. Hence the *SFD* is as shown in Fig. 9.32(*b*).

$$M = -Wx, \text{ linear variation.}$$

At $\quad x = 0, \quad M_A = 0$

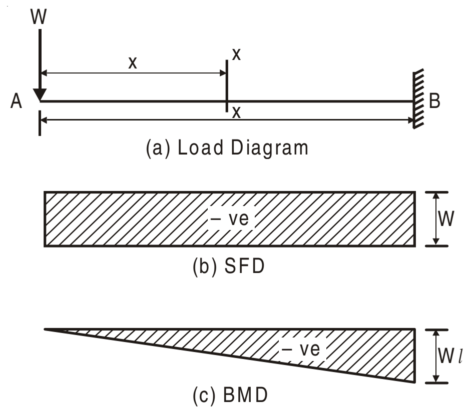

Fig. 9.32

At $\quad x = l, \quad M_B = -Wl$

Hence *BMD* is as shown in Fig. 9.32(*c*).

9.10.2 A Cantilever Subject to UDL Over its Entire Span

Consider the beam shown in Fig. 9.33(*a*).

Considering the left hand side portion of the beam from the section $X–X$ which is at a distance x from the free end A,

$$F = -Wx, \text{ linear variation}$$

At $\quad x = 0, \quad F_A = 0$

At $\quad x = l, \quad F_B = -wl$

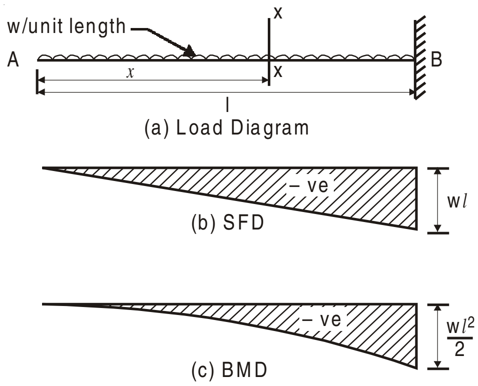

Fig. 9.33

Hence *SFD* is as shown in Fig. 9.33(b).

$$M = -wx\frac{x}{2} = -\frac{wx^2}{2}$$

This is parabolic variation. As magnitude increases at a faster rate with x, it is concave parabola as shown in Fig. 9.33(c), with extreme values as given below:

At $\quad x = 0, M_A = 0,$

At $\quad x = l, \; M_B = -\dfrac{wl^2}{2}$

Simply Supported Beam Subjected to a Concentrated Load

Let W be the concentrated load acting on beam AB at a distance 'a' from the end A as shown in Fig. 9.34(a).

Now $\quad R_A = \dfrac{Wb}{l} \quad \text{and} \quad R_B = \dfrac{Wa}{l}$

Consider the portion AC. At any distance x from A,

$$F = R_A = \frac{Wb}{l}, \text{ constant}$$

$$M = R_A\, x = \frac{Wb}{l}x, \text{ linear variation.}$$

At $\quad x = 0, M_A = 0$

At $\quad x = a, M_C = \dfrac{Wab}{l}$

For portion *AC SFD* and *BMD* can be drawn.

Consider portion *CB*. The expression derived for portion *AC* will not hold good for this portion. Taking a section at a distance *x* from *B* and considering the right hand side segment of the beam,

$$F = -R_B = -\frac{Wa}{l}, \text{ constant}$$

$$M = R_B x = \frac{Wa}{l}x, \text{ linear variation.}$$

At $$x = 0, \quad M_B = 0$$

At $$x = b, \quad M_B = \frac{Wab}{l}$$

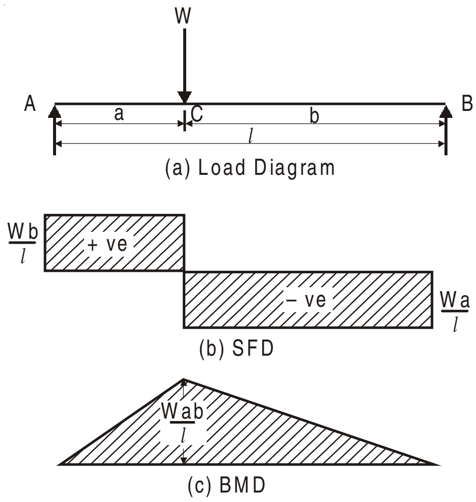

Fig. 9.34

SF and *BM* diagrams for this portion can now be drawn. Hence *SFD* and *BMD* for the beam is as shown in Fig. 9.34(*b*) and 9.34(*c*) respectively.

Particular case:

When $a = b = l/2$

$$F = \frac{Wb}{l} = \frac{W(l/2)}{l} = \frac{W}{2}$$

and moment under the load (centre of span since $a = b = l/2$)

$$M = \frac{Wab}{l} = \frac{W\, l/2 \times l/2}{l} = \frac{Wl}{4}$$

A Simply Supported Beam Subjected to UDL

Let the beam *AB* of span *l* be subjected to uniformly distributed load of intensity w/unit length as shown in Fig. 9.35(*a*).

$$R_A = \frac{wl \cdot l/2}{l} = \frac{wl}{2}$$

$$R_B = \frac{wl}{2}$$

At a section X–X which is at a distance x from A,

$$F = R_A - wx = \frac{wl}{2} - wx, \text{ linear variation}$$

At $\quad x = 0, \quad F_A = \frac{wl}{2}$

At $\quad x = l, \quad F_B = -\frac{wl}{2}$

$\therefore$ *SFD* is as shown in Fig. 9.35(*b*).

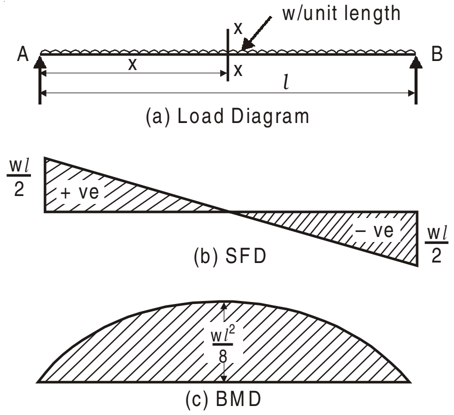

Fig. 9.35

Shear force is zero at x given by

$$0 = \frac{wl}{2} - wx$$

$$x = \frac{l}{2}$$

$\therefore$ Maximum moment occurs at this points.

At section X–X

$$M = R_A x - wx \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2}, \text{ parabolic variation.}$$

As x increases rate of reduction in the value of M is faster. Hence it is convex parabola.

At $\quad x = 0, \quad M_A = 0$

At $\quad x = l, \quad M_B = 0$

Maximum moment occurs at $x = \frac{l}{2}$ where shear force $\left(i.e., \frac{dM}{dx}\right) = 0$

$$M_{max} = \frac{wl}{2} \cdot l/2 - \frac{w(l/2)^2}{2} = \frac{wl^2}{8}$$

Hence *BMD* is as shown in Fig. 9.35(*c*).

Overhanging Beam Subjected to a Concentrated Load at Free End

Consider the overhanging beam *ABC* of span $AB = l$ and overhang $BC = a$, subjected to a concentrated load *W* at free end as shown in Fig. 9.36(*a*).

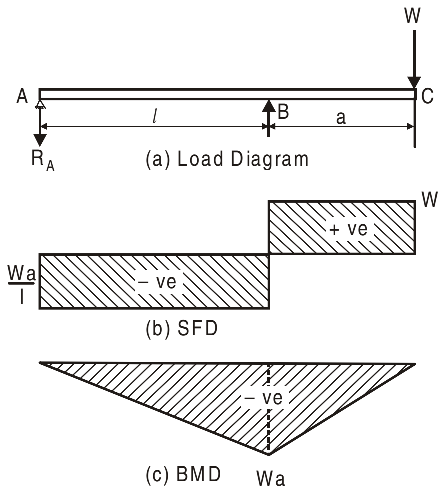

Fig. 9.36

$$R_A = \frac{Wa}{l}, \text{ downward}$$

$$\therefore \quad R_B = W + \frac{Wa}{l} = W\left(l + \frac{a}{l}\right)$$

Portion *AB:*

Measuring *x* from *A* and considering left side of the section.

$$F = -R_A = -\frac{Wa}{l}, \text{ constant}$$

$$M = -R_A x = -\frac{Wax}{l}, \text{ linear variation.}$$

At $\quad x = 0, \quad M_A = 0$

At $\quad x = l, \quad M_B = -Wa$

Portion *BC:*

Measuring *x* from *C*, shear force and bending moments at that section are given by

$$F = W, \text{ constant}$$

$$M = -Wx, \text{ linear variation.}$$

At $\quad x = 0, \quad M_C = 0$

At $\quad x = a, \quad M_B = -Wa.$

SFD and *BMD* for the entire beam is shown in Fig. 9.36(*b*) and 9.36(*c*) respectively.

Example 9.10. *Draw shear force and bending moment diagram for the cantilever beam shown in Fig. 9.37(a).*

Solution: Portion *AB*:

At distance x, from A,

$$F = -20 - 20\,x, \text{ linear variation.}$$

At $x = 0$, $F_A = -20$ kN

At $x = 1$, $F_B = -20 - 20 \times 1 = -40$ kN.

$$M = -20x - 20x \cdot \frac{x}{2}, \text{ parabolic variation}$$

At $x = 0$, $M_A = 0$

At $x = 1$ m, $M_B = -20 - 20 \times 1 \times \frac{1}{2} = -30$ kN-m.

Portion *BC*:

Measuring x from A,

$$F = -20 - 40 - 20x, \text{ linear variation.}$$

At $x = 1$ m, $F_B = -80$ kN

At $x = 3$ m, $F_C = -120$ kN.

$$M = -20x - 40(x - 1) - 20x \cdot \frac{x}{2}, \text{ parabolic variation;}$$

At $x = 1$ m, $M = -30$ kN-m

At $x = 3$ m, $M = -60 - 40 \times 2 - 20 \times 3 \times \frac{3}{2}$

$$= -230 \text{ kN-m}$$

Hence *SFD* and *BMD* are shown in Fig. 9.37(*b*) and 9.37(*c*) respectively.

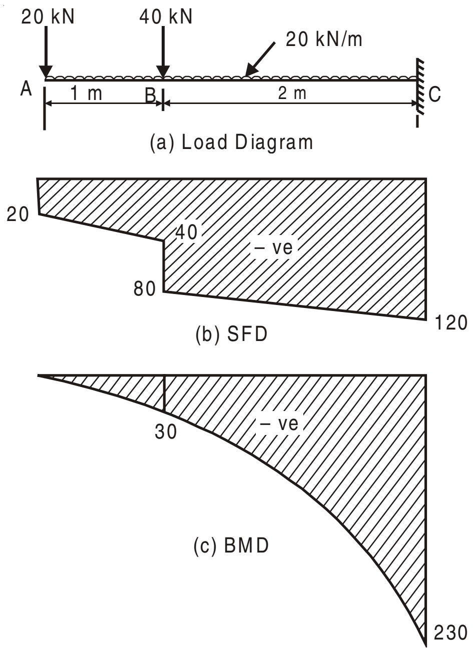

Fig. 9.37

Example 9.11. *Draw the SF and BM diagrams for the beam shown in Fig. 9.38(a) and find out the position and the magnitude of maximum moment.*

Solution: $\Sigma M_A = 0 \rightarrow$

$$R_B \times 10 = 20 \times 5 \times 2.5 + 20 \times 5 + 40 \times 7.5 + 20 \times 8.5$$

$\therefore$ $R_B = 82$ kN.

$\Sigma V = 0 \rightarrow$

$\therefore$ $R_A = 20 \times 5 + 20 + 40 + 20 - 82 = 98$ kN.

Portion *AC:*

Measuring x from A,

$$F = 98 - 20x, \text{ linear variation}$$

At $x = 0$, $F_A = 98$ kN

At $x = 5$ m, $F_B = 98 - 100 = -2$ kN

Points where shear force is zero is given by,

$$0 = 98 - 20x$$

or $x = 4.9$ m

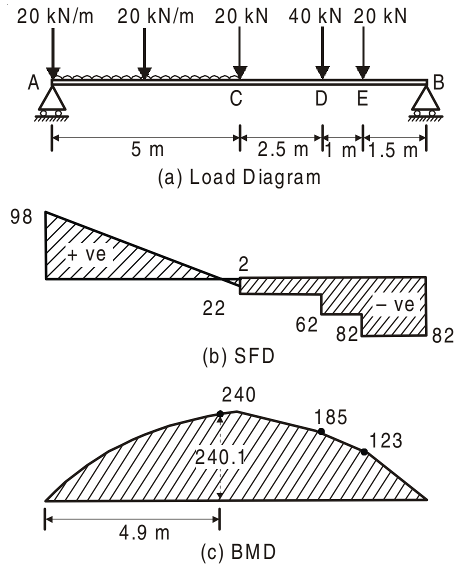

Fig. 9.38

Moment is given by

$$M = 98x - 20x\,\frac{x}{2}, \text{ parabolic variation.}$$

At $x = 0$, $M_A = 0$

At $x = 5$ m, $M_B = 98 \times 5 - \dfrac{20 \times 5^2}{2} = 240$ kN-m

Maximum moment occurs at x = 4.9 m where $F = \frac{dM}{dx} = 0$

$$M_{max} = 98 \times 4.9 - 20 \times \frac{4.9^2}{2} = 240.1 \text{ kN-m}$$

Portion *CD:*

Measuring x from B and considering right hand side segment of the beam,

$$F = -82 + 20 + 40 = -22 \text{ kN, constant}$$

$$M = 82x - 20(x - 1.5) - 40(x - 2.5)$$

$$= 22x + 130, \text{ linear variation.}$$

At $x = 2.5$ m, $M_D = 22 \times 2.5 + 130 = 185$ kN-m

At $x = 5$ m, $M_C = 22 \times 5 + 130 = 240$ kN-m.

Portion *DE:*

Measuring x from B and considering the portion of the beam on the right side of the section,

$$\boldsymbol{F} = -82 + 20 = \mathbf{-62 \text{ kN, constant}}$$

$$M = 82x - 20(x - 1.5) \text{ linear variation}$$

At $x = 1.5$ m,

$$M_E = 82 \times 1.5 = 123 \text{ kN-m.}$$

At $x = 2.5$, $M = 82 \times 2.5 - 20 \times 1 = 185$ kN-m.

Portion *EB:*

Measuring x from B and considering the right side segment,

$$\boldsymbol{F} = \mathbf{-82 \text{ kN, constant}}$$

$$M = 82x, \text{ linear variation.}$$

At $x = 0$, $M_B = 0$

At $x = 1.5$ m, $M_E = 82 \times 1.5 = 123$ kN-m.

SFD and *BMD* are shown in Fig. 9.38(b) and 9.38(c) respectively, for the entire beam.

Example 9.12. *A beam of span 8 m has roller support at A and hinge support at B as shown in Fig. 9.39(a). Draw SF and BM diagrams when the beam is subjected to udl, a concentrated load and an externally applied moment as shown in the Figure.*

Solution: $\sum M_A = 0 \rightarrow$

$$R_B \times 8 - 10 \times 4 \times 2 - 20 \times 4 + 240 = 0$$

$$R_B = -10 \text{ kN} \quad \text{(upwards)}$$

$$= 10 \text{ kN.} \quad \text{(downwards)}$$

$$\sum V = 0 \rightarrow$$

$$R_A = 10 \times 4 + 20 + 10 = 70 \text{ kN.}$$

Portion *AC:*

Measuring x from A and considering left hand side segment of the beam,

$$F = 70 - 10x, \text{ linear variation}$$

At $x = 0$, $F_A = 70$ kN

At $x = 4$ m, $F_C = 70 - 40 = 30$ kN

$$M = 70x - 10x\,\frac{x}{2}, \text{ parabolic variation}$$

At $\quad x = 0, \quad M_A = 0$

At $\quad x = 4$ m, $\quad M_C = 70 \times 4 - 10 \times 4 \times \frac{4}{2} = 200$ kN-m.

Portion *CD:* Measuring x from B,

$$F = 10 \text{ kN, constant}$$

$$M = -10x + 240, \text{ linear variation}$$

$$x = 4 \text{ m}, \quad M_C = 200 \text{ kN-m}$$

At $\quad x = 2$ m, $\quad M_D = -10 \times 2 + 240 = 220$ kN-m.

Portion *DB:*

Measuring x from *B,*

$$F = 10 \text{ kN, constant}$$

$$M = -10x, \quad \text{linear variation}$$

At $\quad x = 0, \quad M_B = 0$

At $\quad x = 2$ m, $\quad M_D = -10 \times 2 = -20$ kN-m.

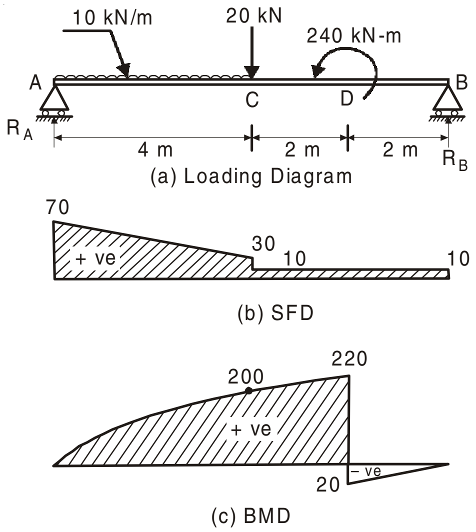

Fig. 9.39

SFD and *BMD* are shown in Fig. 9.39(*b*) and 9.39(*c*).

Note: The bending moment value will be the same at a point, whether calculated by considering left hand side or the right hand side segment of the beam, if there is no external moment acting at that point. If the external moment is acting at a point, there will be sudden change in *BMD* at that point to the extent equal to the magnitude of external bending moment.

Example 9.13. *The overhanging beam ABC is supported at A and B, the span AB being 6 m. The overhang BC is 2 m (see Fig. 3.42 a). It carries a uniformly distributed load of 30 kN/m over a length of 3 m from A and concentrated load of 20 kN at free end. Draw SF and BM diagrams.*

Solution: $\Sigma M_A = 0 \rightarrow$

$$R_B \times 6 = 20 \times 8 + 30 \times 3 \times 1.5$$

$\therefore$ $R_B = 49.167$ kN.

$$\Sigma V = 0 \rightarrow$$

$$R_A = 30 \times 3 + 20 - 49.167 = 60.833 \text{ kN.}$$

Portion *AD:*

Measuring x from A and considering left side segment,

$$F = 60.833 - 30x, \textit{ linear variation}$$

At $x = 0, \quad F_A = 60.833$ kN.

At $x = 3$ m,

$$F_D = 60.833 - 30 \times 3 = -29.167 \text{ kN.}$$

The point of zero shear is given by

$$0 = 60.833 - 30x$$

or, $x = 2.028$ m.

At section X–X, the moment is given by

$$M = 60.833\,x - 30\frac{x^2}{2}, \text{ parabolic variation}$$

At $x = 0, \quad M_A = 0$

At $x = 3$ m,

$$M_D = 60.833 \times 3 - 30 \times \frac{9}{2} = 47.5 \text{ kN-m.}$$

Maximum moment occurs at $x = 2.028$ m. since here $F = \dfrac{dM}{dx} = 0$

$$M_{\max} = 60.833 \times 2.028 - 30 \times \frac{2.028^2}{2} = 61.678 \text{ kN-m.}$$

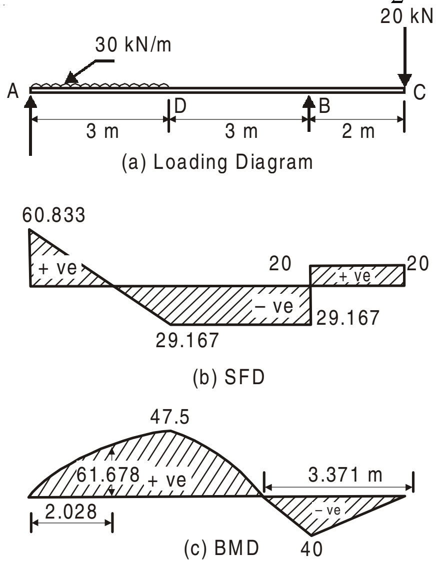

Fig. 9.40

Portion *DB:*

Measuring x from the free end C and considering right hand side segment,

$$F = 20.0 - 49.167$$
$$= 29.167 \text{ kN, constant.}$$
$$M = -20x + 49.167\ (x - 2), \text{ linear variation}$$

At $\quad x = 5$ m, $\quad M_D = -20 \times 5 + 49.167 \times 3 = 47.5$ kN-m.

At $\quad x = 2$ m, $\quad M_B = -20 \times 2 = -40$ kN-m.

In this portion the bending moment changes the sign. The point of contraflexure is given by the expression

$$0 = -20x + 49.167\ (x - 2)$$

i.e., $\quad x = 3.371$ m from free end.

Portion *BC:*

Measuring x from free end,

$$F = 20 \text{ kN, constant}$$
$$M = -20x, \text{ linear variation}$$

At $\quad x = 0,\ M_C = 0$

At $\quad x = 2$ m, $\quad M_B = -40$ kN-m.

Hence *SF* and *BM* diagrams are as shown in Fig. 9.40(*b*) and 9.40(*c*) respectively.

Example 9.14. *Draw BM and SF diagrams for the beam shown in Fig. 9.41(a), indicating the values at all salient points.*

Solution: $\quad \sum M_B = 0 \rightarrow$

$$R_E \times 4 + 20 \times 1 - 30 \times 2 \times 1 - 40 \times 3 - 25 \times 1 \times 4.5 = 0$$

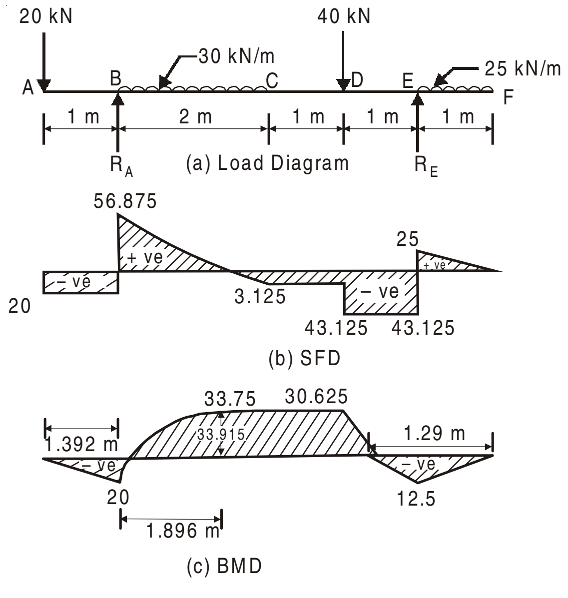

Fig. 9.41

$$R_E = 68.125 \text{ kN.}$$
$$\sum V = 0 \rightarrow$$
$$R_B = 20 + 30 \times 2 + 40 + 25 \times 1 - 68.125$$
$$= 76.875 \text{ kN.}$$

Portion *AB:*

Measuring x from A,

$$F = -20 \text{ kN, constant}$$
$$M = -20x, \text{ linear variation}$$

At $x = 0$, $M_A = 0$

At $x = 1$ m, $M_B = -20$ kN-m

Portion *BC:*

Measuring x from B,

$$F = -20 + 76.875 - 30x, \text{ linear variation}$$

At $x = 0$, $F = 56.875$ kN.

At $x = 2$ m, $F = -3.125$ kN-m.

The point of zero shear force is given by

$$0 = -20 + 76.875 - 30x$$
$$x = 1.896 \text{ m from } B.$$

At distance x from B the moment is given by

$$M = -20(x + 1) + 76.875x - 30x\frac{x}{2}$$
$$= -20 + 56.875x - 15x^2, \text{ parabolic variation}$$

At $x = 0$, $M_B = -20$ kN-m.

At $x = 2$ m, $M_C = -20 + 56.875 \times 2 - 15 \times 4$

$$= 33.75 \text{ kN-m.}$$

Maximum moment occurs where $SF = 0$. *i.e.,* at $x = 1.896$ m.

$\therefore$ $M_{max} = -20 + 56.875 \times 1.896 - 15 \times 1.896^2$

$$= 33.913 \text{ kN-m.}$$

The bending moment is changing its sign in this portion. Hence the point of contraflexure exists in this portion. It is given by

$$0 = -20 + 56.875x - 15x^2$$

$\therefore$ $x = 0.392$ m.

i.e., the point of contraflexure is at 1.392 m from the free end A.

Portion *CD:*

Measuring x from F,

$$\text{Shear force} = 25 \times 1 - 68.125 - 40$$
$$= -3.125 \text{ kN, constant}$$
$$M = -25 \times 1 \times (x - 0.5) + 68.125\,(x - 1) - 40\,(x - 2), \text{ linear variation}$$

At $x = 3$ m, $M_C = 33.75$ kN-m.

At $x = 2$ m, $M_D = -25 \times 1.5 + 68.125$

$$= 30.625 \text{ kN-m.}$$

Portion *DE:*

Measuring x from free end F,

$$\text{Shear force} = 25 - 68.125 = -43.125 \text{ kN, constant}$$

$$M = -25\,(x - 0.5) + 68.125(x - 1), \text{ linear variation}$$

At $x = 2$ m, $M_D = 30.625$ kN-m.

At $x = 1$ m, $M_E = -12.5$ kN-m.

Portion *EF:* $M = 0$, at $x = 1.29$ m

Measuring x from free end F,

$$\text{Shear force } F = 25x, \text{ linear constant}$$

At $x = 0$, $F_F = 0$

At $x = 1$ m, $F_E = 25$ kN

$$M = -25x \cdot \frac{x}{2}, \text{ parabolic variation}$$

At $x = 0$, $M_F = 0$

At $x = 1$ m, $M_E = -12.5$ kN-m.

SF and *BM* diagrams are as shown in Fig. 9.41(*b*) and 9.41(*c*) respectively.