CHAPTER THREE

Methods of Analysis

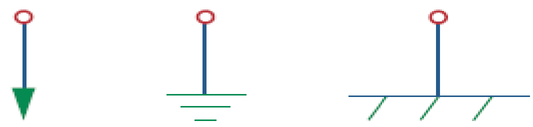
3.1 Nodal Analysis

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. To simplify matters, we shall assume in this section that circuits do not contain voltage sources.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining $n-1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each $n-1$ of the nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages

The reference node should has the on of these samples shown in figure below



(a) (b) (c)

Figure 3.1: type to represent the ground of the circuit

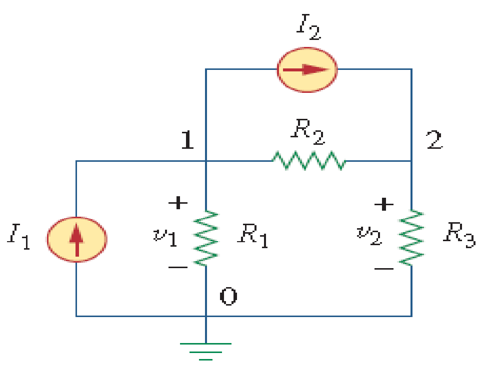
The second important thing should understand this condition with analysis is:-

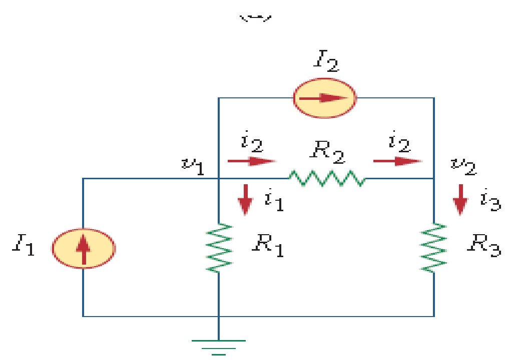
Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}$$

Let analysis this circuit shown below





This circuit is contain three node , one of them is reference , is node 0

- At node 1, applying KCL gives

$$I_1 = I_2 + i_1 + i_2 \quad \text{...........(3.1)}$$

- At node 2,

$$I_2 + i_2 = i_3 \quad \text{.......(3.2)}$$

- Voltage analysis of the circuit based on the **a higher potential to a lower potential in a resistor**

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

.. .. (3.3)

Sub. The equation 3.3 in to 3.1 and 3.2 results

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

❖ Method 2

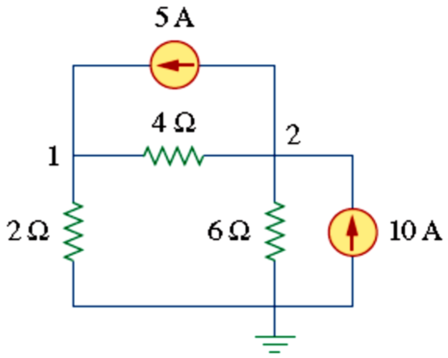
$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2) \quad \text{.... (3.4)}$$

$$I_2 + G_2(v_1 - v_2) = G_3 v_2 \quad \text{......(3.5)}$$

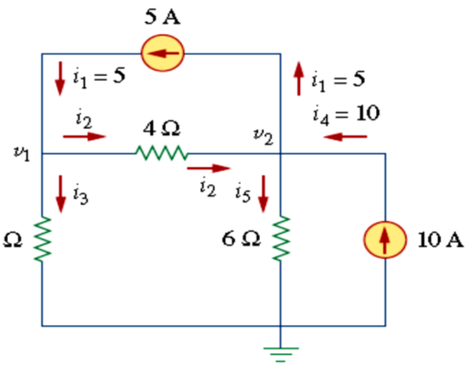
Cramer's rule, or matrix inversion. To use either of the last two methods, one must cast the simultaneous equations in matrix form. For example, Eqs. (3.4) and (3.5) can be cast in matrix form as:

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Example: Calculate the node voltages in the circuit shown in Figure below.



Solution: - Consider Figure shown below for analysis



At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

Multiplying each term in the last equation by 4, we obtain

$$20 = v_1 - v_2 + 2v_1$$

or

$$3v_1 - v_2 = 20$$

At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

$$-3v_1 + 5v_2 = 60$$

❖ **METHOD 1:-** Using the elimination technique,

$$4v_2 = 80 \quad \Rightarrow \quad v_2 = 20\text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \quad \Rightarrow \quad v_1 = \frac{40}{3} = 13.333\text{ V}$$

❖ **METHOD 2:-** To use Cramer's rule,

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

The determinant of the matrix is

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

We now obtain v_1 and v_2 as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333\text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20\text{ V}$$

Note: - giving us the same result as did the elimination method.

Then finds the current values

$$i_1 = 5\text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668\text{ A}, \qquad i_3 = \frac{v_1}{2} = 6.666\text{ A}$$

$$i_4 = 10\text{ A}, \qquad i_5 = \frac{v_2}{6} = 3.333\text{ A}$$

Example:- Determine the voltages at the nodes in Figure 3.2(a)

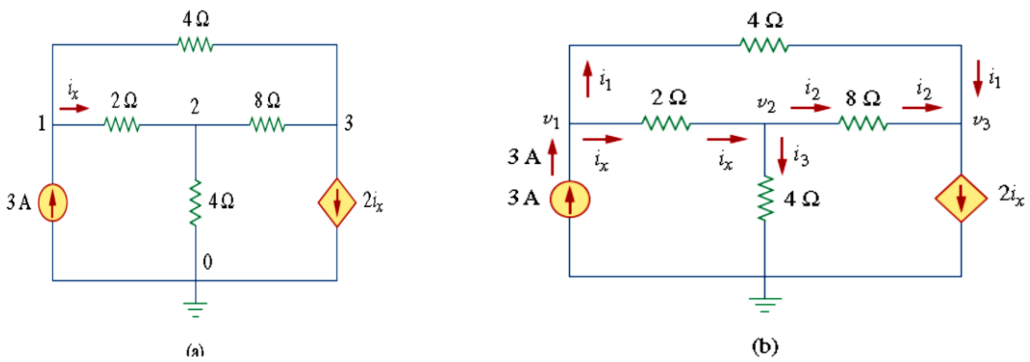


Figure 3.2: (a) original circuit, (b) circuit for analysis.

Solution :- At node 1,

$$3 = i_1 + i_x \quad \Rightarrow \quad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

Multiplying by 4 and rearranging terms, we get

$$3v_1 - 2v_2 - v_3 = 12 \qquad \text{.....(3.6)}$$

At node 2,

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

Multiplying by 8 and rearranging terms, we get

$$-4v_1 + 7v_2 - v_3 = 0 \qquad \text{......(3.7)}$$

At node 3,

$$i_1 + i_2 = 2i_x \quad \Rightarrow \quad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

Multiplying by 8, rearranging terms, and dividing by 3, we get

$$2v_1 - 3v_2 + v_3 = 0 \qquad \text{......(3.8)}$$

We have three simultaneous equations to solve to get the node voltages v_1,v_2 and v_3 We shall solve the equations in three ways.

- **METHOD 1 :-** Using the elimination technique, we add Eqs. (3.6) and (3.7).

$$5v_1 - 5v_2 = 12$$

or

$$v_1 - v_2 = \frac{12}{5} = 2.4 \qquad \text{.....(3.9)}$$

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Adding Eqs. (3.7) and (3.8) gives

$$-2v_1 + 4v_2 = 0 \qquad \Rightarrow \qquad v_1 = 2v_2 \qquad \text{....(3.10)}$$

Substituting Eq. (3.10) into Eq. (3.9) yields

$$2v_2 - v_2 = 2.4 \qquad \Rightarrow \qquad v_2 = 2.4, \qquad v_1 = 2v_2 = 4.8\text{ V}$$

Thus

$$v_1 = 4.8\text{ V}, \qquad v_2 = 2.4\text{ V}, \qquad v_3 = -2.4\text{ V}$$

❖ **METHOD 2** To use Cramer's rule, we put Eqs. (3.6) to (3.8) in matrix form

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

From this, we obtain

$$v_1 = \frac{\Delta_1}{\Delta}, \qquad v_2 = \frac{\Delta_2}{\Delta}, \qquad v_3 = \frac{\Delta_3}{\Delta}$$

where Δ, Δ_1, Δ_2, and Δ_3 are the determinants to be calculated as follows. As explained in Appendix A, to calculate the determinant of a 3 by 3 matrix, we repeat the first two rows and cross multiply.

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{array}{c|ccc|c} & 3 & -2 & -1 & \\ & -4 & 7 & -1 & \\ & 2 & -3 & 1 & \\ - & 3 & -2 & -1 & + \\ - & -4 & 7 & -1 & + \\ - & & & & + \end{array}$$

$$= 21 - 12 + 4 + 14 - 9 - 8 = 10$$

Similarly, we obtain

$$\Delta_1 = \begin{array}{c|ccc|c} & 12 & -2 & -1 & \\ & 0 & 7 & -1 & \\ & 0 & -3 & 1 & \\ - & 12 & -2 & -1 & + \\ - & 0 & 7 & -1 & + \\ - & & & & + \end{array} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{array}{c|ccc|c} & 3 & 12 & -1 & \\ & -4 & 0 & -1 & \\ & 2 & 0 & 1 & \\ - & 3 & 12 & -1 & + \\ - & -4 & 0 & -1 & + \\ - & & & & + \end{array} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{array}{c|ccc|c} & 3 & -2 & 12 & \\ & -4 & 7 & 0 & \\ & 2 & -3 & 0 & \\ - & 3 & -2 & 12 & + \\ - & -4 & 7 & 0 & + \\ - & & & & + \end{array} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

Thus, we find

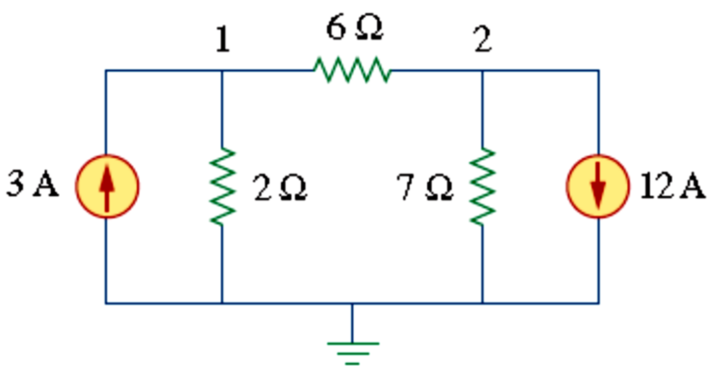
$$v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8 \text{ V}, \qquad v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4 \text{ V}$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4 \text{ V}$$

as we obtained with Method 1.

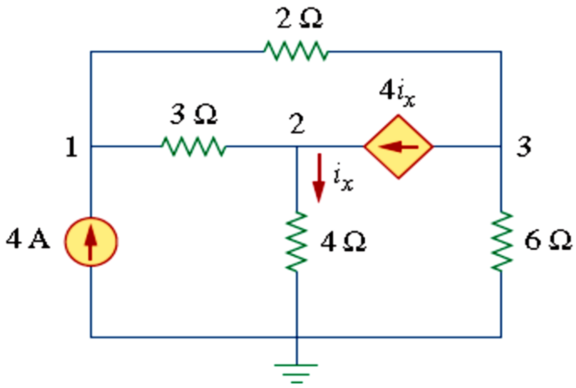
Homework:

❖ Obtain the node voltages in the circuit of Figure shown below



Answer: v_1 =6 V, v_2 =42 V.

❖ Find the voltages at the three no reference nodes in the circuit of Figure shown below .



Answer: $v_1 = 32$ V, $v_2 = -25.6$ V, $v_3 = 62.4$ V.

3.2 Nodal Analysis with Voltage Sources

We now consider how voltage sources affect nodal analysis. We use the circuit in Fig. 3.3 for illustration. Consider the following two possibilities.

■ **CASE 1** If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In Fig. 3.3, for example,

$$v_1 = 10\text{V} \quad \ldots\ldots(3.11)$$

Thus, our analysis is somewhat simplified by this knowledge of the voltage at this node.

■ **CASE 2** If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes

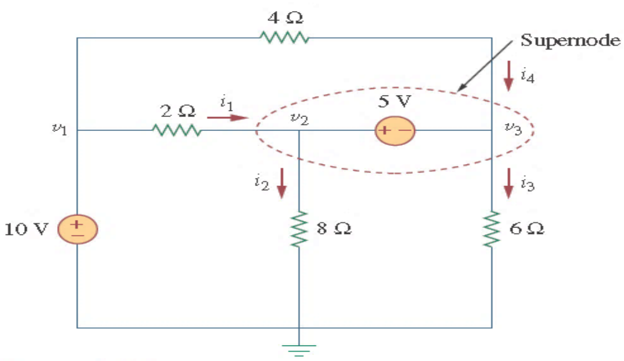


Figure 3.3: A circuit with a supernode.

form a *generalized node* or *supernode*; we apply both KCL and KVL to determine the node voltages.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

However, KCL must be satisfied at a supernode like any other node. Hence, at the supernode in Fig. 3.3,

$$i_1 + i_4 = i_2 + i_3$$

or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6} \qquad ....(3.12)$$

To apply Kirchhoff's voltage law to the supernode in Fig. 3.7, we redraw the circuit as shown in Fig. 3.4. Going around the loop in the clockwise direction gives

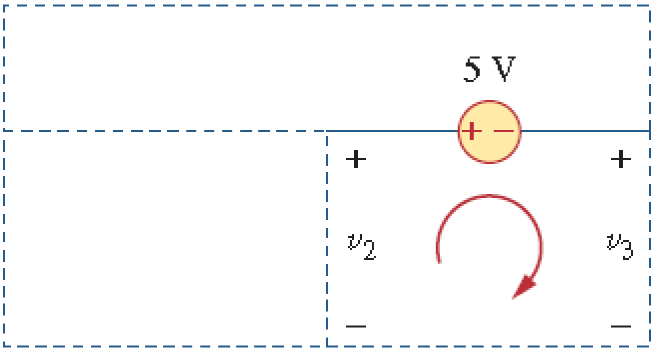


Figure 3.4 : Applying KVL to a supernode.

$$-v_2 + 5 + v_3 = 0 \qquad \Rightarrow \qquad v_2 - v_3 = 5 \qquad ......(3.13)$$

Based on the Eqs. (3.10),(3.11) and (3.12) we find the node voltage .

Note the following properties of a supernode:
1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
2. A supernode has no voltage of its own.
3. A supernode requires the application of both KCL and KVL.

Example :- For the circuit shown in Fig. 3.5, find the node voltages

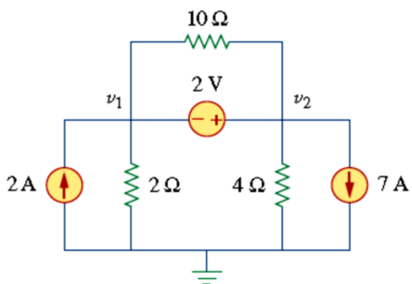


Figure 3.5 :- circuit For Example.

Solution:

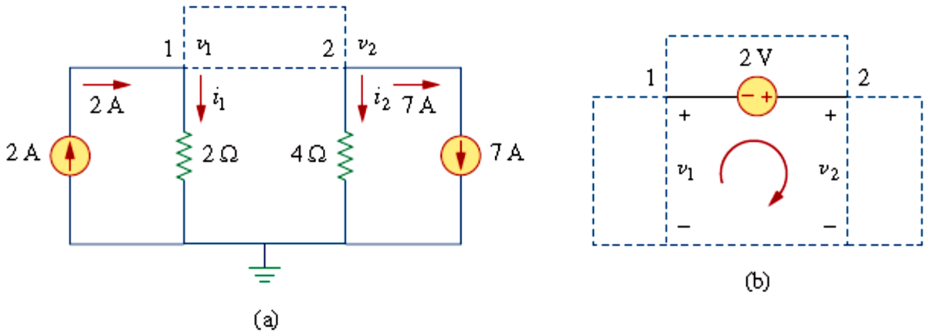


Figure 3.6: (a) KCL to the supernode, (b) KVL to the loop.

The supernode contains the 2-V source, nodes 1 and 2, and the 10Ω resistor. Applying KCL to the supernode as shown in Figure 3.6 (a) gives

$$2 = i_1 + i_2 + 7$$

Expressing i_1 and i_2 in terms of the node voltages

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \quad \Rightarrow \quad 8 = 2v_1 + v_2 + 28$$

or

$$v_2 = -20 - 2v_1 \qquad \text{......(3.14)}$$

To get the relationship between v_1 and v_2 we apply KVL to the circuit in Figure 3.6(b). Going around the loop, we obtain

$$-v_1 - 2 + v_2 = 0 \quad \Rightarrow \quad v_2 = v_1 + 2 \qquad \text{.... (3.15)}$$

From Eqs. (3.14) and (3.15), we write

$$v_2 = v_1 + 2 = -20 - 2v_1$$

or

$$3v_1 = -22 \quad \Rightarrow \quad v_1 = -7.333 \text{ V}$$

And

$$v_2 = v_1 + 2 = -5.333 \text{ V}$$

Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.

Example : Find the node voltages in the circuit of Figure 3.7.

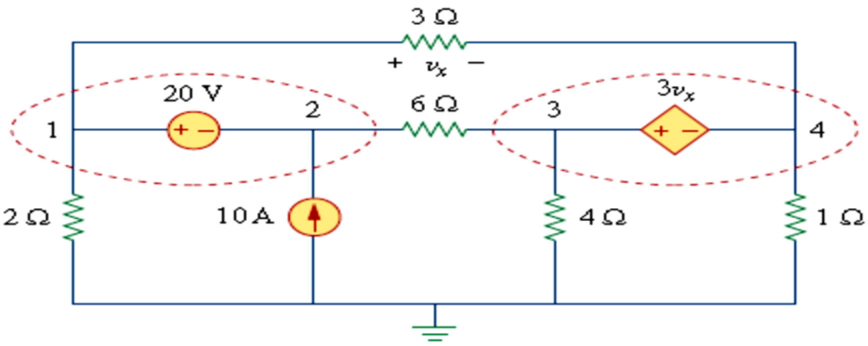


Figure 3.7

Solution:

Nodes 1 and 2 form a supernode; so do nodes 3 and 4. We apply KCL to the two supernodes as in Figure (3.8(a)). At supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

Expressing this in terms of the node voltages,

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

or

$$5v_1 + v_2 - v_3 - 2v_4 = 60 \qquad \text{....(3.16)}$$

At supernode 3-4,

$$i_1 = i_3 + i_4 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

or

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \qquad \text{....(3.17)}$$

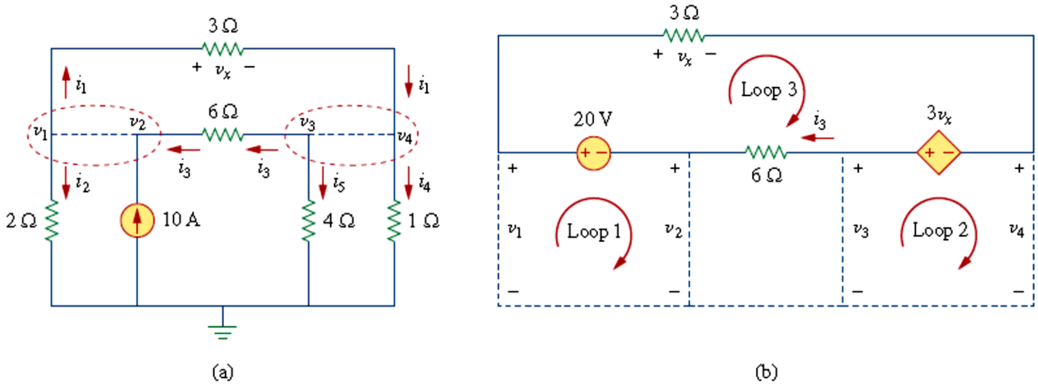


Figure 3.8 : (a) KCL to the two supernodes, (b) KVL to the loops.

We now apply KVL to the branches involving the voltage sources as shown in Figure 3.8(b). For loop 1,

$$-v_1 + 20 + v_2 = 0 \quad \Rightarrow \quad v_1 - v_2 = 20 \qquad ...(3.18)$$

For loop 2,

$$-v_3 + 3v_x + v_4 = 0$$

But $v_x = v_1 - v_4$ so that

$$3v_1 - v_3 - 2v_4 = 0 \qquad ...(3.19)$$

For loop 3,

$$v_x - 3v_x + 6i_3 - 20 = 0$$

But $6i_3 = v_3 - v_2$ and $v_x = v_1 - v_4$. Hence,

$$-2v_1 - v_2 + v_3 + 2v_4 = 20 \qquad ...(3.20)$$

From Eq. (3.18) $v_2=v_1-20$, Substituting this into Eqs. (3.16) and (3.17), respectively, gives

$$6v_1 - v_3 - 2v_4 = 80 \qquad \textbf{.....(3.21)}$$

and

$$6v_1 - 5v_3 - 16v_4 = 40 \qquad \textbf{.....(3.22)}$$

Based on the Eqs. [(3.19),(3.21),(3.22)] can be cast in matrix form as

$$\begin{bmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 40 \end{bmatrix}$$

Using Cramer's rule gives

$$\Delta = \begin{vmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{vmatrix} = -18, \qquad \Delta_1 = \begin{vmatrix} 0 & -1 & -2 \\ 80 & -1 & -2 \\ 40 & -5 & -16 \end{vmatrix} = -480,$$

$$\Delta_3 = \begin{vmatrix} 3 & 0 & -2 \\ 6 & 80 & -2 \\ 6 & 40 & -16 \end{vmatrix} = -3120, \qquad \Delta_4 = \begin{vmatrix} 3 & -1 & 0 \\ 6 & -1 & 80 \\ 6 & -5 & 40 \end{vmatrix} = 840$$

Thus, we arrive at the node voltages as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{-480}{-18} = 26.67 \text{ V}, \qquad v_3 = \frac{\Delta_3}{\Delta} = \frac{-3120}{-18} = 173.33 \text{ V},$$

$$v_4 = \frac{\Delta_4}{\Delta} = \frac{840}{-18} = -46.67 \text{ V}$$

$$v_2 = v_1 - 20 = 6.667 \text{ V}.$$

Homework

❖ Find v and i in the circuit of Figure (3.9)

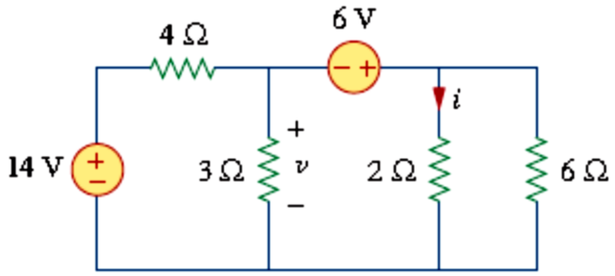


Figure 3.9

Answer: -400 mV, 2.8 A.

❖ Find v_1 and v_2 in the circuit of Figure (3.10) using nodal analysis

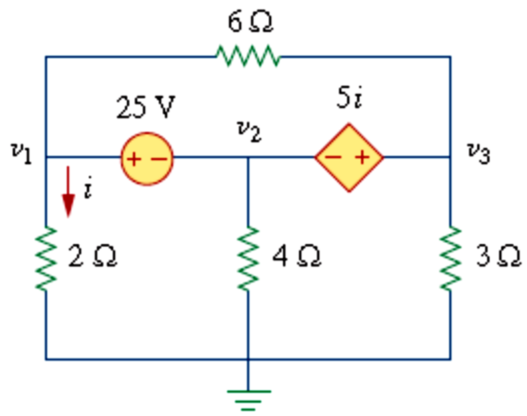


Figure 3.10

Answer: $v_1 = 7.608$ V, $v_2 = -17.39$ V, $v_3 = 1.6305$ V.