Functions and Graphs

Chapter Outline

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

For example, consider the function f, where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$. Since every nonnegative real number has a real-value square root,

For a general function f with domain D, we often use x to denote the input and y to denote the output associated with x. When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x. Using function notation, we write $y = f(x)$, and we read this equation as "y equals f of x." For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using Figure 1.2, Figure 1.3, and Figure 1.4.

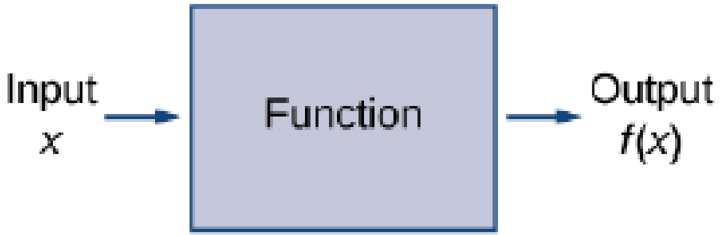

Figure 1.2 A function can be visualized as an input/output device.

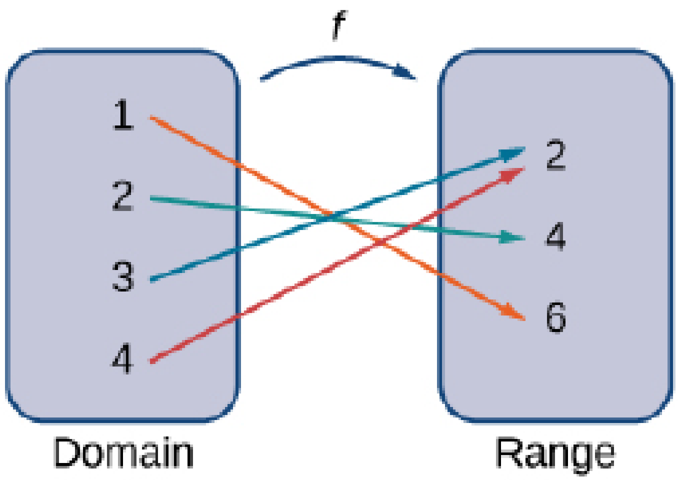

Figure 1.3 A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

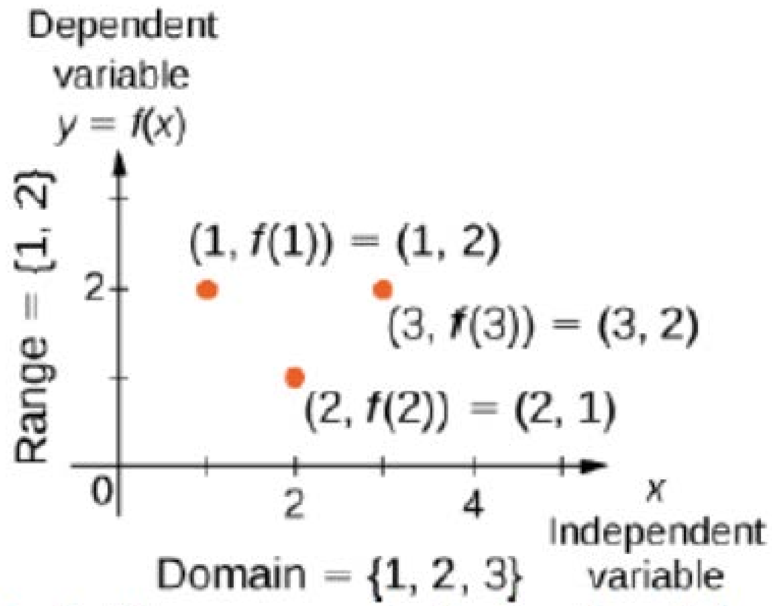

Figure 1.4 In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The **graph of a function** is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In **Figure 1.5**, we plot a graph of this function.

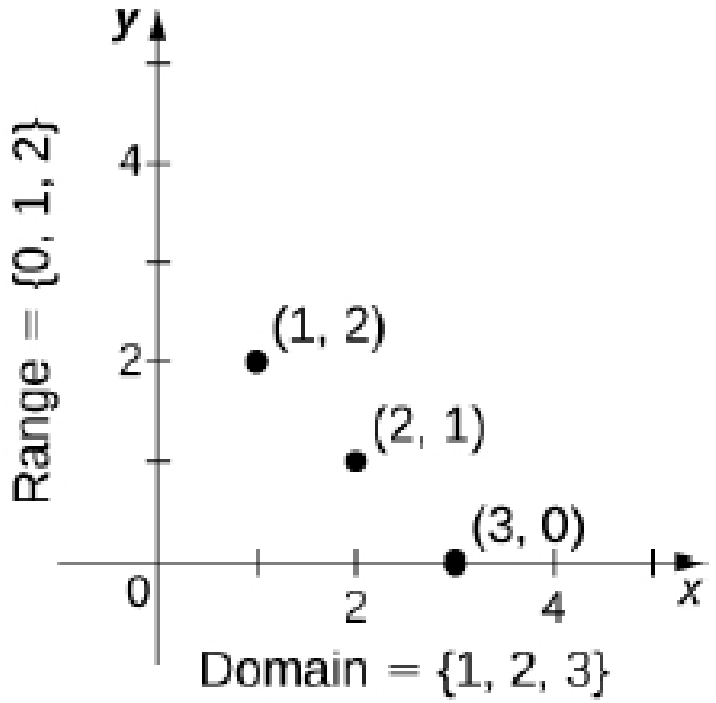

Figure 1.5 Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$,

the domain of $f(x) = x^2$ set of all real numbers.

the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers.

Example 1.1

Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

a. $f(-2)$

b. $f(\sqrt{2})$

c. $f(a+h)$

Solution

Substitute the given value for x in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$

b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$

c.
$$\begin{aligned} f(a+h) = 3(a+h)^2 + 2(a+h) - 1 &= 3\left(a^2 + 2ah + h^2\right) + 2a + 2h - 1 \\ &= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1 \end{aligned}$$

Example 1.2

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x - 4)^2 + 5$

b. $f(x) = \sqrt{3x + 2} - 1$

c. $f(x) = \dfrac{3}{x - 2}$

a. Consider $f(x) = (x - 4)^2 + 5$.

i. Since $f(x) = (x - 4)^2 + 5$ is a real number for any real number x, the domain of f is the interval $(-\infty, \infty)$.

ii. Since $(x - 4)^2 \geq 0$, we know $f(x) = (x - 4)^2 + 5 \geq 5$.

$$y \geq 5$$

b. Consider $f(x) = \sqrt{3x+2} - 1$.

i. To find the domain of f, we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x | x \geq -2/3\}$.

ii. To find the range of f,

$$\sqrt{3x+2} \geq 0,\ f(x) = \sqrt{3x+2} - 1 \geq -1.$$

the range of f is $\{y | y \geq -1\}$.

c. Consider $f(x) = 3/(x - 2)$.

i. Since $3/(x - 2)$ is defined when the denominator is nonzero, the domain is $\{x | x \neq 2\}$.

ii. To find the range of f,

$$\frac{3}{x-2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$.

Thus, the range is $\{y | y \neq 0\}$.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

Hours after Midnight	Temperature (°F)	Hours after Midnight	Temperature (°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

Table 1.1 Temperature as a Function of Time of Day

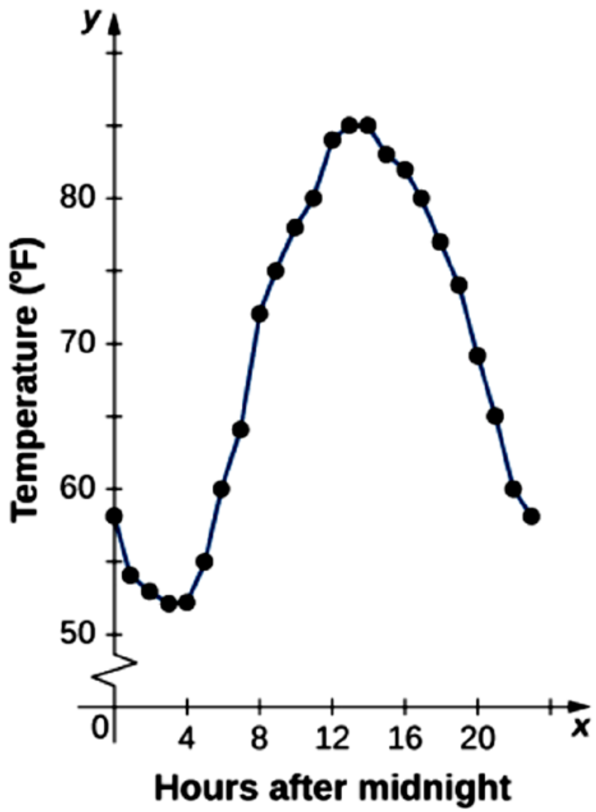

Figure 1.7 Connecting the dots in Figure 1.6 shows the general pattern of the data.

the area of a circle of radius r is given by the formula $A(r) = \pi r^2$.