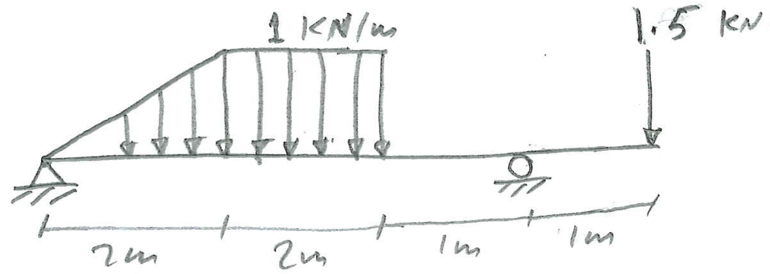
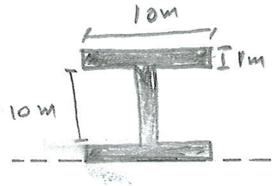
1

Ex: Draw the shear-force and bending-moment diagrams for the loaded beam and find σ_{max} & δ_{max}.

Let: E = 200 GPa



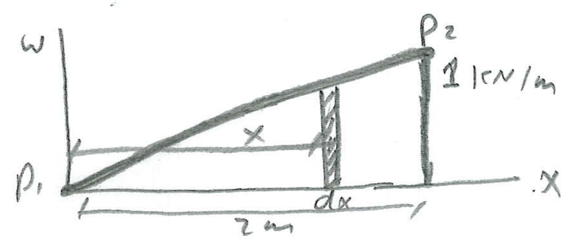


Cross-section area

Sol:- Step one

Determine the equivalent concentrated force and its location for distributed load by using

$f = \int w\,dx \quad , \quad \bar{x} = \dfrac{\int x w\,dx}{\int w\,dx}$



$P_1 = (0,0) \; , \; P_2(2,1)$

$\therefore \dfrac{w_2 - w_1}{x_2 - x_1} = \dfrac{w - w_1}{x - x_1} \Rightarrow \dfrac{1-0}{2-0} = \dfrac{w-0}{x-0}$

$\therefore \dfrac{1}{2} = \dfrac{w}{x} \Rightarrow \boxed{w = \frac{1}{2}x}$ — ①

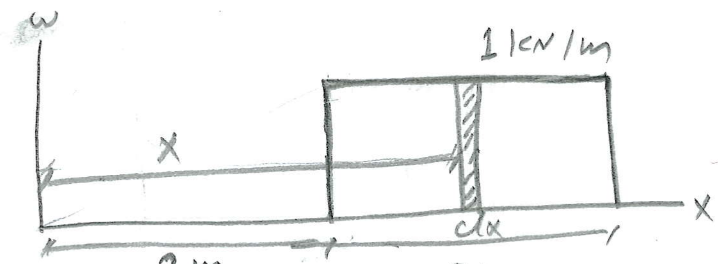
$\therefore f_1 = \int_0^2 \frac{1}{2} x\,dx \Rightarrow f_1 = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$

$\therefore f_1 = \frac{1}{2}\left[\frac{4}{2}\right] \Rightarrow \boxed{f_1 = 1 \text{ KN}}$

$\bar{x} = \dfrac{\int_0^2 x \cdot \frac{1}{2}x\,dx}{1} \Rightarrow \bar{x} = \frac{1}{2}\int_0^2 x^2\,dx$

$\bar{x} = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \Rightarrow \bar{x} = \frac{1}{2}\left[\frac{8}{3}\right]$

$\boxed{\therefore \bar{x} = \frac{4}{3} \text{ m}} \quad \therefore \boxed{\bar{x} = 1.333 \text{ m}}$



$\therefore \boxed{w = 1}$ — ②

$\therefore f_2 = \int_2^4 dx \Rightarrow f_2 = [x]_2^4$

$\therefore \boxed{f_2 = 2 \text{ KN}}$

$\bar{x} = \dfrac{\int_2^4 x \cdot 1 \cdot dx}{2} \Rightarrow \bar{x} = \frac{1}{2}\int_2^4 x\,dx$

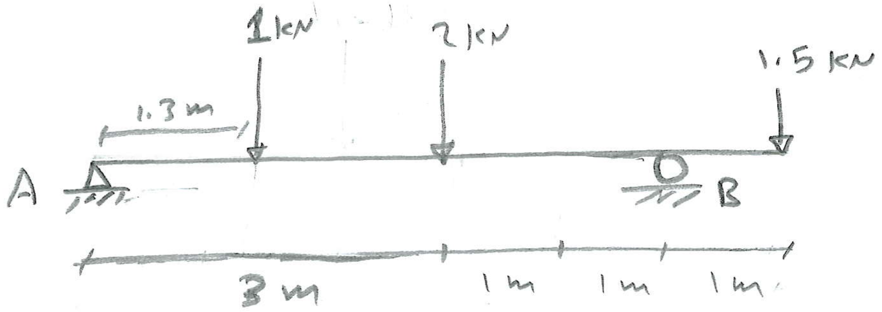
$\bar{x} = \frac{1}{2}\left[\frac{x^2}{2}\right]_2^4 \Rightarrow \bar{x} = \frac{1}{2}\left[\frac{16}{2} - \frac{4}{2}\right]$

$\bar{x} = \frac{1}{2}\left[\frac{12}{2}\right] \Rightarrow \boxed{\bar{x} = 3 \text{ m}}$

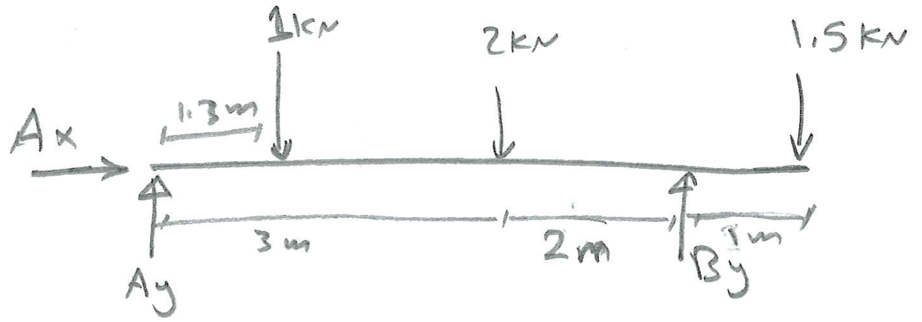
Step two

Determine the reactions of the beam supports

from step one beam became



∴ free body diagram for the beam



$\therefore \sum F_x = 0 \Rightarrow \boxed{A_x = 0}$

$\sum F_y = 0 \Rightarrow A_y - 1 - 2 + B_y - 1.5 = 0 \Rightarrow \boxed{A_y + B_y = 4.5}$ — (3)

$\sum M_A = 0 \Rightarrow -1\times(1.3) - 2(3) + B_y(5) - 1.5(6) = 0$

$\Rightarrow -1.3 - 6 - 9 + 5B_y = 0 \Rightarrow 5B_y = 16.3 \Rightarrow B_y = \frac{16.3}{5} \Rightarrow \boxed{B_y = 3.26\ \text{kN}}$

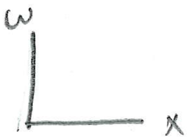
$\therefore$ from (3) $\Rightarrow A_y + 3.26 = 4.5 \Rightarrow \boxed{A_y = 1.24\ \text{kN}}$

Step three

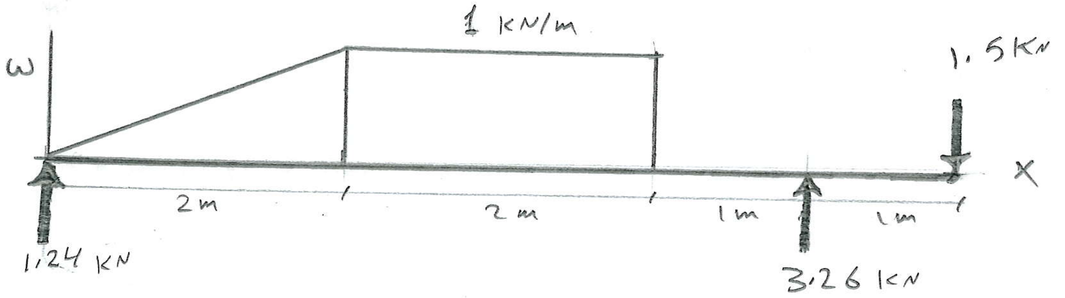
Draw the shear-force diagram (S.F.D)

- From load diagram (L.D) between Load (w) and distance (x), determine the shear-force diagram (S.F.D) between shear force (V) and distance (X) by using:

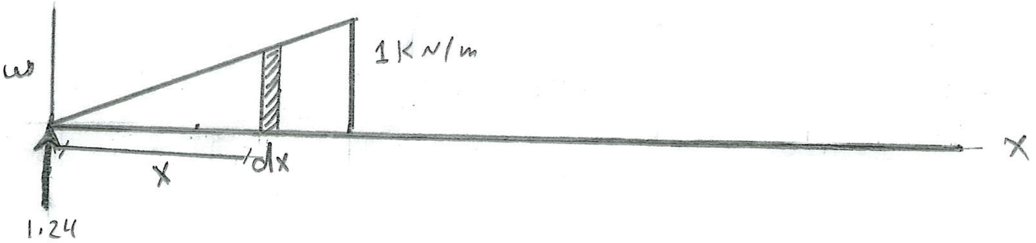
$$\int_{V_0}^{V} dV = -\int_{x_0}^{x} w\,dx$$



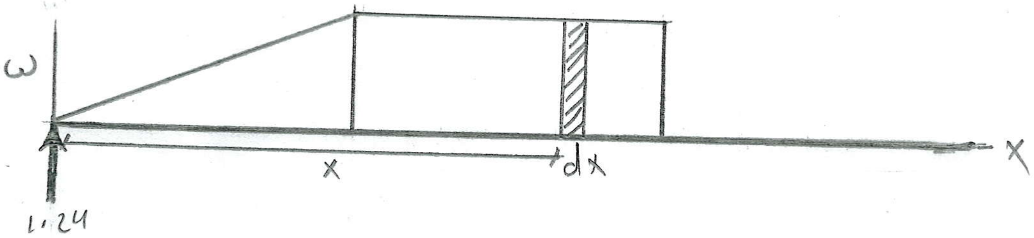
(L.D.)



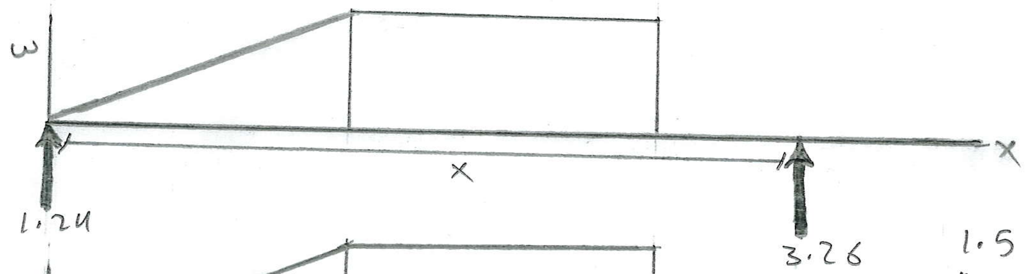
a- (X=0 → X=2)



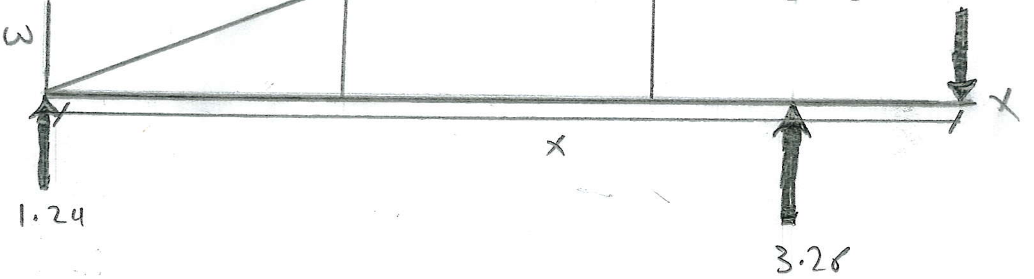
b- (X=2 → X=4)



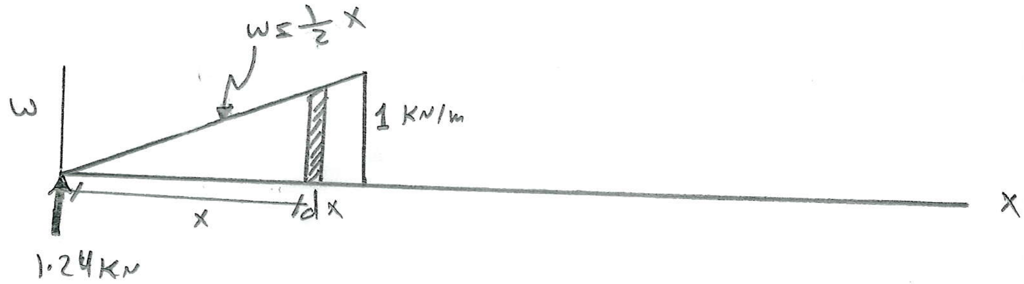
c- (X=4 → X=5)



d- (X=5 → X=6)



a- $x=0 \rightarrow x=2$

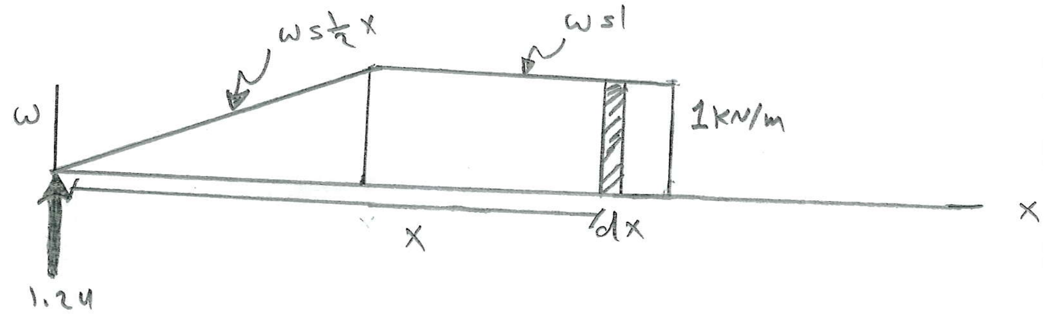


$\because w = \frac{1}{2}x$ from equation (1)

$\therefore \int_{v_0}^{v} dv = -\int_{x_0}^{x} w\,dx \Rightarrow \int_{1.24}^{v} dv = -\int_{0}^{x} \frac{1}{2}x\,dx \Rightarrow [v]_{1.24}^{v} = \frac{-1}{2}\left[\frac{x^2}{2}\right]_0^x$

$\therefore v - 1.24 = \frac{-1}{2}\frac{x^2}{2} \Rightarrow \boxed{v = 1.24 - 0.25x^2}$ (4)

b- $x=2 \rightarrow x=4$



$\therefore \int_{v_0}^{v} dv = -\int w\,dx \Rightarrow \int_{1.24}^{v} dv = -\int_{0}^{2} \frac{1}{2}x\,dx - \int_{2}^{x} 1 \cdot dx$

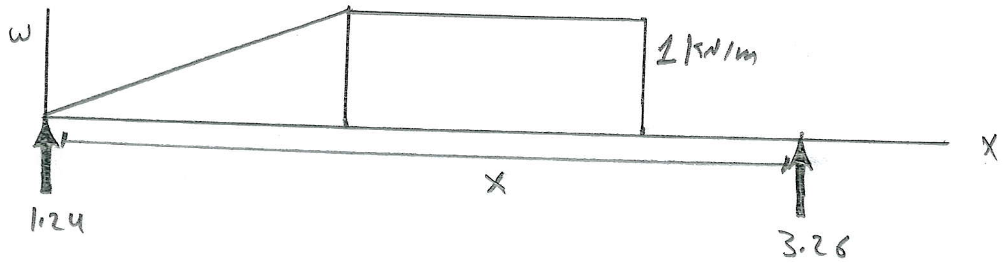
$\therefore v - 1.24 = -\left[\frac{1}{2} \cdot \frac{x^2}{2}\right]_0^2 - [x]_2^x \Rightarrow v - 1.24 = -1 - (x-2)$

$\therefore v - 1.24 = -1 - (x-2) \Rightarrow v - 1.24 = -1 - x + 2$

$v = 1.24 - 1 + 2 - x$

$\therefore \boxed{v = 2.24 - x}$ (5)

c - $x = 4 \rightarrow x = 5$

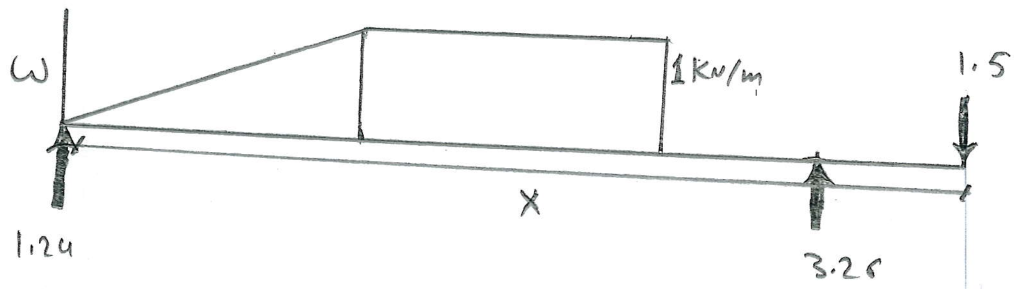


$$\int_{v_0}^{v} dv = -\int_{0}^{2} \frac{1}{2} x\,dx - \int_{2}^{4} dx + 3.26$$

$v - 1.24 = -1 - 2 + 3.26 \Rightarrow v = 1.24 - 1 - 2 + 3.26$

$\therefore \boxed{v = 1.5}$ — 6

d - $x = 5 \rightarrow x = 6$



$$\int_{v_0}^{v} dv = -\int_{0}^{2} \frac{1}{2} x\,dx - \int_{2}^{4} dx + 3.26 - 1.5$$

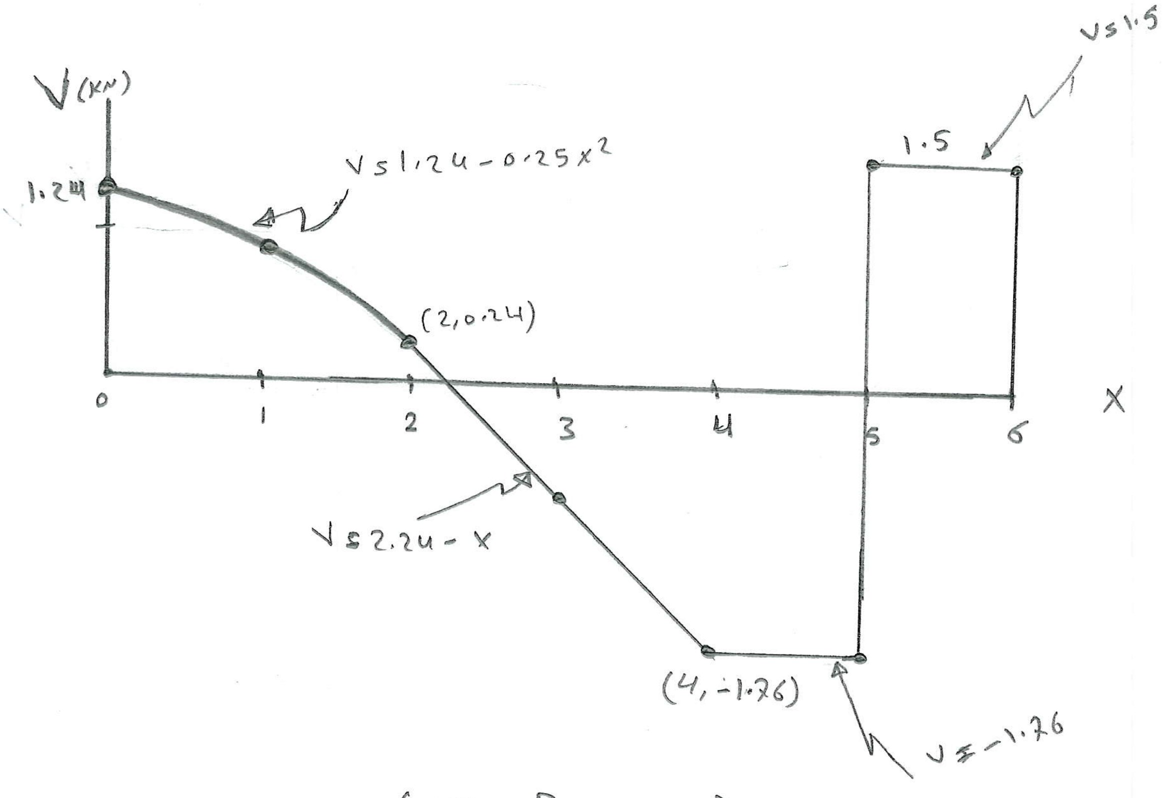
$v - 1.24 = -1 - 2 + 3.26 - 1.5$

$v = 1.24 + 3.26 - 1 - 2 - 1.5$

$\boxed{v = 0}$ — 7

		X	V	
-a- $X=0 \to X=2$	$V=1.24-0.25X^2$	0	1.24	(0, 1.24)
		1	0.99	(1, 0.99)
		2	0.24	(2, 0.24)
-b- $X=2 \to X=4$	$V=2.24-X$	2	0.24	(2, 0.24)
		3	-0.76	(3, -0.76)
		4	-1.76	(4, -1.76)
-c- $X=4 \to X=5$	$V=1.5$	5	1.5	(5, 1.5)
-d- $X=5 \to X=6$	$V=0$	6	0	(6, 0)

∴ shear force diagram (S.F.D)



(S. F. D)

Step four

Draw the bending-moment diagram (B.M.D)

- From shear-force diagram (S.F.D), determine the bending-moment diagram (B.M.D) between bending moment (M) and distance (x) by using:

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} v \, dx$$

$\because M_0 = 0 \Rightarrow M = \int_{x_0}^{x} v\, dx$

a- $x = 0 \rightarrow x = 2$, $v = 1.24 - 0.25x^2$

$\therefore M = \int_0^x (1.24 - 0.25x^2)\, dx \Rightarrow M = \int_0^x 1.24\, dx - \int_0^x 0.25x^2$

$M = 1.24 [x]_0^x - 0.25 \left[\frac{x^3}{3}\right]_0^x \Rightarrow M = 1.24x - \frac{1}{12} x^3$

$\therefore \boxed{M = 1.24x - 0.08333x^3}$ — 8

b- $x = 2 \rightarrow x = 4$

$M = \int_0^2 (1.24x - 0.25x^2)\, dx + \int_2^x (2.24 - x)\, dx$

$M = \int_0^2 1.24x\, dx - \int_0^2 0.25x^2\, dx + \int_2^x 2.24\, dx - \int_2^x x\, dx$

$M = 1.24 \left[\frac{x^2}{2}\right]_0^2 - 0.25 \left[\frac{x^3}{3}\right]_0^2 + 2.24 [x]_2^x - \left[\frac{x^2}{2}\right]_2^x$

$M = 1.24 \left(\frac{4}{2}\right) - 0.25 \left(\frac{8}{3}\right) + 2.24(x-2) - \left(\frac{x^2}{2} - \frac{4}{2}\right)$

$M = 2.48 - 0.666 + 2.24x - 4.48 - \frac{1}{2}x^2 + 2$

$\boxed{M = -0.66 + 2.24x - 0.5x^2}$ — 9 c- $x = 4 \rightarrow x = 5$

$$M = \int_0^2 (1.24 - 0.25x^2)dx + \int_2^4 (2.24 - x)dx + \int_4^x -1.76dx$$

$$M = \int_0^2 1.24\, dx - \int_0^2 0.25x^2 dx + \int_2^4 2.24 dx - \int_2^4 x dx - \int_4^x 1.76 dx$$

$$M = 1.24[x]_0^2 - 0.25\left[\frac{x^3}{3}\right]_0^2 + 2.24[x]_2^4 - \left[\frac{x^2}{2}\right]_2^4 - 1.76[x]_4^x$$

$$M = 1.24(2) - 0.25\left(\frac{8}{3}\right) + 2.24(4-2) - \left(\frac{16}{2} - \frac{4}{2}\right) - 1.76(x-4)$$

$$M = 2.48 - 0.66 + 4.48 - 6 - 1.76x + 7.04$$

$$\boxed{M = 7.34 - 1.76x} \text{ — } (10)$$

d- $x = 5 \rightarrow x = 6$

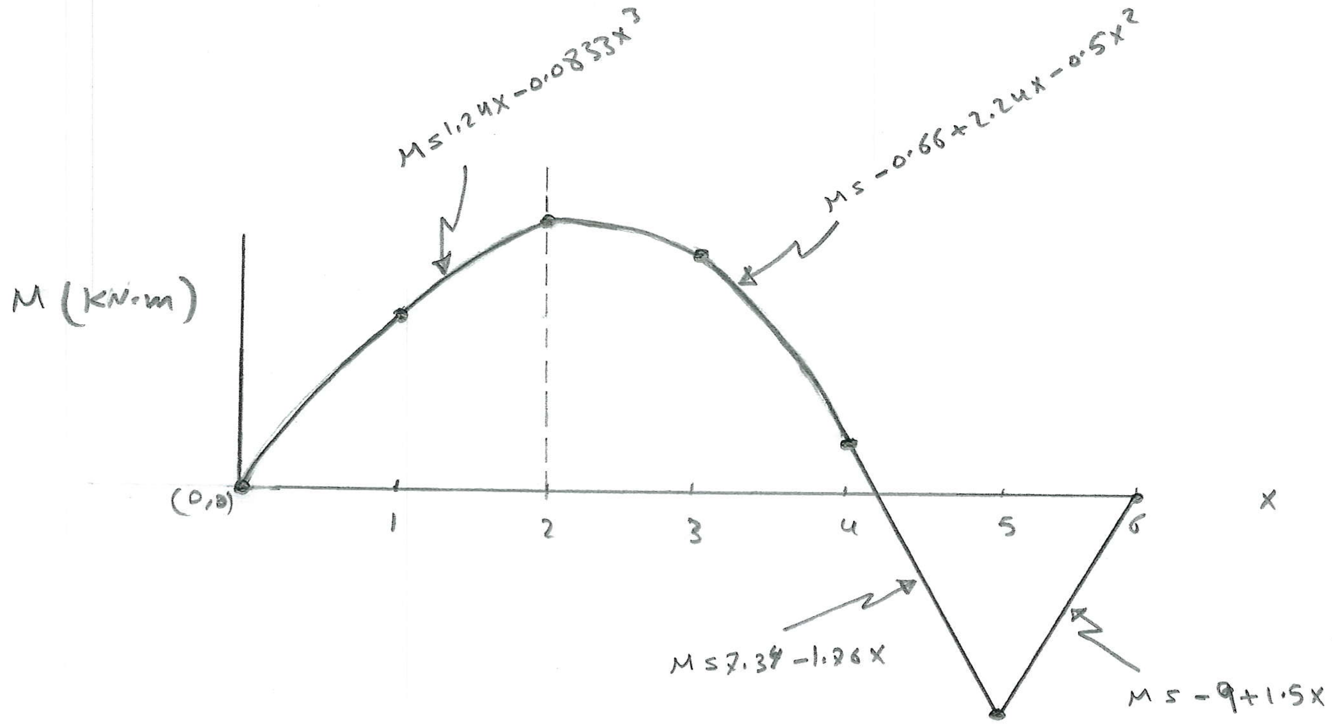
$$M = \int_0^2 (1.24 - 0.25x^2)dx + \int_2^4 (2.24 - x)dx - \int_4^5 1.76dx + \int_5^x 1.5dx$$

$$M = 1.24[x]_0^2 - 0.25\left[\frac{x^3}{3}\right]_0^2 + 2.24[x]_2^4 - \left[\frac{x^2}{2}\right]_2^4 - 1.76[x]_4^5 + 1.5[x]_5^x$$

$$M = 2.48 - 0.66 + 4.48 - 6 - 1.76 + 1.5x - 7.5$$

$$\boxed{M = -9 + 1.5x} \text{ — } (11)$$

		X	M	
-a- $X = 0 \rightarrow X = 2$	$M = 1.24X - 0.0833X^3$	0	0	(0, 0)
		1	1.156	(1, 1.15)
		2	1.813	(2, 1.81)
-b- $X = 2 \rightarrow X = 4$	$M = -0.66 + 2.24X - 0.5X^2$	2	1.813	(2, 1.81)
		3	1.56	(3, 1.56)
		4	0.33	(4, 0.33)
-c- $X = 4 \rightarrow X = 5$	$M = 7.34 - 1.76X$	4	0.33	(4, 0.33)
		5	-1.46	(5, -1.46)
-d- $X = 5 \rightarrow X = 6$	$M = -9 + 1.5X$	5	-1.46	(5, -1.46)
		6	0	(6, 0)



(B. M. D)

The Maximum moment occurs when the shear force curve crosses the zero axis

$\therefore$ from shear force diagram

$$0 = 2.24 - x \Rightarrow \boxed{x = 2.24\,m}$$

and when $x = 5$

But the bending moment at $x = 2.24$ is

$$M = -0.66 + 2.24(2.24) - 0.5(2.24)^2$$

$$\therefore \boxed{M = 1.84 \text{ KN.m}}$$

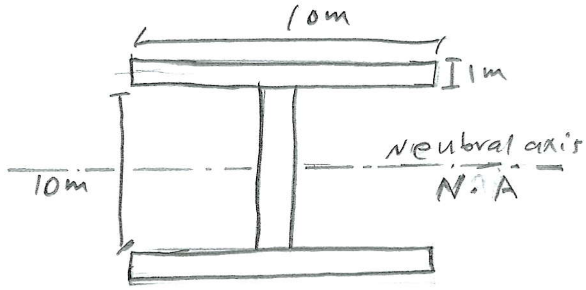
and the bending moment at $x = 5$ is

$$\boxed{M = -1.5}$$

$\therefore$ The maximum bending moment occurs at $x = 2.24m$

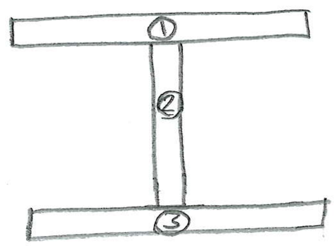
$$\therefore \boxed{M_{max.} = 1.84 \text{ KN.m}}$$

The neutral axis located at the centroid of cross-section area



∴ The centroid of the cross-section area from the base is

$$\bar{y} = \frac{\sum A y_c}{\sum A}$$



∴ $A_1 = 10 * 1 \Rightarrow \boxed{A_1 = 10\ m^2}$

$x_{1c} = 1 + 10 + 0.5 \Rightarrow \boxed{x_{1c} = 11.5\ m}$

$A_2 = 10 * 1 \Rightarrow \boxed{A_2 = 10 m^2}$

$x_{2c} = 1 + 5 \Rightarrow \boxed{x_{2c} = 6\ m}$

$A_3 = 10 * 1 \Rightarrow \boxed{A_3 = 10\ m^2}$

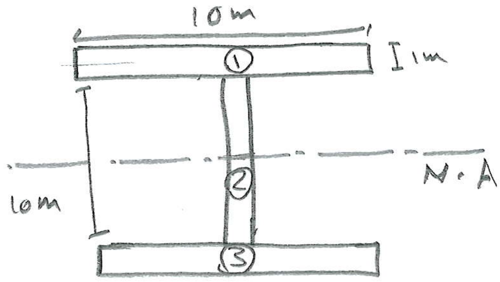
$\boxed{x_{3c} = 0.5\ m}$

$$\therefore \bar{y} = \frac{(10 * 11.5) + (10 * 6) + (10 * 0.5)}{10 + 10 + 10} = \frac{180}{30}$$

$\boxed{\bar{y} = 6\ m}$

The moment of inertia for the cross-section area about the neutral axis is

The moment of inertia for rectangle is $\frac{bh^3}{12}$ about base



Thus: The moment of inertia of the cross-section area about the neutral axis is

$$I_{x(N.A)} = I_{x(base)} + A \cdot d^2$$

$\therefore I_1 x = \frac{10(1)^3}{12} \Rightarrow \boxed{I_1 x = \frac{10}{12} \ m^4}$

$A_1 = 10 * 1 \Rightarrow \boxed{A_1 = 10 \ m^2}$

$\boxed{d_1 = 5.5 \ m}$

$I_2 x = \frac{1(10)^3}{12} \Rightarrow \boxed{I_2 x = \frac{1000}{12} \ m^4}$

$A_2 = 10 * 1 \Rightarrow \boxed{A_2 = 10 \ m^2}$

$\boxed{d_2 = 0 \ m}$

$I_3 x = \frac{10(1)^3}{12} \Rightarrow \boxed{I_3 x = \frac{10}{12} \ m^4}$

$A_3 = 10 * 1 \Rightarrow \boxed{A_3 = 10 m^2}$

$\boxed{d_3 = 5.5 \ m}$

$$\therefore I_{x(N.A)} = \left[\frac{10}{12} + 10 * (5.5)^2\right] + \left[\frac{1000}{12} + 10 * (0)^2\right] + \left[\frac{10}{12} + 10 * (5.5)^2\right]$$

$$= \left(\frac{10}{12} + 55\right) + \left(\frac{1000}{12}\right) + \left(\frac{10}{12} + 55\right)$$

$$\boxed{I_{x(N.A)} = 195 \ m^4}$$

The maximum bending stress is

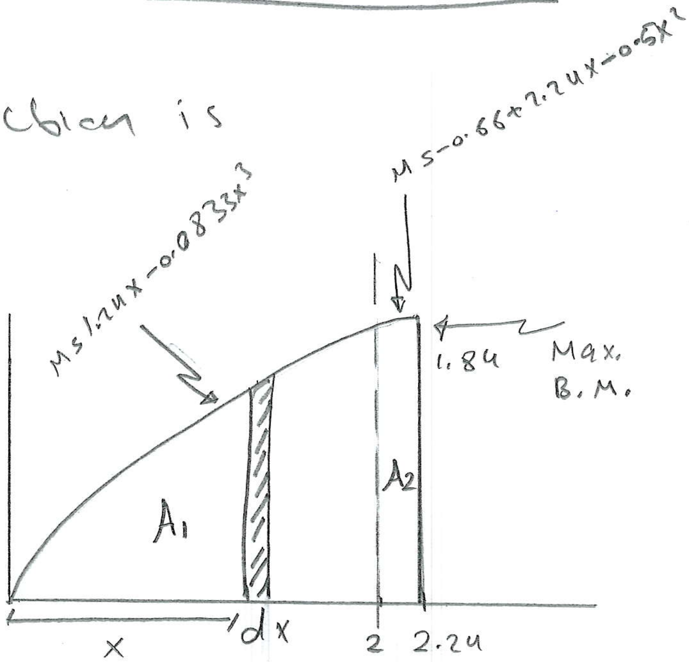
$$\sigma_{max.} = \frac{M_{max.} \cdot y}{I}$$

$$\therefore \sigma_{max.} = \frac{1.84 * 6}{195} \Rightarrow \sigma_{max.} = 0.05661 \ kN/m^2$$

$$\therefore \sigma_{max.} = 56.61 \ \frac{N}{m^2} \Rightarrow \boxed{\sigma_{max.} = 56.61 \ Pa} \ \text{Ans.}$$

The maximum bending deflection is

$$\delta_{max.} = \frac{A \bar{x}}{E I}$$



(B.M.D)

$$A_1 = \int_0^2 (1.24x - 0.0833x^3)\,dx$$

$$= \int_0^2 1.24x\,dx - \int_0^2 0.0833x^3\,dx$$

$$A_1 = 1.24\left[\frac{x^2}{2}\right]_0^2 - 0.0833\left[\frac{x^4}{4}\right]_0^2$$

$$A_1 = 1.24 * 2 - 0.0833 * 4$$

$$A_1 = 2.48 - 0.3532$$

$$\boxed{A_1 = 2.126 \ m^2}$$

$$A_2 = \int_2^{2.24} (-0.66 + 2.24X - 0.5X^2)\,dX \Rightarrow A_2 = \int_2^{2.24} -0.66\,dx + \int_2^{2.24} 2.24x\,dx - \int_2^{2.24} 0.5x^2\,dx$$

$$A_2 = -0.66\,[x]_2^{2.24} + 2.24\left[\frac{x^2}{2}\right]_2^{2.24} - 0.5\left[\frac{x^3}{3}\right]_2^{2.24}$$

$$A_2 = -0.158 + 1.39 - 0.54$$

$$\therefore \boxed{A_2 = 0.69 \ m^2}$$

$$\bar{x} = \frac{\int x\,dA}{\int dA}$$

$$\int x\,dA_1 = \int_0^2 x\,(1.24x - 0.0833x^3)\,dx$$

$$= \int_0^2 (1.24x^2 - 0.0833x^4)\,dx$$

$$= \int_0^2 1.24x^2\,dx - \int_0^2 0.0833x^4\,dx$$

$$= 1.24\left[\frac{x^3}{3}\right]_0^2 - 0.0833\left[\frac{x^5}{5}\right]_0^2$$

$$= 1.24\left(\frac{2^3}{3}\right) - 0.0833\left(\frac{2^5}{5}\right)$$

$$= 3.3 - 0.533$$

$$\boxed{\therefore \int x\,dA_1 = 2.76\ \text{m.m}^2}$$

$$\int x\,dA_2 = \int_2^{2.24} x\,(-0.66 + 2.24x - 0.5x^2)\,dx$$

$$= \int_2^{2.24} -0.66x\,dx + \int_2^{2.24} 2.24x^2\,dx - \int_2^{2.24} 0.5x^3\,dx$$

$$= -0.66\left[\frac{x^2}{2}\right]_2^{2.24} + 2.24\left[\frac{x^3}{3}\right]_2^{2.24} - 0.5\left[\frac{x^4}{4}\right]_2^{2.24}$$

$$= -0.335 + 2.41 - 1.14$$

$$\boxed{\int x\,dA_2 = 0.92\ \text{m.m}^2}$$

$$\therefore \bar{x} = \frac{2.76 + 0.92}{2.126 + 0.69}$$

$$\therefore \boxed{\bar{x} = 1.3\ \text{m}}$$

$$\therefore \delta_{max.} = \frac{(2.126 + 0.69) * 1.3}{200 * 195} = 0.0000938\ \text{m}$$

$$\therefore \boxed{\delta_{max} = 0.093\ \text{mm}}$$