Limits

If $\lim\limits_{x \to a} f(x) = L$

(1.) L قيمة معرفة

رقم $, \infty + \infty = \infty$, $\infty +$ رقم $= \infty$, $\frac{\text{رقم}}{\infty} = 0$

$\frac{\text{رقم}}{0} = \infty$

(2.) L قيمة غير معرفة

$\frac{\infty}{\infty}$, $\infty - \infty$, $\frac{0}{0}$, 1^{∞}

(a) $x^2 - a^2 = (x-a)(x+a)$ فرق بين مربعين

(b) $x^2 + bx = x(x+b)$ سحب عامل مشترك

(c) $ax^2 + bx + c = 0$ [تجربة / دستور $x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

(d.) $x^3 - a^3 = (x-a)(x^2 + a.x + a^2)$ فرق بين مكعبين

(e) $\sqrt{\ }$, $1 - \cos x$ نضرب بالمرافق

$\sqrt[4]{\ }$ نضرب بالمرافق مرتين

Ex: Find

① $\lim_{x\to 5} (x^2-4x+3)$

$= 5^2 - 4*5 + 3 = 8$

② $\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = \frac{0}{2} = 0$

③ $\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$

$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 1+1 = 2$

④ $\lim_{x\to 5} \frac{x^2-25}{x^2-7x+10} = \frac{5^2-25}{5^2-7*5+10} = \frac{0}{0}$

$= \lim_{x\to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x-5)} = \frac{5+5}{5-5} = \frac{10}{0} = \infty$

⑤ $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-5x+6} = \frac{0}{0}$

$\lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{(x-3)\cancel{(x-2)}} = \frac{2-1}{2-3} = -1$

Ex: ⑥ $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2} = \dfrac{0}{0}$

$= \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}} = 2^2 + 2 \times 2 + 4 = 12$

Ex: ⑦ $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x^2 - 1} = \dfrac{0}{0}$

$= \lim\limits_{x \to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x-1)(x+1)(\sqrt{x}+1)}$

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$= \lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)(\sqrt{x}+1)} = \dfrac{1}{(1+1)(\sqrt{1}+1)} = \dfrac{1}{4}$

Ex: ⑧ $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4} = \dfrac{0}{0}$

$= \lim\limits_{x \to 4} \dfrac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x-4)(\sqrt{x}+2)}$

$= \lim\limits_{x \to 4} \dfrac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)} = \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{4}$

Ex:- 9 $\lim\limits_{x \to 2} \dfrac{4-x^2}{3-\sqrt{x^2+5}}$

$$= \lim_{x \to 2} \frac{(2-x)(2+x)\,(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})}$$

$$= \lim_{x \to 2} \frac{(2-x)(2+x)(3+\sqrt{x^2+5})}{(9-(x^2+5))}$$

$$= \lim_{x \to 2} \frac{\cancel{(2-x)}\cancel{(2+x)}\,(3+\sqrt{x^2+5})}{\cancel{(2-x)}\cancel{(2+x)}} = 3+\sqrt{2^2+5} = 6$$

Ex:- 10 $\lim\limits_{x \to 1} \dfrac{x^2-x}{\sqrt{x+1}-\sqrt{2x}}$

$$= \lim_{x \to 1} \frac{x(x-1)\,(\sqrt{x+1}+\sqrt{2x})}{(\sqrt{x+1}-\sqrt{2x})(\sqrt{x+1}+\sqrt{2x})}$$

$$= \lim_{x \to 1} \frac{x(x-1)(\sqrt{x+1}+\sqrt{2x})}{(x+1-2x)}$$

$$= \lim_{x \to 1} \frac{x\overset{-1}{\cancel{(x-1)}}(\sqrt{x+1}+\sqrt{2x})}{\cancel{(1-x)}}$$

$$= 1 * -1 * (\sqrt{1+1} + \sqrt{2*1})$$

$$= -2\sqrt{2}$$

Ex: (11) $\lim\limits_{x \to 16} \dfrac{16 - x}{\sqrt[4]{x} - 2} = \dfrac{0}{0}$

$= \lim\limits_{x \to 16} \dfrac{(16 - x)(\sqrt[4]{x} + 2)}{(\sqrt[4]{x} - 2)(\sqrt[4]{x} + 2)}$

$= \lim\limits_{x \to 16} \dfrac{(16 - x)(\sqrt[4]{x} + 2)}{(\sqrt{x} - 4)}$

$= \lim\limits_{x \to 16} \dfrac{(16 - x)(\sqrt[4]{x} + 2)(\sqrt{x} + 4)}{(\sqrt{x} - 4)(\sqrt{x} + 4)}$

$= \lim\limits_{x \to 16} \dfrac{\overset{-1}{\cancel{(16 - x)}}(\sqrt[4]{x} + 2)(\sqrt{x} + 4)}{\cancel{(x - 16)}}$

$= -1 \, (\sqrt[4]{16} + 2)(\sqrt{16} + 4)$

$= -(4 * 8) = -32$

Ex: (12) $\lim\limits_{x \to 2} \dfrac{\sqrt[3]{x} - \sqrt[3]{2}}{x - 2} = \dfrac{0}{0}$

$= \lim\limits_{x \to 2} \dfrac{\cancel{(\sqrt[3]{x} - \sqrt[3]{2})}}{\cancel{(\sqrt[3]{x} - \sqrt[3]{2})}\left((\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{2} + (\sqrt[3]{2})^2\right)}$

$= \dfrac{1}{(\sqrt[3]{2})^2 + (\sqrt[3]{2})^2 + (\sqrt[3]{2})^2}$

$= \dfrac{1}{3(\sqrt[3]{2})^2}$

Ex: $\boxed{13}$ $\lim\limits_{x \to 3} \dfrac{x^5 - 243}{x - 3} = \dfrac{0}{0}$

$$
\begin{array}{r|l}
 & x^4 + 3x^3 + 9x^2 + 27x + 81 \\ \hline
x - 3 & x^5 - 243 \\
 & \mp x^5 \pm 3x^4 \\ \hline
 & 3x^4 - 243 \\
 & \mp 3x^4 \pm 9x^3 \\ \hline
 & 9x^3 - 243 \\
 & \mp 9x^3 \pm 27x^2 \\ \hline
 & 27x^2 - 243 \\
 & \mp 27x^2 \pm 81x \\ \hline
 & 81x - 243 \\
 & \mp 81x \pm 243 \\ \hline
 & 0 \quad 0
\end{array}
$$

$$= \lim_{x \to 3} \frac{(x^4 + 3x^3 + 9x^2 + 27x + 81)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 3^4 + 3 * 3^3 + 9 * 3^2 + 27 * 3 + 81$$

$$= 405$$

Ex: (14) $\lim\limits_{x \to 1} \dfrac{x^3 + x^2 - 5x + 3}{x^3 - 3x + 2}$

$$
\begin{array}{r|l}
 & x^2 + 2x - 3 \\ \hline
x - 1 & \cancel{x^3} + x^2 - 5x + 3 \\
 & \mp \cancel{x^3} \pm x^2 \\ \hline
 & 2x^2 - 5x + 3 \\
 & \mp 2x^2 \pm 2x \\ \hline
 & -3x + 3 \\
 & \pm 3x \mp 3 \\ \hline
 & 0 \quad 0
\end{array}
\qquad
\begin{array}{r|l}
 & x^2 + x - 2 \\ \hline
x - 1 & \cancel{x^3} - 3x + 2 \\
 & \mp \cancel{x^3} \pm x^2 \\ \hline
 & x^2 - 3x + 2 \\
 & \mp x^2 \pm x \\ \hline
 & -2x + 2 \\
 & \pm 2x \mp 2 \\ \hline
 & 0 \quad 0
\end{array}
$$

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + 2x - 3)}{\cancel{(x-1)}(x^2 + x - 2)} = \lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{(x+2)\cancel{(x-1)}}$$

$$= \frac{1+3}{1+2} = \frac{4}{3}$$

Infinity as a Limit

Ex:- ① $\lim_{x \to \infty} \frac{8x^2 + x}{2x^2 + 1} = \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{8x^2}{x^2} + \frac{x}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{8 + \cancel{\frac{1}{x}}^{0}}{2 + \cancel{\frac{1}{x^2}}^{0}} = \frac{8}{2} = 4$

Ex:- ② $\lim_{x \to \infty} \frac{x^4 + x}{x^2 + 1} = \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{x^4}{x^4} + \frac{x}{x^4}}{\frac{x^2}{x^4} + \frac{1}{x^4}}$

$= \lim_{x \to \infty} \frac{1 + \cancel{\frac{1}{x^3}}^{0}}{\cancel{\frac{1}{x^2}}^{0} + \cancel{\frac{1}{x^4}}^{0}}$

$= \frac{1}{0} = \infty$

Ex:- 3

$$\lim_{x \to \infty} \frac{x^2 + 2x}{x^3 + 6} = \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^3} + \frac{2x}{x^3}}{\frac{x^3}{x^3} + \frac{6}{x^3}} = \lim_{x \to \infty} \frac{\cancel{\frac{1}{x}}^{0} + \cancel{\frac{2}{x^2}}^{0}}{1 + \cancel{\frac{6}{x^3}}^{0}}$$

$$= \frac{0}{1} = 0$$

Ex:- (4) $\lim_{x \to \infty} \sqrt{x^2 + x} - x = \infty - \infty$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{(\sqrt{x^2 + x} + x)}$$

$$= \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x}$$

$$= \lim_{x \to \infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}} + \frac{x}{x}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1 + \cancel{\frac{1}{x}}_{0}} + 1} = \frac{1}{2}$$

(1) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

(2) $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

(3.) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x} = 0$

Ex:- (1) $\lim\limits_{x \to 0} \frac{\sin 2x}{x} = \frac{0}{0}$

$\lim\limits_{x \to 0} \cancelto{1}{\frac{\sin 2x}{2x}} * 2 = 2$

(2) $\lim\limits_{x \to 0} \frac{\sin^2 3x}{x^2} = \frac{0}{0}$

$\lim\limits_{x \to 0} \cancelto{1}{\frac{\sin 3x}{3x}} \cdot \cancelto{1}{\frac{\sin 3x}{3x}} * 3 * 3 = 9$

(3) $\lim\limits_{x \to 0} \frac{\tan 8x}{\sin 2x} = \frac{0}{0}$

$= \lim\limits_{x \to 0} \cancelto{1}{\frac{\tan 8x}{8x}} \cdot 8x \cdot \cancelto{1}{\frac{2x}{\sin 2x}} \cdot \frac{1}{2x}$

$= \frac{8}{2} = 4$

Ex:- (4) $\lim_{x \to 3} \frac{\sin(x^2-9)}{(x-3)} = \frac{0}{0}$

$= \lim_{x \to 3} \frac{\sin[(x-3)\cancel{(x+3)}]}{\cancel{[(x-3)(x+3)]}} * (x+3)$ 1

$= 3+3 = 6$

(5.) $\lim_{x \to 2} \frac{\sin(x-2)}{x^2-4} = \frac{0}{0}$

$\lim_{x \to 2} \frac{\cancel{\sin(x-2)}}{\cancel{(x-2)}(x+2)} = \frac{1}{2+2} = \frac{1}{4}$ 1

(6.) $\lim_{x \to 0} x . \tan(\frac{1}{x})$

$= \lim_{x \to 0} x . \frac{\cancel{\tan(\frac{1}{x})}}{\cancel{\frac{1}{x}}} . \frac{1}{x} = 1$ 1

(7.) $\lim_{x \to \infty} x . \tan(\frac{1}{x})$

let $y = \frac{1}{x} \to y \to 0$

$x = \frac{1}{y}$

$\lim_{y \to 0} \frac{1}{y} . \tan y = \lim_{y \to 0} \frac{\tan y}{y} = 1$

(8.) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2} = \dfrac{0}{0}$

$= \lim\limits_{x \to 0} \dfrac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$= \lim\limits_{x \to 0} \dfrac{1-\cos^2 x}{x^2 \cdot (1+\cos x)} = \lim\limits_{x \to 0} \dfrac{\overset{1}{\cancel{\sin^2 x}}}{\cancel{x^2} \cdot (1+\cos x)}$

$= \dfrac{1}{1+\cos 0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

(9.) $\lim\limits_{x \to 0} \dfrac{x^2}{1-\sqrt{\cos x}} = \dfrac{0}{0}$

$= \lim\limits_{x \to 0} \dfrac{x^2 \cdot (1+\sqrt{\cos x})}{(1-\sqrt{\cos x})(1+\sqrt{\cos x})}$

$= \lim\limits_{x \to 0} \dfrac{x^2 \cdot (1+\sqrt{\cos x})}{(1-\cos x)} * \dfrac{1+\cos x}{1+\cos x}$

$= \lim\limits_{x \to 0} \dfrac{x^2(1+\sqrt{\cos x})(1+\cos x)}{1-\cos^2 x}$

$= \lim\limits_{x \to 0} \dfrac{\overset{1}{\cancel{x^2}}(1+\sqrt{\cos x})(1+\cos x)}{\cancel{\sin^2 x}}$

$= (1+\sqrt{\cos 0})(1+\overset{1}{\cancel{\cos 0}}) = 2*2 = 4$

Ex:- (10) $\lim_{x \to \pi} \frac{\pi - x}{\sin x} = \frac{0}{0}$

let $y = \pi - x \Rightarrow y \to 0$

$x = \pi - y$

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$\sin(\pi - y) = \sin y$

$= \lim_{y \to 0} \frac{y}{\sin(\pi - y)}$

$= \lim_{y \to 0} \frac{y}{\sin y} = 1$

Ex:- (11) $\lim_{x \to \infty} \frac{2x + x.\sin x}{5x^2 - 2x + 1} = \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{2x}{x^2} + \frac{x.\sin x}{x^2}}{\frac{5x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{\cancel{\frac{2}{x}}^{0} + \cancel{\frac{\sin x}{x}}^{0}}{5 - \cancel{\frac{2}{x}}^{0} + \cancel{\frac{1}{x^2}}^{0}} = \frac{0}{5} = 0$